Two theorems on functions in two complex variables

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The properties of functions in one complex variable, given by their function element about the center, have been studied extensively.

Much less effort has been spent on the corresponding questions in the theory of functions in two complex variables. This second questions is all the more interesting as a direct transfer of the methods of the theory of functions in one complex variables to the theory of several variables poses considerable difficulties.

In the present work, we shall derive two theorems from this field of problems on functions in two complex variables by means of the algebra of infinitely many variables.

At this point, I wish to thank Herr Hammerstein for his valuable suggestions in composing this work.

§ 1

In the present work, we shall use the following notation: ξ, η, ζ, τ denote the real Cartesian coordinates of four-dimensional space; $X \equiv \xi + \mathrm{i}\eta$, $Z \equiv \zeta + \mathrm{i}\tau$ denote the complex variables.

$\mathfrak{B}$ denotes a simply connected domain in four-dimensional space that contains the coordinate origin in its interior. The boundary of $\mathfrak{B}$ consists of piecewise regular three-dimensional manifolds.

$\mathfrak{K}$ denotes a circular domain about the origin that is completely contained in the interior of $\mathfrak{B}$.

$a_{\nu\mu}$ is an abbreviation for the sequence $\frac{1}{\sqrt{\iiiint_{\mathfrak{K}} |X^\nu Z^\mu|^2 \mathrm{d}\omega}}$,[1)] where $\mathrm{d}\omega \equiv \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta \mathrm{d}\tau$ is the four-dimensional volume element.

[1)] Since in the present work we will study functions in two complex variables, we will frequently

$G(X, Z)$ denotes a function in two complex variablex X, Z that is regular on the closure of $\mathfrak{K}$, whose function element is

$$\sum_{(\nu\ \mu)=1}^{\infty} b_{\nu\ \mu} X^{\nu} Z^{\mu}. \tag{2}$$

We have:

Theorem. *For every domain $\mathfrak{B}$ we can find a sequence of Hermitian forms in infinitely many variables*

$$f_{(n,m)}(\alpha_{\nu\mu}, \overline{\alpha}_{\nu'\mu'}) \equiv \sum_{(\nu\ \mu)=1}^{\infty} \sum_{(\nu'\ \mu')=1}^{\infty} A^{(n\ m)}_{\nu\ \mu,\nu'\ \mu'} \alpha_{\nu\mu} \overline{\alpha}_{\nu'\mu'} \tag{3}$$

with the following properties: If $G(X, Z)$ is regular on the interior of $\mathfrak{B}$ and the integral[2)]

$$\iiiint_{\mathfrak{B}} G(X, Z)\overline{G(X, Z)} \mathrm{d}\omega,$$

exists, then the limit

$$\lim_{(n\ m)\to\infty} f_{(n,m)} \left(\frac{b_{\nu\mu}}{a_{\nu\mu}}, \overline{\frac{b_{\nu'\mu'}}{a_{\nu'\mu'}}} \right) \tag{4}$$

exists. Conversely, if the limit (4) *exists, then the function $G(X, Z)$ on $\mathfrak{K}$ given by the function element* (2) *can be extended to all of $\mathfrak{B}$, and in $\mathfrak{B}$ the following inequality holds*

$$|G(X, Z)| \leq \frac{\sqrt{2}}{\pi \gamma(X, Z)^2} \sqrt{\lim_{(n\ m)\to\infty} f_{(n,m)} \left(\frac{b_{\nu\mu}}{a_{\nu\mu}}, \overline{\frac{b_{\nu'\mu'}}{a_{\nu'\mu'}}} \right)} \tag{5}$$

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use double indices. For the following, we will fix the order

$$(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots \tag{1}$$

By $(\nu\ \mu) = k$ we will mean the double indices $\nu\ \mu$ of the k-th term in the sequence (1). By $(\nu^0\ \mu^0)$, we will mean the immediate predecessor of $(\nu\ \mu)$ in the sequence.

A power series in two complex variables whose terms have the order (1) is called a *diagonal power series*.

According to Reinhardt (*Über die Abbildungen durch analytische Funktionen zweier Veränderlichen*, Mathematische Annalen 83, §5, p. 211-255), the circular domain is defined as a convex domain whose boundary is given by the equation $\sqrt{\xi^2 + \eta^2} = g(\sqrt{\zeta^2 + \tau^2})$, where g is a continuous real function.

The corresponding theorems for the case of one complex variable were stated in §5 of the article mentioned in footnote 7), p. 12.

[2)]By $\iiiint_{\mathfrak{B}} |v(X, Z)|^2 \mathrm{d}\omega$ we mean $\lim_{\mathfrak{B}_n \to \mathfrak{B}} \iiiint_{\mathfrak{B}_n} |v(X, Z)|^2 \mathrm{d}\omega$ in the usual sense, where $\mathfrak{B}_n$ is a sequence of subdomains of $\mathfrak{B}$ in the interior of $\mathfrak{B}$.

where $\gamma(X, Z)$ is the distance of the point (X, Z) to the boundary of $\mathfrak{B}$.

