On the kernel function of a domain and its behavior at the boundary

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Part I

§ 1

The Cartesian (rectangular) coordinates of a point P of the four-dimensional Euclidean space $\mathfrak{R}$[1)] are denoted by x_1, y_1, x_2, y_2 and we put

$$z_k = x_k + \mathrm{i}y_k, \quad \overline{z}_k = x_k - \mathrm{i}y_k \quad (k = 1, 2).$$

For a point $P(x_1, y_1, x_2, y_2)$ we also write $P(z_1, z_2, \overline{z}_1, \overline{z}_2)$. In the same manner we replace the real components $\xi_1, \eta_1, \xi_2, \eta_2$ of a vector by the complex ones $\zeta_1, \zeta_2, \overline{\zeta}_1, \overline{\zeta}_2$, where $\zeta_k = \xi_k + \mathrm{i}\eta_k$, $\overline{\zeta}_k = \xi_k - \mathrm{i}\eta_k$ $(k = 1, 2)$, and by a *unit vector* $(\zeta_1, \zeta_2, \overline{\zeta}_1, \overline{\zeta}_2)$ we mean one that satifies

$$|\zeta_1| = 1, \quad |\zeta_2| = 1.$$

The angle θ between two unit vectors $\alpha_1, \alpha_2, \overline{\alpha}_1, \overline{\alpha}_2$ and $\beta_1, \beta_2, \overline{\beta}_1, \overline{\beta}_2$ is given by

$$\cos(\theta) = \frac{1}{2}(\alpha_1\overline{\beta}_1 + \overline{\alpha}_1\beta_1 + \alpha_2\overline{\beta}_2 + \overline{\alpha}_2\beta_2).$$

For a real or complex function φ in the four real variables x_1, y_1, x_2, y_2 we write $\varphi(z_1, z_2, \overline{z}_1, \overline{z}_2)$, or $\varphi(z, \overline{z})$ for short. By

$$\mathrm{d}\varphi = \sum_{k=1}^{2}\left(\frac{\partial\varphi}{\partial x_k}\mathrm{d}x_k + \frac{\partial\varphi}{\partial y_k}\mathrm{d}y_k\right) = \sum_{k=1}^{2}(A_k\mathrm{d}z_k + B_k\mathrm{d}\overline{z}_k) \tag{1.1}$$

[1)]In the following, we will use boldface upper case German letters for four-dimensional manifolds, boldface lower case German letters for three-dimensional manifolds, ordinary upper case German letters for two-dimensional manifolds, and ordinary lower-case letters for one-dimensional manifolds.

Regarding the notation used in the following, note that $\mathfrak{B}[f(z_1, z_2) = \gamma]$ denotes the intersection of the domain $\mathfrak{B}$ with the surface $f(z_1, z_2) = \gamma$.

we define the derivatives

$$\frac{\partial \varphi}{\partial x_k} = A_k, \quad \frac{\partial \varphi}{\partial \overline{z}_k} = B_k,$$

and thus obtain[2)]

$$\frac{\partial \varphi}{\partial z_k} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x_k} - \mathrm{i}\frac{\partial \varphi}{\partial y_k}\right), \quad \frac{\partial \varphi}{\partial \overline{z}_k} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x_k} + \mathrm{i}\frac{\partial \varphi}{\partial y_k}\right).$$

The function φ is called an *analytic* function of the two complex variables z_1, z_2 on a connected domain $\mathfrak{B}$ if it is continously differentiable there and moreover satisfies

$$\frac{\partial \varphi}{\partial \overline{z}_k} = 0 \quad (k = 1, 2), \tag{1.2}$$

the Cauchy-Riemann differential equations. In this case we write

$$\varphi(z_1, z_2, \overline{z}_1, \overline{z}_2) = \varphi(z_1, z_2).$$

In the same way we call a continuously differentiable function on a connected domain $\mathfrak{B}$ analytic in $\overline{z}_1, \overline{z}_2$ if

$$\frac{\partial \varphi}{\partial z_k} = 0 \quad (k = 1, 2),$$

and write accordingly

$$\varphi(z_1, z_2, \overline{z}_1, \overline{z}_2) = \varphi(\overline{z}_1, \overline{z}_2).$$

If $\varphi(z_1, z_2)$ is an analytic function of z_1, z_2, then $\overline{\varphi(z_1, z_2)}$ is an analytic function of $\overline{z}_1, \overline{z}_2$, and we write

$$\overline{\varphi(z_1, z_2)} = \overline{\varphi}(\overline{z}_1, \overline{z}_2).$$

