The Auslander conjecture
Where is Auslander's mistake?

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§ 1 Background

Flat affine manifolds. A *flat affine manifold* is a differentiable manifold M equipped with an affine connection whose torsion and curvature tensors vanish everywhere. If we further assume that M is geodesically complete, then it turns out that M can be written as a quotient manifold $\mathbb{R}^n/\Gamma$, where $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$ is a group of affine transformations of $\mathbb{R}^n$ that acts freely and properly discontinously on $\mathbb{R}^n$ (see Thurston's book [16] for the definitions and a detailed discussion of these properties). In particular, Γ is the fundamental group of M. Of particular interest is the special case where M is compact. In this case we say that Γ acts *cocompactly* on $\mathbb{R}^n$. There are two very readable surveys on this topic, one by Abels [1] from 2001, and one by Goldman [10] from 2014. Therefore, we shall not go into the details of the theory any further.

The Milnor conjecture. Milnor studied the fundamental groups of flat affine manifolds in his seminal 1977 paper [14]. He observed that a discrete subgroup of a Lie group with finitely many connected components is *either virtually polycyclic or contains a free subgroup with two generators*. This follows directly from a result by Tits [17]. Milnor's main result is [14, Theorem 1.2] stating that *a torsion-free virtually polycyclic group can be realized as the fundamental group of some complete flat affine manifold*. It seemed natural to conjecture the converse, that in fact *the fundamental group of every complete flat affine manifold is virtually polycyclic*. However this conjecture turned out to be false, as Margulis's counterexample in [13] shows. However, this counterexample is a free group Γ acting non-cocompactly by affine Lorentz transformations on $\mathbb{R}^3_1$, the three-dimensional affine space equipped with a Lorentzian scalar product. This allows the more restricted conjecture that *the fundamental group group of every compact complete flat affine manifold is virtually polycyclic*. This has somewhat generously become known as *Auslander's conjecture*.

Auslander's claim. More than a decade before Milnor's paper, Auslander published a paper [6] in 1964 that stated Milnor's original conjecture as a theorem, namely that the fundamental group of every complete flat affine manifold is virtually polycyclic, without the assumption of compactness. In light of Margulis's counterexample, the statement is false in this generality, so there is a mistake in the proof of the theorem or one of the lemmas leading up to it. This will be discussed in § 2 below. When Milnor wrote his article, he was well aware of Auslander's work, as the bibliography in [14] shows, but did not cite Auslander's paper [6]. So it seems reasonable to assume that Milnor noticed a flaw in Auslander's argument or found its proof too confusing to be credible, but did not address this issue in his own text.

A counterexample by Margulis. Margulis constructed a counterexample to Milnor's conjecture in the non-compact case in 1984 [13]. The construction is rather intricate, and its details are not of much interest to us right now. A sketch is given in Abels's survey [1, Section 8]. The counterexample is a free group in two generators $\Gamma \subset \mathbf{Aff}(\mathbb{R}^3_1)$ whose linear part is contained in the Lorentz group $\mathbf{SO}_{2,1}$. The group Γ acts properly discontinuously on $\mathbb{R}^3_1$. It worth noting for later that if $\mathrm{L} : \mathbf{Aff}(\mathbb{R}^3_1) \to \mathbf{SO}_{2,1}$ denotes the projection to the linear part of an affine transformation, then $\mathrm{L}(\Gamma) \cong \Gamma$, since the kernel of L is the abelian group $\mathbb{R}^3_1$ and thus $\Gamma \cap \mathbb{R}^3_1 = \mathbf{1}$. In addition, $\mathrm{L}(\Gamma)$ is discrete in $\mathbf{SO}_{2,1}$. In fact, a theorem of Wang [19, Appendix] implies that the identity component $(\overline{\mathrm{L}(\Gamma)})^\circ$ of the closure of the linear part is solvable since Γ° is trivial, and since $\mathrm{L}(\Gamma)$ is a free group, $(\overline{\mathrm{L}(\Gamma)})^\circ$ is necessarily trivial.

State of the conjecture. At the time of this writing (September 12, 2017), the Auslander conjecture has been confirmed in several special cases. In dimensions up to three, it follows from the classification of affine crytallographic groups due to Fried and Goldman [8] from 1983. In 2016 Tomanov [18] published a proof for Auslander conjecture up to dimension five. Abels, Margulis and Soifer announced that the conjecture holds at least up to dimension seven, but their paper containing the proof was withdrawn from arXiv.org. It is also known that the Auslander conjecture holds if the flat affine structure comes from a flat Lorentzian metric or a flat metric of signature $(n-2, 2)$. The first fact was proved by Goldman and Kamishima [9], see also the classification results by Grunewald and Margulis [11], and the second one was proved by Abels, Margulis and Soifer [2].