PROOF: As a first step in the proof, we prove the existence of a doubly orthogonal system $\varphi_{\nu\mu}(X, Z)$. By *doubly orthogonal* we mean that the orthogonality relations holds with respect to the domain $\mathfrak{B}$,

$$\iiiint_{\mathfrak{B}} \varphi_{\nu\mu}\overline{\varphi}_{\nu'\mu'}\mathrm{d}\omega = k_{\nu\mu} \qquad \text{if } (\nu'\ \mu') = (\nu\ \mu) \tag{6a}$$

$$\iiiint_{\mathfrak{B}} \varphi_{\nu\mu}\overline{\varphi}_{\nu'\mu'}\mathrm{d}\omega = 0 \qquad \text{if } (\nu'\ \mu') \neq (\nu\ \mu) \tag{6b}$$

as well as with respect to the angular domain $\mathfrak{K}$,

$$\iiiint_{\mathfrak{K}} \varphi_{\nu\mu}\overline{\varphi}_{\nu'\mu'}\mathrm{d}\omega = 1 \qquad \text{if } (\nu'\ \mu') = (\nu\ \mu) \tag{7a}$$

$$\iiiint_{\mathfrak{K}} \varphi_{\nu\mu}\overline{\varphi}_{\nu'\mu'}\mathrm{d}\omega = 0 \qquad \text{if } (\nu'\ \mu') \neq (\nu\ \mu) \tag{7b}$$

The existence of the system $\varphi_{\nu\mu}$ is shown inductively, by posing the following variational problem (once the existence of the preceding solutions φ_{00}, φ_{10}, φ_{01}, …, $\varphi_{\nu^0\mu^0}$ has been shown): Among all regular functions on the open domain $\mathfrak{B}$, find the one that minimizes

$$\iiiint_{\mathfrak{B}} v(X, Z)\overline{v(X, Z)}\mathrm{d}\omega \tag{8}$$

when substituted for v, and subject to the conditions

$$\begin{aligned}
&\iiiint_{\mathfrak{B}} v(X, Z)\overline{\varphi_{00}(X, Z)}\mathrm{d}\omega = 0,\\
&\iiiint_{\mathfrak{B}} v(X, Z)\overline{\varphi_{10}(X, Z)}\mathrm{d}\omega = 0,\\
&\vdots\\
&\iiiint_{\mathfrak{B}} v(X, Z)\overline{\varphi_{\nu^0\mu^0}(X, Z)}\mathrm{d}\omega = 0,\\
&\iiiint_{\mathfrak{B}} v(X, Z)\overline{v(X, Z)}\mathrm{d}\omega = 1.
\end{aligned} \tag{9}$$

The full proof for the existence of the solution of this variational problem will only be shown for φ_{00}; the proof for arbitrary $\varphi_{\nu\mu}$ is analogous.

The integral (8) has a finite lower bound k_{00}. (Clearly $1 < k_{00} < \frac{\mathrm{vol}(\mathfrak{B})}{\mathrm{vol}(\mathfrak{K})}$.) Therefore, by a theorem proved earlier[3)], for sufficiently late v in a minimal sequence for given ε,

$$v(X,Z) \leq \frac{\sqrt{2\iiiint_{\mathfrak{B}} v(X,Z)\overline{v(X,Z)}\mathrm{d}\omega}}{\pi\gamma(X,Z)^2} = \frac{\sqrt{2(k_{00}+\varepsilon)}}{\pi\gamma(X,Z)^2} \tag{10}$$

is bounded at every inner point of $\mathfrak{B}$.

Choose a sequence of domains $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \ldots \subset \mathfrak{B}_l \subset \ldots \subset \mathfrak{B}$ with limit $\lim_{l\to\infty} \mathfrak{B}_l = \mathfrak{B}$. In $\mathfrak{B}_1$, choose a sequence of inner points $P_1, P_2, \ldots$ that accumulates at an inner point P. By the well-known diagonal method we can pick a subsequence of the minimal sequence that converges on the set $P_1, P_2, \ldots$. Now Vitali's double sequence ansatz yields the uniform convergence of this sequence in every $\mathfrak{B}_l$, since it is bounded there by (10). This defines a boundary function φ_{00}, of which we claim that it has the desired minimal property, that is,

$$\iiiint_{\mathfrak{B}} \varphi_{00}(X,Z)\overline{\varphi_{00}(X,Z)}\mathrm{d}\omega = k_{00}, \tag{11}$$

$$\iiiint_{\mathfrak{K}} \varphi_{00}(X,Z)\overline{\varphi_{00}(X,Z)}\mathrm{d}\omega = 1. \tag{11*}$$