Now let $\mathfrak{B}$ be an open, simple domain of $\mathfrak{R}$ of which we shall assume that it can be mapped to a bounded domain by an analytic map. By $F_{\mathfrak{B}}$ we denote the totality of all regular analytic functions $f(z_1, z_2)$ on $\mathfrak{B}$ for which the integral

$$(f, f)_{\mathfrak{B}} = \int_{\mathfrak{B}} |f(z_1, z_2)|^2 \mathrm{d}\omega \tag{1.3}$$

[2)]Regarding this notation, cf. Wirtinger, *Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen*, Mathematische Annalen 97 (1927), p. 357, and Kneser, *Die singulären Kanten bei analytischen Funktionen mehrerer Veränderlichen*, Mathematische Annalen 106 (1932), p. 656.

is finite, where $d\omega = dx_1 dy_1 dx_2 dy_2$ is the four-dimensional volume element. If the distance between two functions f and g in $F_{\mathfrak{B}}$ is defined by expression

$$d(f,g) = \sqrt{(f-g, f-g)_{\mathfrak{B}}} \tag{1.4}$$

then $F_{\mathfrak{B}}$ forms a metric linear space. As inner product (f,g) of two functions f and g in $F_{\mathfrak{B}}$ we define[3)]

$$(f,g)_{\mathfrak{B}} = \int_{\mathfrak{B}} f(z_1,z_2)\overline{g(z_1,z_2)} d\omega. \tag{1.5}$$

The function f, g are orthogonal if

$$(f,g)_{\mathfrak{B}} = 0. \tag{1.6}$$

For every domain $\mathfrak{B}$ there exists a complete orthonormal system.[4)] If

$$\varphi_1(z_1,z_2), \quad \varphi_2(z_1,z_2), \quad \varphi_3(z_1,z_2), \ldots$$

denotes a complete, normalized orthonormal system, such that

$$(\varphi_\mu, \varphi_\nu) = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases},$$

then every function of $F_{\mathfrak{B}}$ can be expaned into a series

$$f(z_1,z_2) = \sum_{\nu=1}^{\infty} c_\nu \varphi_\nu(z_1,z_2) \tag{1.7}$$

where $c_\nu = (f, \varphi_\nu)_{\mathfrak{B}}$, and the series converges uniformly in the interior of $\mathfrak{B}$.[5)] The latters follows from the fact that the series

$$\mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \sum_{\nu=1}^{\infty} |\varphi_\nu(z_1,z_2)|^2 \tag{1.8}$$

[3)]Compare also (1.3) for this definition.

[4)]The completeness of the system φ_ν means that for every f in $F_{\mathfrak{B}}$, the relation $(f,f)_{\mathfrak{B}} = \sum_{\nu=1}^{\infty} |(f,\varphi_\nu)_{\mathfrak{B}}|^2$ holds, compare Bergmann, *Zwei Sätze über Funktionen von zwei komplexen Veränderlichen*, Mathematische Annalen 100 (1928), p. 399, as well as Bochner, *Über orthogonale Systeme analytischer Funktionen*, Mathematische Zeitschrift 14 (1922), p. 180. More recently, Herr Hammerstein proved, using the theory of complex orthogonal functions, that for a very general class of simply connected domains the orthonormal system is already complete. For these domains, an effective computation of the kernel function is possible. His work appears in Sitzungsberichte Berlin der preußischen Akademie der Wissenschaften, mathematisch-physikalische Klasse, 1933.

[5)]Bergmann, *Über unendliche Hermitesche Formen, die zu einem Bereiche gehören, nebst Anwendungen auf Fragen der Abbildung durch Funktionen von zwei komplexen Veränderlichen.*, Mathematische Zeitschrift 29 (1929), p. 640.

converges to a finite value for every inner point z_1, z_2 of $\mathfrak{B}$. $\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)$, the *kernel function* of the domain $\mathfrak{B}$, depends on the choice of orthonormal system. More generally we introduce the function $\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{t}_1, \overline{t}_2)$ via

$$\mathsf{K}_{\mathfrak{B}}(z; \overline{t}) = \mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{t}_1, \overline{t}_2) = \sum_{\nu=1}^{\infty} \varphi_\nu(z_1, z_2)\overline{\varphi_\nu(t_1, t_2)}, \tag{1.9}$$

where z_1, z_2 and t_1, t_2 are taken from the domain $\mathfrak{B}$. Then

$$\mathsf{M}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1^0, \overline{z}_2^0) = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1^0, \overline{z}_2^0)}{\mathsf{K}_{\mathfrak{B}}(z_1^0, z_2^0; \overline{z}_1^0, \overline{z}_2^0)} \tag{1.10}$$