§ 2 Auslander's original claim

For easier reference, we reproduce the critical parts of Auslander's original paper verbatim, fixing only obvious typos. These are Lemma 1, Lemma 2 and the proof of Theorem 1 in §1 of Auslander's 1964 paper.

Note that references in the numbered lines below refer to Auslander's original article and not to the references in this text. To avoid confusion of Auslander's references with our own, we replace the references $[n]$ in Auslander's original text by [An].

First, we recall some of Auslander's notation. A semidirect product of two groups S and H is written $G = S \cdot H$, where S is the normal subgroup. If π_S denotes the projection map to S, then Auslander writes $A(g)(s) = \pi_S(gs)$ for the action of an element $g \in G$ by an automorphism of S.

The maximal solvable normal subgroup (not necessarily connected) of a given Lie group G is called the *radical* of G by Auslander and denoted by $r(\Gamma)$.

Lemma 1 *Let $G \cdot H$, where N is a simply connected nilpotent analytic group and H is a subgroup of the group of continuous automorphisms of N. Let $\Gamma \subset G$ be such that $A(\Gamma)$ operates properly discontinuously on N. Let $\Gamma_1 = \Gamma \cap N$ and let $N_1 \subset N$ be the analytic subgroup of N which is uniquely determined by the properties that $N_1 \supset \Gamma_1$ and N_1/Γ_1 is compact. Assume further that Γ/Γ_1 contains no elements of finite order. Then $A(\Gamma/\Gamma_1)$ operates properly discontinuously on N/N_1.*

PROOF: It is trivial to verify that under $A(\Gamma)$ the cosets $N_1 n_0$, $n_0 \in N$, are permuted amongst themselves. It is also easily verified that the image of each coset of N_1 in $N/A(\Gamma)$ is homeomorphic to N_1/Γ_1 and hence compact. The image of a coset $N_1 n_0$ in $N/A(\Gamma)$ will be called a sheet. Our problem is to prove that this sheeting of $N/A(\Gamma)$ actually gives rise to a fiber bundle structure of $N/A(\Gamma)$ pver $N/A(N_1\Gamma)$ with fiber N_1/Γ. The conclusion of our lemma would follow trivially from this. The proof that the sheeting gives rise to a fiber bundle structure of the type indicated requires only that we prove the existence of local cross sections.

Clearly the sheeting of $M = N/A(\Gamma)$ gives rise to an involutive distribution and hence we may apply the Frobenius theorem. let $m_0 \in M$ and let $Y(m_0)$ be the sheet through m_0. Let U_i, $i = 1, \ldots, k$, be local coordinates which cover $Y(m_0)$

and are such that if $(X_{i1}, \ldots, X_{in})$ are the coordinates on U_i, then

1. $X_{i1} = C_1, \ldots, X_{ik} = C_k$ correspond to the connected components of the intersection of a sheet with U_i.
2. $Y(m_0) \cap U_i$ is given by $X_{i1} = 0, \ldots, X_{ik} = 0$.
3. If a sheet Y meets $\bigcap U_i$, then $Y \subset \bigcap U_i$.
4. Consider $y_0 \in Y \subset \bigcap U_i$. Let $y_0 \in U_{j_0}$, say. Then y_0 determines a connected component of $Y \cap U_{j_0}$. Call this $W(y_0)$. We may consider $W(y_0) \cap U_i, i = 1, \ldots, k$.

   We may also consider $U_{j_0} \cap Y(m_0)$ and the collection of sets $U_{j_0} \cap Y(m_0) \cap U_i$.

   We will require that $W(y_0) \cap U_i$ be empty if and only if $U_{j_0} \cap Y(m_0) \cap U_i)$ is empty.

It is clear that coordinate neighborhoods exist with these four conditions. We will have proven our lemma once we have shown that each sheet $Y \subset \bigcap U_i$ meets each U_i in exactly one connected component.

Let $U = \bigcap U_i$ and let U^* be a connected lift of U into N. Then we can delete from U^* a cut set G^* such that G^* is invariant under Γ_1 and for each coset of N_1 in U^*, X, $X \backslash X \cap G^*$ is a collection of disjoint fundamental domains for N_1/Γ_1. We now let G be the image of G^* in U and consider $U \backslash G$. We will use Y^1 to denote the restriction of sheets in U to $U \backslash G$.