(11*) is clear; (11) is seen as follows: For sufficiently large n,

$$\begin{aligned}
&\iiiint_{\mathfrak{B}} \varphi_{00}(X,Z)\overline{\varphi_{00}(X,Z)}\mathrm{d}\omega \\
&\leq \iiiint_{\mathfrak{B}_l} v_n(X,Z)\overline{v_n(X,Z)}\mathrm{d}\omega + \varepsilon \quad (\text{volume of } \mathfrak{B}_l) \\
&\leq \iiiint_{\mathfrak{B}} v_n(X,Z)\overline{v_n(X,Z)}\mathrm{d}\omega + \varepsilon \quad (\text{volume of } \mathfrak{B}) \\
&\leq k_{00} + \varepsilon' + \varepsilon \quad (\text{volume of } \mathfrak{B})
\end{aligned} \tag{12}$$

where ε' and ε are given small numbers. As the inequality (12) holds for any $\mathfrak{B}_l$, ε', ε, it holds for $\mathfrak{B}$. Since on the other hand

$$\iiiint_{\mathfrak{B}} \varphi_{00}\overline{\varphi}_{00}\mathrm{d}\omega \geq k_{00},$$

[3)]Compare the following articles: *Über Hermitesche Formen, die zu einem Bereich gehören* (preliminary note), Sitzungsberichte der Berliner Mathematischen Gesellschaft 26, 1927, p. 178-184 and *Über unendliche Hermitesche Formen, die zu einem Bereich gehören, nebst Anwendungen auf Fragen der Abbildung durch Funktionen von zwei komplexen Veränderlichen*, Mathematische Zeitschrift 29, 1929, 641-677, Corollary to Hilfssatz 1.

the relation (11) follows.

As stated before, the proof of existence of the other $\varphi_{\nu\mu}$ is analogous.

Now it remains to show that the system $\varphi_{\nu\mu}$ satisfies the relation (6b). To see this, form the expression

$$\psi \equiv a_1\varphi_{\nu\mu} + a_2\varphi_{\nu'\mu'} \quad (\nu'\ \mu') > (\nu\ \mu), \tag{13}$$

where a_1, a_2 satisfy the relation

$$|a_1|^2 + |a_2|^2 = 1. \tag{14}$$

Evidently, ψ satisfies the first $(\nu^0\ \mu^0)$ orthogonal relations (see footnote 1) for $(\nu^0\ \mu^0)$)

$$\iiiint_{\mathfrak{K}} \psi\overline{\varphi}_{00}\mathrm{d}\omega = 0, \iiiint_{\mathfrak{K}} \psi\overline{\varphi}_{10}\mathrm{d}\omega = 0, \ldots, \iiiint_{\mathfrak{K}} \psi\overline{\varphi}_{\nu^0\mu^0}\mathrm{d}\omega = 0, \tag{15}$$

$$\iiiint_{\mathfrak{K}} \psi\overline{\psi}\mathrm{d}\omega = 1. \tag{16}$$

Hence

$$\iiiint_{\mathfrak{B}} \psi\overline{\psi}\mathrm{d}\omega \geq k_{\nu\mu}. \tag{17}$$

On the other hand

$$\iiiint_{\mathfrak{K}} \psi\overline{\psi}\mathrm{d}\omega = |a_1|^2 k_{\nu\mu} + |a_2|^2 k_{\nu'\mu'} + a_1\overline{a}_2 A + a_1\overline{a}_2\overline{A} \tag{18}$$

$$= k_{\nu\mu} + (k_{\nu'\mu'} - k_{\nu\mu})|a_2|^2 + a_1\overline{a}_2 A + \overline{a}_1 a_2\overline{A} \tag{19}$$

(where $A \equiv A_1 + \mathrm{i}A_2 \equiv \iiiint_{\mathfrak{B}} \varphi_{\nu\mu}\overline{\varphi}_{\nu'\mu'}\mathrm{d}\omega$). If we substitute sufficiently small real values for a_2, expand a_1 according to (14) in a neighborhood of $a_2 = 0$ and substitute the thus obtained power series for a_1 in (19), then

$$\iiiint_{\mathfrak{B}} \psi\overline{\psi}\mathrm{d}\omega = k_{\nu\mu} + 2a_2A_1 + a_2^2(k_{\nu'\mu'} - k_{\nu\mu}) + a_2^2A_1 + \ldots. \tag{20}$$

A_1 has to be zero, otherwise we obtain a contradiction to inequality (17). In the same way, A_2 has to be zero.

We will now derive some additional properties of the system $\varphi_{\nu\mu}$.