is the *minimal function* of the domain $\mathfrak{B}$ with respect to the point z_1^0, z_2^0, that is, the function in $F_{\mathfrak{B}}$ that assumes the value 1 in z_1^0, z_2^0 and minimizes $(f, f)_{\mathfrak{B}}$.[6)] This minimum is then given by

$$(\mathsf{M}_{\mathfrak{B}}, \mathsf{M}_{\mathfrak{B}})_{\mathfrak{B}} = \frac{1}{\mathsf{K}_{\mathfrak{B}}(z_1^0, z_2^0; \overline{z}_1^0, \overline{z}_2^0)}, \tag{1.11}$$

which leads to a new definition of the kernel function: *The kernel function can also be defined as the upper bound $|h(z_1^0, z_2^0)|^2$, where $h(z_1, z_2)$ runs through all functions that satisfy*

$$(h, h)_{\mathfrak{B}} \leq 1.$$

This readily implies the following fundamental relation

$$\mathsf{K}_{\mathfrak{B}}(z; \overline{z}) \geq \mathsf{K}_{\mathfrak{B}^*}(z; \overline{z}), \tag{1.12}$$

where

$$\mathfrak{B}^* \subset \mathfrak{B}.$$

[6)]The kernel function and the minimal function were introduced to the theory of functions in one complex variable in Bergmann, *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Mathematische Annalen 86 (1922), p. 237. They are related to the circle map of the domain $\mathfrak{B}$ in a simple manner: If $\mathfrak{B}$ is simply connected and a an inner point of $\mathfrak{B}$, then the circle map $w(z)$ normalized in the point a ($\frac{\mathrm{d}w(z)}{\mathrm{d}z}|_{z=a} = 1$) of $\mathfrak{B}$ is $w(z) = \int_a^z \mathsf{M}_{\mathfrak{B}}(z; \overline{a})\mathrm{d}z$, whereas $\mathsf{K}_{\mathfrak{B}}(a; \overline{a}) = \frac{1}{\pi \mathsf{P}_{\mathfrak{B}}^2(a)}$, where $\mathsf{P}_{\mathfrak{B}}(a)$ denotes the mapping radius of $\mathfrak{B}$ with respect to a.

For multiply connected domains one obtains different kernel and minimal functions, depending on whether one chooses $F_{\mathfrak{B}}$ to be the class of one-valued analytic functions $f(z)$, or whether one further restricts the class by requiring that the integrals $\int_a^z f(z)\mathrm{d}z$ of the functions in the class should be one-valued as well. Compare the work of Zarankiewicz, *Über ein numerisches Verfahren zur konformen Abbildung einfach zusammenhängender Gebiete*, Zeitschrift für angewandte Mathematik und Mechanik 14 (1934), p. 97.

Another property of the kernel function follows from the last definition. Consider the intersection $\mathfrak{B}(z_2 = \gamma)$.[7)] Then *the kernel function assumes its maximum on the boundary of* $\mathfrak{B}(z_2 = \gamma)$. Namely, for every inner point z_1^0 of $\mathfrak{B}(z_2 = \gamma)$ there is a function $f(z_1, \gamma)$ such that

$$|f(z_1^0, \gamma)|^2 > \mathsf{K}_{\mathfrak{B}}(z_1^0, \gamma; \overline{z}_1^0, \overline{\gamma}) - \varepsilon, \qquad \varepsilon > 0. \tag{1.13}$$

This function $f(z_1, \gamma)$ is a function in one complex variable z_1. Its absolute value on the boundary thus assumes a value

$$|f(\zeta, \gamma)|^2 \geq |f(z_1^0, \gamma)^2| \geq \mathsf{K}_{\mathfrak{B}}(z_1^0, \gamma; \overline{z}_1^0, \overline{\gamma}),$$

which implies the claim.

An analytic transformation of the domain is realized by

$$z_k^* = g_k(z_1, z_2) \quad (k = 1, 2) \tag{1.14}$$

where the g_k are regular analytic in $\mathfrak{B}$ and the transformation (1.14) maps the domain $\mathfrak{B}$ bijectively to a simple domain $\mathfrak{B}^*$. The map (1.14) has an inverse

$$z_k = h_k(z_1^*, z_2^*) \quad (k = 1, 2)$$

where the h_k are regular analytic functions in $\mathfrak{B}^*$. The functional determinants

$$D(z_1, z_2) = \frac{\partial(z_1^*, z_2^*)}{\partial(z_1, z_2)}, \quad E(z_1^*, z_2^*) = \frac{\partial(z_1, z_2)}{\partial(z_1^*, z_2^*)}$$