Now $Y^1(m_0)$ is contractable. Let U_i^1 be the covering of $U \backslash G$ induced by the U_i. Then, since $Y^1(m_0)$ is contractable, we may contract $Y(m_0)$ to a point by contracting $Y^1(m_0) \cap U_{j_0}^1$ first until what is left is covered by the remaining U_i^1. By this method we can contract $Y^1(m_0)$ to a point. But we may do the same for each sheet of Y meeting $U_{j_0}^1$ and since the incidence relations are the same we obtain that each sheet Y^1 can be contracted to a point. We next note that any point in Y beginning and ending in G must be in the same sheet on Y in U. Hence Y must meet U in exactly one sheet. This proves our lemma. $\diamond$

Lemma 2 *Let $G = S \cdot H$ where S is a simply connected solvable Lie group, H is an analytic group and the dot denotes the semidirect product. Let $\Gamma \subset G$ be such that*

1. *$A(\Gamma)$ operates properly discontinuously on S.*

2. *The image of Γ i G/S is discrete and isomorphic to Γ.*

Under the above conditions, Γ is abelian.

PROOF: Let $h(\Gamma)$ denote the image of Γ in H. Consider $G/\Gamma \to G/\Gamma S$. This is a fiber bundle, since ΓS is closed in G with fiber homeomorphic to S and hence to euclidean space. Hence we have a fiber bundle over $H/h(\Gamma)$ with fiber euclidean space. We may let $c : H/h(\Gamma) \to G/\Gamma$ denote a cross section in this bundle. Now we also have the fiber bundle

$$\begin{tikzcd} G/\Gamma \arrow[d, "\pi"'] \\ S\backslash G/\Gamma \simeq S/A(\Gamma) \end{tikzcd}$$

Let $p(t)$, $0 \le t \le 1$, be a closed path in $H/h(\Gamma)$ realizing the fundamental group element $\gamma_0 \in \Gamma$. Then let $p^*(t) = c(p(t))$. Then $\pi(p^*(t))$ realizes the fundamental group element γ_0 in $S/A(\Gamma)$. Now through each point of $p^*(t)$ there passes a fiber of $G/\Gamma \to G/\Gamma S$, say $X(t)$, where $X(t)$ is topologically a euclidean space. Further, $\pi X(t)$ is the covering map of $X(t)$ onto $S/A(\Gamma)$. Consider $p^*(0) \in X(0)$ and let $m_0 = \pi(p^*(0))$. Then Γ has a unique lifting to an action of $X(0)$, once the point m_0 and $p^*(0)$ are specified. Then for each value of $t = t_0$, the path $\pi p^*(t)$ for $0 \le t \le t_0 \le 1$ determines a unique action of Γ on $X(t_0)$ by $\{\pi p^*(t)\}\gamma\{\pi p^*(t)\}^{-1}$ for $0 \le t \le t_0 \le 1$. Then for $t_0 = 1$ we have the lift of Γ to $X(0)$ given by

$$\gamma_0\gamma\gamma_0^{-1}, \quad \gamma \in \Gamma.$$

But since the action of Γ on $X(0)$ is determined uniquely by m_0 and $p^*(1) = p^*(0)$ we have that

$$\gamma_0\gamma\gamma_0^{-1} = \gamma, \quad \gamma \in \Gamma.$$

Hence γ_0 is in the center of Γ. Byt γ_0 was arbitrary and hence our lemma is proven. $\diamond$

Important Observation. This lemma does not depend on the groups explicitly appearing in the statement of the lemma, but rather on the existence of certain fiber bundles with certain mappings.

Lemma 2' *Let X be a topologically euclidean space and let π be a properly discontinuous group of transformations of X. Assume there exists a fiber bundle $\varrho : B \to X/\pi$ with a continuous fiber F such that*

1. *π operates properly discontinuously on F.*

2. *B is a fiber bundle over F/π with euclidean spaces V as fiber.*

3. *For each $V \subset B$, $\varrho|V$ is a covering map of V onto X/π.*

Then π is abelian.