I. The quantities $k_{\nu\mu}$ clearly form a non-decreasing sequence. But it can happen that several $k_{\nu\mu}$ are identical. We will now show that no infinitely many $k_{\nu\mu}$ can be identical. The functions $\frac{\varphi_{\nu\mu}(X,Z)}{\sqrt{k_{\nu\mu}}}$ form a normalized orthogonal system on $\mathfrak{B}$. In the work cited in footnote 3), it was shown that the kernel of a normalized orthogonal system, that is, the sequence

$$\sum_{(\nu\ \mu)=1}^{\infty} \left| \frac{\varphi_{\nu\mu}(X,Z)}{\sqrt{k_{\nu\mu}}} \right|^2 \tag{21}$$

converges uniformly on every closed inner subdomain of $\mathfrak{B}$. In particular, it is bounded on the closure of $\mathfrak{K}$, and the integral

$$\begin{aligned} &\iiiint_{\mathfrak{K}} \sum_{(\nu\ \mu)=1}^{\infty} \frac{\varphi_{\nu\mu}(X,Z)\overline{\varphi_{\nu\mu}(X,Z)}}{k_{\nu\mu}} \mathrm{d}\omega \\ &\equiv \sum_{(\nu\ \mu)=1}^{\infty} \frac{1}{k_{\nu\mu}} \iiiint_{\mathfrak{K}} \varphi_{\nu\mu}(X,Z)\overline{\varphi_{\nu\mu}(X,Z)} \mathrm{d}\omega \equiv \sum_{(\nu\ \mu)=1}^{\infty} \frac{1}{k_{\nu\mu}} \end{aligned} \tag{22}$$

exists, which follows from (7a). This implies that there cannot exist infinitely many $k_{\nu\mu}$ that are identical. q.e.d.

II. The system $\frac{\varphi_{\nu\mu}}{\sqrt{k_{\nu\mu}}}$ is the solution sequence of the following maximum problem: Determine the function $w(X,Z)$ that maximizes the integral

$$\iiiint_{\mathfrak{K}} w(X,Z)\overline{w(X,Z)} \mathrm{d}\omega, \tag{23}$$

subject to the conditions

$$\begin{aligned} &\iiiint_{\mathfrak{B}} w(X,Z)\frac{\varphi_{00}(X,Z)}{\sqrt{k_{00}}} \mathrm{d}\omega = 0, \iiiint_{\mathfrak{B}} w(X,Z)\frac{\varphi_{10}(X,Z)}{\sqrt{k_{10}}} \mathrm{d}\omega = 0, \dots, \\ &\iiiint_{\mathfrak{B}} w(X,Z)\frac{\varphi_{\nu^0\mu^0}(X,Z)}{\sqrt{k_{\nu^0\mu^0}}} \mathrm{d}\omega = 0, \iiiint_{\mathfrak{B}} w(X,Z)\overline{w(X,Z)} \mathrm{d}\omega = 0, \dots \end{aligned} \tag{24}$$

III. The system $\varphi_{\nu\mu}$ is complete on the domain $\mathfrak{B}$, that is, every regular function $h(X,Z)$ on the open domain $\mathfrak{B}$ with finite integral $\iiiint_{\mathfrak{B}} h(X,Z)\overline{h(X,Z)}\mathrm{d}\omega$ can be expanded into the $\varphi_{\nu\mu}(X,Z)$ on the interior of $\mathfrak{B}$.

To prove this, we assume that there exists a function $H(X,Z)$ with the stated properties that does not satisfy this. Let $E_{\nu\mu}$ denote

$$E_{\nu\mu} \equiv \iiiint_{\mathfrak{B}} H(X,Z)\overline{H(X,Z)} \mathrm{d}\omega - \sum_{(n\ m)=1}^{(\nu\ \mu)} \frac{\left| \iiiint_{\mathfrak{B}} H(X,Z)\overline{\varphi_{\nu\mu}(X,Z)} \mathrm{d}\omega \right|^2}{k_{nm}}. \tag{25}$$

Then $\lim_{(\nu\ \mu)\to\infty} E_{\nu\mu} = 0$ or > 0.

1. If now $\lim_{(\nu\ \mu)\to\infty} E_{\nu\mu} = 0$, then for every given ε we can find a finite $(\nu\ \mu)$ such that

$$E_{\nu\mu} < \varepsilon. \tag{26}$$

But since

$$\iiiint_{\mathfrak{B}} \left| H(X,Z) - \sum_{(n\ m)=1}^{(\nu\ \mu)} \frac{\varphi_{nm} \iiiint_{\mathfrak{B}} H(X,Z)\overline{\varphi_{nm}(X,Z)}\mathrm{d}\omega}{k_{nm}} \right|^2 \mathrm{d}\omega$$

$$= \iiiint_{\mathfrak{B}} H(X,Z)\overline{H(X,Z)}\mathrm{d}\omega - \sum_{(n\ m)=1}^{(\nu\ \mu)} \frac{\left|\iiiint_{\mathfrak{B}} H(X,Z)\overline{\varphi_{nm}(X,Z)}\mathrm{d}\omega\right|^2}{k_{nm}} < \varepsilon, \tag{27}$$

it follows from the theorem mentioned in footnote 3) that

$$\left| H(X,Z) - \sum_{(n\ m)=1}^{(\nu\ \mu)} \frac{\varphi_{nm} \iiiint_{\mathfrak{B}} H(X,Z)\overline{\varphi_{nm}(X,Z)}\mathrm{d}\omega}{k_{nm}} \right| \le \frac{\varepsilon\sqrt{2}}{\pi\gamma(X,Z)^2}, \tag{28}$$

in contradiction to our assumption.