are different from 0 and ∞ on $\mathfrak{B}$ and $\mathfrak{B}^*$, as $D(z_1, z_2)E(z_1^*, z_2^*) = 1$. If (1.14) maps the domain $\mathfrak{A} \subset \mathfrak{B}$ to the domain $\mathfrak{A}^* \subset \mathfrak{B}^*$, then

$$\int_{\mathfrak{A}} \mathrm{d}\omega = \int_{\mathfrak{A}^*} |E(z_1^*, z_2^*)|^2 \mathrm{d}\omega^*,$$

where $\mathrm{d}\omega^* = \mathrm{d}x_1^* \mathrm{d}y_1^* \mathrm{d}x_2^* \mathrm{d}y_2^*$ is the volume element of $\mathfrak{B}^*$. Hence

$$(f, g)_{\mathfrak{B}} = (f(h_1, h_2)E, g(h_1, h_2)E)_{\mathfrak{B}^*}.$$

Via the map

$$f(z_1, z_2) \mapsto f^*(z_1^*, z_2^*) = f(h_1, h_2)E(z_1^*, z_2^*)$$

[7)] $\mathfrak{B}(z_2 = \gamma)$ denotes the intersection of the domain $\mathfrak{B}$ with the plane $z_2 = \gamma$.

the class $F_{\mathfrak{B}}$ of functions is mapped to $F_{\mathfrak{B}^*}$, and the inner product is preserved under this map. In particular, the complete orthonormal system $\varphi_\nu(z_1, z_2)$ is mapped to the complete orthonormal system for the domain $\mathfrak{B}^*$

$$\varphi_\nu^*(z_1^*, x_2^*) = \varphi_\nu(z_1, z_2)E(z_1^*, z_2^*),$$

and the expansion (1.7) becomes

$$f^*(z_1^*, x_2^*) = \sum_{\nu=1}^{\infty} c_\nu \varphi_\nu^*(z_1^*, z_2^*).$$

This further implies that for the kernel function of $\mathfrak{B}^*$,

$$\begin{aligned}
&\mathsf{K}_{\mathfrak{B}^*}(z_1^*, z_2^*; \overline{z}_1^*, \overline{z}_2^*) = \sum_{\nu=1}^{\infty} |\varphi_\nu^*(z_1^*, z_2^*)|^2 \\
&= \mathsf{K}_{\mathfrak{B}}(h_1(z_1^*, z_2^*), h_2(z_1^*, z_2^*); \overline{h_1(z_1^*, z_2^*)}, \overline{h_2(z_1^*, z_2^*)}) \cdot |E(z_1^*, z_2^*)|^2.
\end{aligned} \tag{1.15}$$

The kernel function is thus a relative invariant for analytic maps.

If we set

$$\mathsf{T} = \ln(\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)) \quad \text{and} \quad \mathsf{T}_{z_k \overline{z}_m} = \frac{\partial^2 \mathsf{T}}{\partial z_k \partial \overline{z}_m}$$

then the Hermitian differential form

$$\mathrm{d}s^2 = \sum_{k,m=1}^{2} \mathsf{T}_{z_k \overline{z}_m} \mathrm{d}z_k \mathrm{d}\overline{z}_m \tag{1.16}$$

is positive definite and invariant under analytic transformations $z_1^* = g_k(z_1, z_2)$ ($k = 1, 2$). Considering that by (1.2) and (1.15)

$$\begin{aligned}
&\frac{\partial^2 \ln(\mathsf{K}_{\mathfrak{B}}(z_1^*, z_2^*; \overline{z}_1^*, \overline{z}_2^*))}{\partial z_k^* \partial \overline{z}_m^*} \\
&= \frac{\partial^2 \ln(\mathsf{K}_{\mathfrak{B}}(h_1, h_2; \overline{h}_1, \overline{h}_2))}{\partial z_k^* \partial \overline{z}_m^*} + \frac{\partial^2 \ln(E(z_1^*, z_2^*))}{\partial z_k^* \partial \overline{z}_m^*} + \frac{\partial^2 \ln(\overline{E}(\overline{z}_1^*, \overline{z}_2^*))}{\partial z_k^* \partial \overline{z}_m^*} \\
&= \frac{\partial^2 \ln(\mathsf{K}_{\mathfrak{B}}(h_1, h_2; \overline{h}_1, \overline{h}_2))}{\partial z_k^* \partial \overline{z}_m^*}
\end{aligned}$$