Theorem 1 *Let Γ be a finitely generated fundamental group of a complete locally affine manifold M. Then the radical of Γ, $r(\Gamma)$, is of finite index in Γ.*

PROOF: Let G_1 be the identity component of the algebraic hull of Γ and let G_2 be the group generated by G_1 and the group of pure translations in $A(n)$. Let $G = [G_1, G_1]$ and $\Gamma^* = \Gamma \cap G$. Then if we can prove the theorem for Γ^* we will have proven the theorem for Γ. Now G has the property that its radical N is nilpotent and must consist of unipotent matrices.

Now $N \cap \Gamma^*$ contains $r(\Gamma^*)$ to within finite index and the image of Γ^* in G/N must be discrete and isomorphic to $\Gamma^*/\Gamma^* \cap N$. This follows from the work in [A1]. The crucial point to observe is that $\Gamma^*/\Gamma^* \cap N$ is finitely generated, as the homomorphic image of a finitely generated group, and is a matrix group. Hence there is a normal subgroup of finite index which is torsion free [A15]. Call Γ_1^* the subgroup of Γ^* which maps onto the above normal subgroup in $\Gamma^*/\Gamma^* \cap N$. Then Γ_1^* is normal and of finite index in Γ^*. Once we see that if Γ_1^* has a radical of finite index then Γ^* has a finite index and hence Γ has.

Hence we have the following.

1. $G \supset \Gamma_1^*$ as discrete subgroup.

2. The radical N of G consists of unipotent matrices.

3. $\Gamma_1^*/\Gamma_1^* \cap N$ is torsion free.

4. If N_1 is the algebraic hull of $\Gamma_1^* \cap N$ then $N_1/\Gamma_1^* \cap N$ is compact and $\gamma_1^* N_1 \gamma_1^{*-1} = N_1$.

5. With no essential loss of generality we may also assume that $G = N \cdot H$ where H is contained in the group of continuous automorphisms of N. Clearly $A(\Gamma_1^*)$ acts properly discontinuously on N, since N contains the pure translations.

Hence by Lemma 1 $A(\Gamma_1^*/\Gamma_1^* \cap N)$ acts properyl discontinuously on N/N_1. We may now apply Lemma 2' to conclude that $\Gamma_1^*/\Gamma_1^* \cap N$ is abelian. To see this we take $X = N/N_1$, π equal to $\Gamma_1^*/\Gamma_1^* \cap N$, B to be $N_1\backslash N \cdot H/\Gamma_1^*$, F will be H, and F/π is given by $H/(\text{Image of } \Gamma_1^* \text{ in } H)$. This completes the proof of the theorem. ◇

§ 3 General remarks

We look at some of the issues in Auslander's proofs, as there are many points that are confusing or not obviously correct. It seems to me that, for all the confusion in the proof, the crucial mistake is simply an unwarrented claim on the lifting of loops to the covering space in line 65 of Auslander's proof of his Lemma 2.

Line 19, Line 21: The index k is used with different meaning in these two lines.

Line 46: At the end of the proof of Lemma 1, line 46 and the following, Auslander probably means "connected component" rather than "sheet".

Line 53: This statement contradicts the existence of the Margulis counterexample; take $S = \mathbb{R}^3_1$, $H = \mathbf{SO}_{2,1}$ and let $\Gamma \subset \mathbf{SO}_{2,1} \ltimes \mathbb{R}^3_1 \subset \mathbf{Aff}(\mathbb{R}^3_1)$ be a free group in two generators acting properly discontinuously on $\mathbb{R}^3_1$. Then Γ satisfies the requirements of Lemma 2, compare the discussion in §1, but is clearly not abelian. So Lemma 2 must already be wrong.

Line 58: The appearance of $S\backslash G/\Gamma$ in the diagram below line 58 does not make sense. What Auslander means is $H\backslash G/\Gamma$.

Line 60: The cross section $c : H/h(\Gamma) \to HS/\Gamma S$ need not be globally defined. So for some $\gamma \in \Gamma$ and their corresponding curves $p(t)$, the embedding $p^*(t) = c(p(t))$ might not even be defined.

Note that since $G = HS$ is a semidirect product, there is a global section $\sigma : H/H \cap \Gamma \to G/\Gamma$ coming from the semidirect splitting of groups. However, in general $H/h(\Gamma) \neq H/H \cap \Gamma$, and it is not clear that the former can have a global section into G/Γ.

For example, suppose $c : H/h(\Gamma) \to G/\Gamma$ is only locally defined, on a collection of simply connected neighborhoods covering $H/h(\Gamma)$. Then any loop $p(t)$ can only realize the identity element of the fundamental group Γ, and accordingly for $p^*(t) = c(p(t))$ in G/Γ.