2. If $\lim_{(\nu\ \mu)\to\infty} E_{\nu\mu} > 0$, then $\frac{1}{E_{\nu\mu}}$ is bounded and the function

$$\theta_{\nu\mu}(X,Z) \equiv \frac{1}{\sqrt{E_{\nu\mu}}} \left(H(X,Z) - \sum_{(n\ m)=1}^{(\nu\ \mu)} \frac{\varphi_{nm} \iiiint_{\mathfrak{B}} H(X,Z)\overline{\varphi_{nm}(X,Z)}\mathrm{d}\omega}{k_{nm}} \right) \tag{29}$$

satisfies the first $(\nu\ \mu)$ orthogonality relations

$$\iiiint_{\mathfrak{B}} \theta_{\nu\mu}(X,Z)\frac{\overline{\varphi_{00}(X,Z)}}{\sqrt{k_{00}}}\mathrm{d}\omega = 0, \iiiint_{\mathfrak{B}} \theta_{\nu\mu}(X,Z)\frac{\overline{\varphi_{10}(X,Z)}}{\sqrt{k_{10}}}\mathrm{d}\omega = 0, \ldots,$$
$$\iiiint_{\mathfrak{B}} \theta_{\nu\mu}(X,Z)\frac{\overline{\varphi_{\nu^0\mu^0}(X,Z)}}{\sqrt{k_{\nu^0\mu^0}}}\mathrm{d}\omega = 0, \iiiint_{\mathfrak{B}} \theta_{\nu\mu}(X,Z)\overline{\theta_{\nu\mu}(X,Z)}\mathrm{d}\omega = 1. \tag{30}$$

By II we have

$$\iiiint_{\mathfrak{B}} \theta_{\nu\mu}(X,Z)\overline{\theta_{\nu\mu}(X,Z)}\mathrm{d}\omega \le \frac{1}{k_{\nu\mu}}. \tag{31}$$

Since $\lim_{(\nu\ \mu)\to\infty} \theta_{\nu\mu}(X,Z) = 0$, it follows from the theorem used above that $\lim_{(\nu\ \mu)\to\infty} \theta_{\nu\mu}(X,Z) = 0$, in contradiction to the original assumption.

IV. We will show that the function element of of $\varphi_{\nu\mu}$ has the form

$$\sum_{(\nu\ \mu)=1}^{\infty} O_{\nu\mu,nm} a_{nm} X^n Z^m, \tag{32}$$

where

$$\| O_{\nu\mu,nm} \| \tag{33}$$

is an orthogonal matrix. For the definition of a_{nm}, see p. 1.

By Hartog's Theorem[4)], $\varphi_{\nu\mu}$ can be expanded on $\mathfrak{K}$ into a diagonal series. Since both the system $\varphi_{\nu\mu}$ as well as the system $a_{\nu\mu} X^\nu Z^\mu$ is orthogonal on $\mathfrak{K}$, the matrix (33) is orthogonal.

V. Similarly, we conclude that, if $\mathfrak{B}$ can be mapped bijectively to a circilar domain in V, W-space via a pair of functions $V(X, Z)$ and $W(X, Z)$, and $|V|$ and $|W|$ are less than 1 on $\mathfrak{K}$, then $\varphi_{\nu\mu}$ can also be represented in the form

$$\varphi_{\nu\mu} = \sum_{(n\ m)=1}^{\infty} \Omega_{\nu\mu,nm} V^n W^m \frac{\partial(V, W)}{\partial(X, Z)}, \tag{34}$$

where

$$\| \Omega_{\nu\mu,nm} \| \tag{35}$$

is also an orthogonal matrix.

For proceed with the proof of the first statement. By III, every regular function $G(X, Z)$ on an open domain with finite $\iiiint_{\mathfrak{B}} G(X, Z)\overline{G(X, Z)} \mathrm{d}\omega$ can be expanded into the complete orthogonal system $\varphi_{\nu\mu}$. Hence on $\mathfrak{B}$

$$G(X, Z) = \sum_{(\nu\ \mu)=1}^{\infty} \varphi_{\nu\mu}(X, Z) \frac{1}{k_{\nu\mu}} \iiiint_{\mathfrak{B}} G \overline{\varphi}_{\nu\mu} \mathrm{d}\omega \tag{36}$$

and moreover on $\mathfrak{K}$

$$G(X, Z) = \sum_{(\nu\ \mu)=1}^{\infty} \varphi_{\nu\mu}(X, Z) \iiiint_{\mathfrak{K}} G \overline{\varphi}_{\nu\mu} \mathrm{d}\omega. \tag{37}$$

As the expansion into orthogonal functions is unique, it follows that

$$\iiiint_{\mathfrak{B}} G \overline{\varphi}_{\nu\mu} \mathrm{d}\omega = k_{\nu\mu} \iiiint_{\mathfrak{K}} G \overline{\varphi}_{\nu\mu} \mathrm{d}\omega. \tag{38}$$

[4)] F. Hartog, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Mathematische Annalen 62, 1906, p. 1-88.