(with $h_k = h_k(z_1^*, z_2^*)$), this is easy to see by a formal computation. Thus $\mathrm{d}s^2$ we have an arc element that yields a metric that is invariant under analytic transformations. That the determinant $N = \begin{vmatrix} \mathsf{T}_{z_1\overline{z}_1} & \mathsf{T}_{z_2\overline{z}_1} \\ \mathsf{T}_{z_1\overline{z}_2} & \mathsf{T}_{z_2\overline{z}_2} \end{vmatrix}$ does not vanish at any inner point $z_1 = z_1^0, z_2 = z_2^0$ of $\mathfrak{B}$ can be seen as follows: If we put

$$\mathsf{T}_{z_k\overline{z}_m} = \frac{\partial^2 \ln(\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2))}{\partial z_k \partial \overline{z}_m} = \frac{1}{\mathsf{K}_{\mathfrak{B}}^2}\left(\mathsf{K}_{\mathfrak{B}}\frac{\partial^2 \mathsf{K}_{\mathfrak{B}}}{\partial z_k \partial \overline{z}_m} - \frac{\partial \mathsf{K}_{\mathfrak{B}}}{\partial z_l}\frac{\partial \mathsf{K}_{\mathfrak{B}}}{\partial \overline{z}_m}\right),$$

then we can write

$$N = \begin{vmatrix} K_{\mathfrak{B}} & \mathsf{K}_{z_1} & \mathsf{K}_{z_2} \\ \mathsf{K}_{\overline{z}_1} & \mathsf{K}_{z_1\overline{z}_1} & \mathsf{K}_{z_2\overline{z}_1} \\ \mathsf{K}_{\overline{z}_2} & \mathsf{K}_{z_1\overline{z}_2} & \mathsf{K}_{z_2\overline{z}_2} \end{vmatrix} \quad \left(\text{where } \mathsf{K}_{z_k} = \frac{\partial \mathsf{K}_{\mathfrak{B}}}{\partial z_k}, \mathsf{K}_{z_k\overline{z}_m} = \frac{\partial^2 \mathsf{K}_{\mathfrak{B}}}{\partial z_k \partial \overline{z}_m}\right).$$

Since $\mathsf{K}_{\mathfrak{B}} = \sum_{\nu=1}^{\infty} \varphi_\nu(z_1, z_2)\overline{\varphi}_\nu(\overline{z}_1, \overline{z}_2)$, it further follows (by the generalized Schwarz relation) that

$$N = \frac{1}{6\mathsf{K}_{\mathfrak{B}}^2}\sum_{\kappa=1}^{\infty}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} |D_{\kappa\mu\nu}(z_1, z_2)|^2,$$

where

$$D_{\kappa\mu\nu}(z_1, z_2) = \begin{vmatrix} \varphi_\kappa(z_1, z_2) & \frac{\partial\varphi_\kappa(z_1,z_2)}{\partial z_1} & \frac{\partial\varphi_\kappa(z_1,z_2)}{\partial z_2} \\ \varphi_\mu(z_1, z_2) & \frac{\partial\varphi_\mu(z_1,z_2)}{\partial z_1} & \frac{\partial\varphi_\mu(z_1,z_2)}{\partial z_2} \\ \varphi_\nu(z_1, z_2) & \frac{\partial\varphi_\nu(z_1,z_2)}{\partial z_1} & \frac{\partial\varphi_\nu(z_1,z_2)}{\partial z_2} \end{vmatrix}.$$

In the last expression for N, all summands are positive. Thus it remains to show that there is one among them that does not vanish. But we can always find an orthonormal system whose first three functions $\varphi_1, \varphi_2, \varphi_3$ can be expanded as follows on a neighborhood of a point z_1^0, z_2^0:

$$\begin{aligned} \varphi_1(z_1, z_2) &= \alpha_0 + \alpha_1(z_1 - z_1^0) + \alpha_2(z_2 - z_2^0) + \dots & \alpha_0 &\neq 0 \\ \varphi_1(z_1, z_2) &= \beta_1(z_1 - z_1^0) + \beta_2(z_2 - z_2^0) + \dots & \beta_1 &\neq 0 \\ \varphi_1(z_1, z_2) &= \gamma_2(z_2 - z_2^0) + \dots & \gamma_2 &\neq 0 \end{aligned}$$