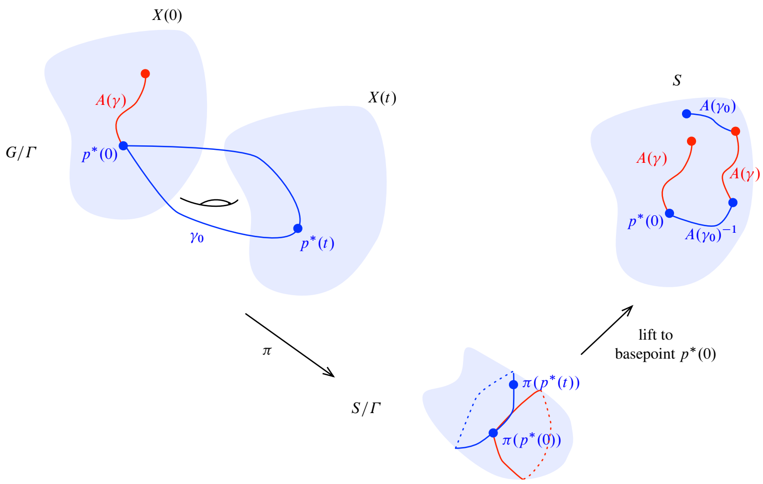
Anyway, it is not clear what purpose the cross section c has, since the fact that the curve $p^*(t)$ in G/Γ is defined as $c(p(t))$ does not seem to be of any relevance in the following arguments.

Line 60, Line 61: It is not true in general that $\pi(p^*(t))$ realizes the fundamental group element $A(\gamma_0)$ of $S/A(\Gamma)$. However, it seems this can be remedied simply by considering the double quotient $\Gamma\backslash G/H \approx A(\Gamma)\backslash S$ instead. So, for the sake of argument, let us assume that $\pi : G/\Gamma \to S/A(\Gamma)$ restricted to the fibers $X(t)$ is indeed the covering map for the action $A(\Gamma)$ on S (and we stick with Auslander's notation of writing $A(\Gamma)$ on the right).

Line 65: This seems to be the crucial mistake in the proof. It is not clear what Auslander wants to say here. Since $X(0) \approx S$ and Γ is the fundamental group, the action of $A(\gamma)$ on S can of course be identified with a curve from $p^*(0)$ to $A(\gamma).p^*(0)$, which is uniquely defined. Auslander states that a Γ-action on $X(t)$ is defined by

$$\pi(p^*(t))\gamma\pi(p^*(t))^{-1}.$$

This presumably means that $\pi(p^*(t))\gamma\pi(p^*(t))^{-1}$ refers to a loop at $\pi(p^*(0))$ in $S/A(\Gamma)$ obtained by traversing $\pi(p^*(\cdot))$ backwards to $\pi(p^*(t))$, then traversing the loop corresponding to γ at this point, and then traverse $\pi(p^*(\cdot))$ again to get back to $\pi(p^*(0))$. If we choose $t = 1$, then this is a concatenation of three loops at $\pi(p^*(0))$ that corresponds to the fundamental group element $\gamma_0\gamma\gamma_0^{-1}$.



Auslander's claims that the "uniqueness" of the lift of Γ to $X(0) \approx S$ means that $\gamma_0\gamma\gamma_0^{-1}$ must be γ. It is not clear to me why this should be the case (or what he even really means by this "uniqueness"), given that none of the above gives us a reason to assume that the loop given by $\gamma_0\gamma\gamma_0^{-1}$ lifts to the same curve in S as the loop given by γ.

Line 86: The group G_2 in the proof of Theorem 1 is never used.

Line 92: Auslander's reference [A1] is his own paper [6] from 1963 on radicals of discrete subgroups of Lie groups. The result from this paper that is used here is [5, Theorem 1], which requires Γ^* to be a uniform subgroup. But this is not assumed in the proof of Auslander's Theorem 1. (Note that Auslander's [5, Theorem 1] is also proved in a more general and clearer form by Wang [19, Appendix].) However, this part of the proof does not contradict Margulis's example, as the properties of Γ^* that allegedly follow from [A1] are satisfied if Γ is a free group in two generators, since then $r(\Gamma^*)$ and $N \cap \Gamma^*$ are trivial.

Line 94: The reference [A15] is Selberg's paper [15] (this paper is hard to find). An elemantary proof was given by Alperin [3].

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