As moreover G and φ_{nm} can both be represented by a diagonal power series on the closure of $\mathfrak{K}$, it follows that (38) equals the expression

$$
\begin{aligned}
&k_{\nu\mu}\iiiint_{\mathfrak{K}}\left(\sum_{(n\ m)=1}^{\infty}\frac{b_{nm}}{a_{nm}}X^nZ^m\right)\left(\sum_{(n\ m)=1}^{\infty}O_{\nu\mu,nm}a_{nm}X^nZ^m\right)\mathrm{d}\omega\\
&=k_{\nu\mu}\sum_{(n\ m)=1}^{\infty}\frac{b_{nm}}{a_{nm}}\overline{O}_{\nu\mu,nm}.
\end{aligned}
\tag{39}
$$

From the existence of

$$
\iiiint_{\mathfrak{B}}G(X,Z)\overline{G(X,Z)}\mathrm{d}\omega=\sum_{(n\ m)=1}^{\infty}\frac{|\iiiint_{\mathfrak{B}}G\overline{\varphi}_{\nu\mu}\mathrm{d}\omega|^2}{k_{\nu\mu}}
\tag{40}
$$

follows the existence of

$$
\lim_{(N\ M)\to\infty}\sum_{(\nu\ \mu)=1}^{(N\ M)}\frac{|\iiiint_{\mathfrak{B}}G\overline{\varphi}_{\nu\mu}\mathrm{d}\omega|^2}{k_{\nu\mu}}.
\tag{41}
$$

If we substitute the value obtained in (39) for each $\iiiint_{\mathfrak{B}}G\overline{\varphi}_{\nu\mu}\mathrm{d}\omega$ in (41), then (41) becomes

$$
\begin{aligned}
&\lim_{(N\ M)\to\infty}\sum_{(\nu\ \mu)=1}^{(N\ M)}k_{\nu\mu}\left|\sum_{(\varrho\ \sigma)=1}^{\infty}\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}}\overline{O}_{\nu\mu,\varrho\sigma}\right|\\
&=\lim_{(N\ M)\to\infty}\sum_{(\varrho\ \sigma)=1}^{\infty}\sum_{(\varrho'\ \sigma')=1}^{\infty}\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}}\overline{\frac{b_{\varrho'\sigma'}}{a_{\varrho'\sigma'}}}\left(\sum_{(\nu\ \mu)=1}^{(N\ M)}k_{\nu\mu}O_{\nu\mu,\varrho\sigma}\overline{O}_{\nu\mu,\varrho'\sigma'}\right).
\end{aligned}
\tag{42}
$$

If we write

$$
A_{\varrho\sigma,\varrho'\sigma'}^{(N\ M)}\equiv\sum_{(\nu\ \mu)=1}^{(N\ M)}k_{\nu\mu}O_{\nu\mu,\varrho\sigma}\overline{O}_{\nu\mu,\varrho'\sigma'}
$$

and

$$
f_{(N\ M)}\left(\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}},\overline{\frac{b_{\varrho'\sigma'}}{a_{\varrho'\sigma'}}}\right)\equiv\sum_{(\varrho\ \sigma)=1}^{\infty}\sum_{(\varrho'\ \sigma')=1}^{\infty}A_{\varrho\sigma,\varrho'\sigma'}^{(N\ M)}\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}}\overline{\frac{b_{\varrho'\sigma'}}{a_{\varrho'\sigma'}}},
$$

then (41) becomes

$$
\lim_{(N\ M)\to\infty}f_{(N\ M)}\left(\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}},\overline{\frac{b_{\varrho'\sigma'}}{a_{\varrho'\sigma'}}}\right),
\tag{43}
$$

which proves the first claim of the theorem.

We now turn to the proof of the second claim. If $G(X,Z)$ is regular on $\mathfrak{K}$, then by (37) it can be represented in the form

$$\sum_{(\nu\ \mu)=1}^{\infty} \frac{\varphi_{\nu\mu}(X,Z)}{\sqrt{k_{\nu\mu}}} \sqrt{k_{\nu\mu}} \iiiint_{\mathfrak{K}} G\overline{\varphi}_{\nu\mu} \mathrm{d}\omega. \tag{44}$$