Then

$$D_{\kappa\mu\nu}(z_1^0, z_2^0) = \alpha_0\beta_1\gamma_2 \neq 0.$$

The invariant metric allows us to easily find new invariants. In our unitary geometry there is an important invariant, namely

$$\boldsymbol{I}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)}{\begin{vmatrix} \mathsf{T}_{z_1\overline{z}_1} & \mathsf{T}_{z_2\overline{z}_1} \\ \mathsf{T}_{z_1\overline{z}_2} & \mathsf{T}_{z_2\overline{z}_2} \end{vmatrix}} = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)^4}{\begin{vmatrix} \mathsf{K}_{\mathfrak{B}} & \mathsf{K}_{z_1} & \mathsf{K}_{z_2} \\ \mathsf{K}_{\overline{z}_1} & \mathsf{K}_{z_1\overline{z}_1} & \mathsf{K}_{z_1\overline{z}_2} \\ \mathsf{K}_{\overline{z}_2} & \mathsf{K}_{z_1\overline{z}_2} & \mathsf{K}_{z_2\overline{z}_2} \end{vmatrix}} \tag{1.17}$$

which we investigate in the following.

Let it be remarked that the Hermitian differential form (1.16) does not lead to the most general case of unitary geometry. For (in Schouten's notation)[8] the following relations hold:

$$S_{mp}^{\cdot\cdot n} = 0, \quad -\sum_n R_{\overline{m}np}^{\cdots n} = -\sum_n R_{\overline{m}pn}^{\cdots n} = -\frac{\partial^2 \boldsymbol{I}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)}{\partial \overline{z}_m \partial z_p} + \mathsf{T}_{\overline{z}_m z_p} \tag{1.18}$$

(and similarly for the conjugate quantities).

Instead of (1.16) we can use any differential form conformal to it (arising from it by multiplication of (1.16) by an invariant).

The kernel function $\mathsf{K}_{\mathfrak{B}}(z, \overline{z})$ of a domain $\mathfrak{B}$ is a infinitely differentiable function on the interior of $\mathfrak{B}$; in the following we will study the behavior of this function for a point z_1, z_2 approaching a boundary point Q. In general, the kernel function then becomes infinite. In the following paragraph we will make the mode of approaching the boundary more precise and define the order of becoming infinite.

Let $B(z, \overline{z})$ be a non-negative function on $\mathfrak{B}$. By the class of *generalized square-integrable functions* with weight[9] $B(z, \overline{z})$ we mean those regular analytic functions $f(z_1, z_2)$ on $\mathfrak{B}$ for which the integral (1.3) is finite, where now the measure $\mathrm{d}\omega$ denotes the four-dimensional (possibly non-Euclidean) volume element multiplied by the weight $B(z, \overline{z})$. (For example, we can choose $B(z, \overline{z}) = \mathsf{K}_{\mathfrak{B}}(z, \overline{z})^{-n}$, $n > 0$, or $B(z, \overline{z}) = \mathrm{e}^{-\mathsf{K}_{\mathfrak{B}}(z, \overline{z})}$.)

[8] Compare Schouten's *Über unitäre Geometrie*, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen 32 (1929), p. 457, and Schouten and Danzig, *Über unitäre Geometrie*, Mathematische Annalen 103 (1930), p. 319.

If the line element is given by $\sum_{m,n} g_{\overline{m}n} \mathrm{d}z_n \mathrm{d}\overline{z}_m$, then the associated linear connection is characterized by the vanishing of the covariant differential of $g_{\overline{m}n}$. Then

$$\Gamma_{mp}^n = \sum_r g^{\overline{r}n} \frac{\partial g_{\overline{r}m}}{\partial z_p}, \quad S_{mp}^{\cdot\cdot n} = \Gamma_{mp}^n = \Gamma_{mp}^n$$

and for the components of the curvature tensor

$$R_{\overline{m}np}^{\cdots k} = -R_{n\overline{m}p}^{\cdots k} = -\frac{\partial \Gamma_{pn}^k}{\partial \overline{z}_m}, \dots$$

[9] *Translator's note:* This is “Belegungsfunktion” in the German original.

§ 2

Given the boundary point Q of $\mathfrak{B}$ at which we will study the behavior of the kernel function, we will assume that in a sufficiently small neighborhood of Q the inner points of $\mathfrak{B}$ are given by

$$\Phi(z_1, z_2, \overline{z}_1, \overline{z}_2) > 0, \tag{2.1}$$

whereas the boundary of $\mathfrak{B}$ is formed by the hypersurface

$$\Phi(z_1, z_2, \overline{z}_1, \overline{z}_2) = 0. \tag{2.2}$$

Here, Φ denotes a continously differentiable function in a neighborhood of the boundary point Q that we choose as coordinate origin. Now, if not all derivatives

$$\frac{\partial \Phi}{\partial z_k}, \quad \frac{\partial \Phi}{\partial \overline{z}_k} \quad (k = 1, 2)$$

vanish at Q, then the tangent hyperplane exists at $Q(0, 0)$,

$$\overline{a}_1 z_1 + a_1 \overline{z}_1 + \overline{a}_2 z_2 + a_2 \overline{z}_2 = 0, \tag{2.3}$$

where

$$\overline{a}_k = \left(\frac{\partial \Phi}{\partial z_k}\right)_{\substack{z_1=0\\ z_2=0}}, \quad a_k = \left(\frac{\partial \Phi}{\partial \overline{z}_k}\right)_{\substack{z_1=0\\ z_2=0}} \quad (k = 1, 2).$$

The normal directions at Q are

$$\alpha_k = \frac{a_k}{|a_k|}, \quad \overline{\alpha}_k = \frac{\overline{a}_k}{|a_k|} \quad (k = 1, 2).$$

If we now determine a real τ such that

$$\tau \overline{a}_1 = \mathrm{e}^{\mathrm{i}\gamma_1} \cos(\theta), \quad \tau \overline{a}_2 = \mathrm{e}^{\mathrm{i}\gamma_2} \sin(\theta),$$

and apply the orthogonal transformation

$$\begin{aligned} z_1^* &= \mathrm{e}^{\mathrm{i}\gamma_1} \cos(\theta) z_1 + \mathrm{e}^{\mathrm{i}\gamma_2} \sin(\theta) z_2 \\ z_2^* &= \mathrm{e}^{\mathrm{i}\delta_1} \sin(\theta) z_1 + \mathrm{e}^{\mathrm{i}\delta_2} \cos(\theta) z_2 \end{aligned}, \qquad \gamma_1 - \delta_1 = \gamma_2 - \delta_2, \tag{2.4}$$

the the tangent hyperplane (2.3) is written in the new coordinates

$$z_1^* + \overline{z}_1^* = 0. \tag{2.5}$$

We choose the sign such that the inner normal corresponds to

$$x_1^* \geq 0. \tag{2.6}$$

The coordinatex z_1^* and z_2^* are called *normal* for the point Q. If we reintroduce the variables x_1, y_1 for normal coordinates[10] $z_1, \overline{z}_1$, then by solving for x_1 we can express the boundary hypersurface in the neighborhood of Q by

$$2x_1 = \psi(y_1, z_2, \overline{z}_2), \tag{2.7}$$

where

$$\left(\frac{\partial\psi}{\partial y_1}\right)_{\substack{y_1=0\\z_2=0}} = 0, \quad \left(\frac{\partial\psi}{\partial z_2}\right)_{\substack{y_1=0\\z_2=0}} = 0, \quad \left(\frac{\partial\psi}{\partial \overline{z}_2}\right)_{\substack{y_1=0\\z_2=0}} = 0.$$

The interior points in the neighborhood of Q are then given by the inequality

$$2x_1 > \psi(y_1, z_2, \overline{z}_2). \tag{2.8}$$

When investigating the behavior of the kernel function at the boundary point $Q(0,0)$, we will distinguish between different modes of approaching the boundary:

1. Approach A^{I}*:* By this we mean the convergence

$$z_1 \to 0, \quad z_2 \to 0, \tag{2.9}$$

such that the point $P(z_1, z_2)$ in $\mathfrak{B}$ remains within a cone Ω_α that consists of all rays passing through Q and have an angle less than α with the inner normal. If we use normal coordinates at Q, the the cone Ω_α is given by the inequalities

$$\begin{aligned} x_1 &> 0, \\ \frac{\varrho}{x_1} &= \frac{\sqrt{|z_1|^2 + |z_2|^2}}{x_1} < \frac{1}{\cos(\alpha)}. \end{aligned} \tag{2.10}$$

2. Approach $\mathsf{A}^{\mathrm{II}}(\alpha)$*:* Let z_1, z_2 denote the normal coordinates. In addition to (2.9), we require

$$\operatorname{arc}(z_1) \to \alpha, \quad |\alpha| < \frac{\pi}{2}, \tag{2.11}$$

to hold.

[10] We omit the $*$ from now on.

3. Approach $\mathsf{A}^{\mathrm{III}}(\alpha)$*:* It further restricts $\mathsf{A}^{\mathrm{II}}(\alpha)$. Instead of (2.11) we require

$$\operatorname{arc}(z_1) = \text{const.} = \alpha, \quad |\alpha| < \frac{\pi}{2}. \tag{2.12}$$

4. Approach $\mathsf{A}^{\mathrm{IV}}(a_1, a_2)$*:* Let $P_0(a_1, a_2)$ be an inner point of $\mathfrak{B}$, and assume the segment QP_0 is contained in the interior of $\mathfrak{B}$ with the exception of the point Q. Then $\mathsf{A}^{\mathrm{IV}}(a_1, a_2)$ means the approach along the segment QP_0, that is, along the straight line of points $P(ta_1, ta_2)$, where $t \to 0$.