(44) is regular on $\mathfrak{B}$, as by the Schwarz inequality, we have for all X, Z in $\mathfrak{B}$

$$\begin{aligned} &\left| \sum_{(\nu\ \mu)=1}^{(N\ M)} \frac{\varphi_{\nu\mu}(X,Z)}{\sqrt{k_{\nu\mu}}} \sqrt{k_{\nu\mu}} \iiiint_{\mathfrak{K}} G\overline{\varphi}_{\nu\mu} \mathrm{d}\omega \right| \\ &\leq \sqrt{\left(\sum_{(\nu\ \mu)=1}^{(N\ M)} \frac{|\varphi_{\nu\mu}(X,Z)|^2}{k_{\nu\mu}} \right) \left(\sum_{(\nu\ \mu)=1}^{(N\ M)} k_{\nu\mu} \left| \iiiint_{\mathfrak{K}} G\overline{\varphi}_{\nu\mu} \mathrm{d}\omega \right|^2 \right)}. \end{aligned} \tag{45}$$

Now the first factor of the square root expression is bounded in the interior of $\mathfrak{B}$ by a theorem from the article cited in footnote 3), and for every $(N\ M)$ the inequality

$$\sqrt{\sum_{(\nu\ \mu)=1}^{(N\ M)} \frac{|\varphi_{\nu\mu}(X,Z)|^2}{k_{\nu\mu}}} \leq \frac{\sqrt{2}}{\pi\gamma(X,Z)^2} \tag{46}$$

holds. By (39) and (42), the second factor $\sum_{(\nu\ \mu)=1}^{(N\ M)} k_{\nu\mu} \left| \iiiint_{\mathfrak{K}} G\overline{\varphi}_{\nu\mu} \mathrm{d}\omega \right|^2$ has the value $f_{(N\ M)}\left(\frac{b_{\varrho\sigma}}{a_{\varrho\sigma}}, \overline{\frac{b_{\varrho'\sigma'}}{a_{\varrho'\sigma'}}} \right)$, and by assumption we may take the limit $(N\ M) \to \infty$. This implies the uniform convergence of (44) in each closed inner subdomain of $\mathfrak{B}$, which proves the second claim.

§ 2

Let $V(X,Z)$, $W(X,Z)$ be a normalized[5)] pair of functions that is regular on a domain $\mathfrak{B}$. Assume the pair of functions V, W maps the domain $\mathfrak{B}$ bijectively to

[5)] By a *normalized* pair of functions we mean a pair with the properties

$$V(0,0) = 0, \quad \left(\frac{\partial V(X,Z)}{\partial X} \right)_{X,Z=0} = 1, \quad \left(\frac{\partial V(X,Z)}{\partial Z} \right)_{X,Z=0} = 0,$$

$$W(0,0) = 0, \quad \left(\frac{\partial W(X,Z)}{\partial X} \right)_{X,Z=0} = 0, \quad \left(\frac{\partial W(X,Z)}{\partial Z} \right)_{X,Z=0} = 1.$$

a circular domain in V, W-space. By Hartog's Theorem, $V(X,Z)$ and $W(X,Z)$ can be expanded on the circular domain $\mathfrak{K}$ (from §1) into a diagonal power series

$$V(X,Z) \equiv X + \sum_{(\nu\ \mu)=(2\ 0)}^{\infty} c_{\nu\mu} X^{\nu} Z^{\mu}, \qquad W(X,Z) \equiv Z + \sum_{(\nu\ \mu)=(2\ 0)}^{\infty} e_{\nu\mu} X^{\nu} Z^{\mu}. \tag{47}$$

The functional determinant $\frac{\partial(V,W)}{\partial(X,Z)}$ has the following form on $\mathfrak{K}$:[6)]

$$\frac{\partial(V,W)}{\partial(X,Z)} \equiv 1 + \sum_{(\nu\ \mu)=(1\ 0)}^{\infty} d_{\nu\mu} X^{\nu} Z^{\mu}. \tag{48}$$

Then the following theorem holds:

Theorem. *The coefficients $d_{\nu\mu}$ in* (48) *satisfy the inequality*

$$|d_{\nu\mu}| \leq \text{const. } a_{\nu\mu}, \qquad a_{\nu\mu} \equiv \frac{1}{\sqrt{\iiiint_{\mathfrak{K}} |X^{\nu} Z^{\mu}|^2 \mathrm{d}\omega}}. \tag{49}$$

PROOF: To prove (49), consider the Hermitian form belonging to the domain $\mathfrak{B}$,

$$\sum_{(\nu\ \mu)=1}^{\infty} \sum_{(\nu'\ \mu')=1}^{\infty} a_{\nu\mu} \overline{\alpha}_{\nu'\mu'} \iiiint_{\mathfrak{B}} X^{\nu} Z^{\mu} \overline{X^{\nu'} Z^{\mu'}} \mathrm{d}\omega. \tag{50}$$

Without loss of generality, we may assume that $\mathfrak{B}$ is completely contained in the interior of the unit hypersphere. Hence (50) is a fully continuous, positive definite Hermitian form. By the well-known theorem of Hilbert, (50) can be written in the form

$$\sum_{(\nu\ \mu)=1}^{\infty} \mathsf{K}_{\nu\mu} \left(\sum_{(n\ m)=1}^{\infty} o_{\nu\mu,nm} \alpha_{nm} \right) \overline{\left(\sum_{(\nu\ \mu)=1}^{\infty} o_{\nu\mu,nm} \alpha_{nm} \right)}. \tag{51}$$

To find an estimate necessary for the proof, we begin again with a variational problem (different from the one in §1). Namely, determine the minimum of the form (50) under the condition

$$\alpha_{\nu\mu} = 1 \tag{52}$$

[6)]Here, we make the assumption that the function $\frac{\partial(V,W)}{\partial(X,Z)}$ can be uniformly approximated by polynomials in every interior subdomain of $\mathfrak{B}$.

for abritrary but fixed $(\nu\ \mu)$. If we introduce the vectors $\mathfrak{v}_{nm}$ (in Hilbert space) with components

$$\mathfrak{v}_{nm} \equiv \left(\frac{o_{00,nm}}{\sqrt{\mathsf{K}_{00}}}, \frac{o_{10,nm}}{\sqrt{\mathsf{K}_{10}}}, \frac{o_{01,nm}}{\sqrt{\mathsf{K}_{01}}}, \ldots\right) \tag{53}$$

and denote by $*$ the inner product, then we can show: The system of quantities $\beta_{nm}^{(\nu\ \mu)}$ that solves the above variational problem can be written in the form

$$\beta_{nm}^{(\nu\ \mu)} = \frac{\mathfrak{v}_{\nu\mu} * \overline{\mathfrak{v}}_{nm}}{\mathfrak{v}_{\nu\mu} * \overline{\mathfrak{v}}_{\nu\mu}}. \tag{54}$$

The minimum $\lambda_{\nu\mu}$ of (50) under the condition (52) becomes

$$\lambda_{\nu\mu} = \frac{1}{\mathfrak{v}_{\nu\mu} * \overline{\mathfrak{v}}_{\nu\mu}}. \tag{55}$$

For the derivation of the formulas (54) and (55), see the article *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*[7], §2.

Now we give a lower bound for $\lambda_{\nu\varrho}$. By Lemma 2 of the article in footnote 3),

$$\lambda_{\nu\mu} \geq \iiiint_{\mathfrak{K}} |X^\nu Z^\mu|^2 \mathrm{d}\omega = \frac{1}{a_{\nu\mu}^2}. \tag{56}$$

By Satz 3 and 4 in the second article cited in footnote [?], the functional determinant $\frac{\partial(V,W)}{\partial(X,Z)}$ of a pair of functions that maps the domain $\mathfrak{B}$ to a circular domain has the following property: If we substitute the functional determinant $\frac{\partial(V,W)}{\partial(X,Z)}$ for v in the integral

$$\iiiint_{\mathfrak{B}} |v(X,Z)|^2 \mathrm{d}\omega, \tag{57}$$

then this integral is minimized among all regular functions on $\mathfrak{B}$ subject to the condition

$$v(0,0) = 1. \tag{58}$$

Since by assumption $\frac{\partial(V,W)}{\partial(X,Z)}$ can be approximated by polynomials on every subdomain in the interior of $\mathfrak{B}$, the coefficients of the function element of $\frac{\partial(V,W)}{\partial(X,Z)} \equiv D(X,Z)$ are nothing other than the $\beta_{nm}^{(0\ 0)}$, that is,

$$D(X,Z) = 1 + \sum_{(n\ m)=(1\ 0)}^{\infty} d_{nm} X^n Z^m = 1 + \sum_{(n\ m)=(1\ 0)}^{\infty} \beta_{nm}^{(0\ 0)} X^n Z^m. \tag{59}$$

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[7] Mathematische Annalen 86, 1922, p. 238-271.

From (54) it further follows that

$$
\begin{aligned}
|d_{nm}| = |\beta_{nm}^{(0\ 0)}| = \left| \frac{\mathfrak{v}_{00} * \overline{\mathfrak{v}}_{nm}}{\mathfrak{v}_{00} * \overline{\mathfrak{v}}_{00}} \right| &\leq \frac{\sqrt{(\overline{\mathfrak{v}}_{00} * \mathfrak{v}_{00})(\mathfrak{v}_{nm} * \overline{\mathfrak{v}}_{nm})}}{\mathfrak{v}_{00} * \overline{\mathfrak{v}}_{00}} \\
&= \sqrt{\frac{\mathfrak{v}_{nm} * \overline{\mathfrak{v}}_{nm}}{\mathfrak{v}_{00} * \overline{\mathfrak{v}}_{00}}}.
\end{aligned}
\tag{60}
$$

Now, by (55),

$$
|d_{nm}| \leq \frac{a_{nm}}{\sqrt{\mathfrak{v}_{00} * \overline{\mathfrak{v}}_{00}}} \leq \frac{a_{nm}}{\text{vol(circular domain in } V, W\text{-space)}} \equiv \frac{a_{nm}}{\text{const.}}, \tag{61}
$$

which was to be shown.

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