For the approach A^{I} we let $\overline{\lambda}^{\mathrm{I}}(Q; \mathfrak{B})$ (or $\overline{\lambda}^{\mathrm{I}}(Q)$ or $\overline{\lambda}^{\mathrm{I}}$ for short) denote the *upper order* and $\underline{\lambda}^{\mathrm{I}}(Q; \mathfrak{B})$ (or $\underline{\lambda}^{\mathrm{I}}(Q)$ or $\underline{\lambda}^{\mathrm{I}}$) the *lower order* of the kernel function becoming infinite at $Q(0, 0)$, which mean the lower and upper bound, respectively, of those numbers r and s for which

$$\varrho_{\mathrm{n}}^{r} \mathsf{K}_{\mathfrak{B}}(z, \overline{z}) \leq M < \infty \tag{2.13}$$

and

$$\varrho_{\mathrm{n}}^{s} \mathsf{K}_{\mathfrak{B}}(z, \overline{z}) \geq m > 0 \tag{2.14}$$

holds, if z_1, z_2 converge to $Q(0, 0)$ in the sense of A^{I} and ϱ_{n} denotes the projection of $\varrho = \sqrt{|z_1|^2 + |z_2|^2}$ to the inner normal.

If $r > \overline{\lambda}^{\mathrm{I}}$ and $s < \underline{\lambda}^{\mathrm{I}}$, then clearly $s < r$ and hence generally

$$\underline{\lambda}^{\mathrm{I}} \leq \overline{\lambda}^{\mathrm{I}}.$$

If $\underline{\lambda}^{\mathrm{I}} = \overline{\lambda}^{\mathrm{I}} = \lambda^{\mathrm{I}}$, then we simply speak of the *order* λ^{I}.

We say that the order λ^{I} is *attainable* from above or below, respectively, if

$$\varrho_{\mathrm{n}}^{\lambda^{\mathrm{I}}} K_{\mathfrak{B}}(z, \overline{z}) \leq M < \infty \quad \text{or} \quad \varrho_{\mathrm{n}}^{\lambda^{\mathrm{I}}} K_{\mathfrak{B}}(z, \overline{z}) \geq m > 0, \tag{2.15}$$

respectively. The relations (2.15) for the attainability of the order can be written as

$$\overline{L}^{\mathrm{I}\lambda^{\mathrm{I}}}(Q) = \overline{\lim}\, \varrho_{\mathrm{n}}^{\lambda^{\mathrm{I}}} K_{\mathfrak{B}}(z, \overline{z}) = \overline{L} < \infty,$$
$$\underline{L}^{\mathrm{I}\lambda^{\mathrm{I}}}(Q) = \underline{\lim}\, \varrho_{\mathrm{n}}^{\lambda^{\mathrm{I}}} K_{\mathfrak{B}}(z, \overline{z}) = \underline{L} > 0.$$

We call $\overline{L}$ and $\underline{L}$ the *upper* and *lower limit* at the point Q, respectively. We speak of a *limit of order* λ^{I} if $\underline{L} = \overline{L} = L$, that is, if

$$L_{\mathfrak{B}}^{\mathrm{I}\lambda^{\mathrm{I}}}(Q) = \lim \varrho_{\mathrm{n}}^{\lambda^{\mathrm{I}}} K_{\mathfrak{B}}(z, \overline{z}) = L \tag{2.16}$$

exists and is different from zero. If z_1, z_2 are normal coordinates at the point Q, then

$$\varrho_{\mathrm{n}} = x_1 = \frac{z_1 + \overline{z}_1}{2}. \tag{2.17}$$

In the case of approaches of type $\mathsf{A}^{\mathrm{II}}(\alpha)$ and $\mathsf{A}^{\mathrm{III}}(\alpha)$, in the definition of the limit values we replace ϱ_{n} by the distance ϱ_{t} to the analytic plane that lies in the tangent hyperplane. If we use normal coordinates again, then we obtain

$$\varrho_{\mathrm{t}} = |z_1|. \tag{2.18}$$

The orders $\overline{\lambda}^{\mathrm{II}}$, $\underline{\lambda}^{\mathrm{II}}$, λ^{II} and $\overline{\lambda}^{\mathrm{III}}$, $\underline{\lambda}^{\mathrm{III}}$, λ^{III} and in particular their limit values now depend on the angle α, and in particular we use the notation

$$L_{\mathfrak{B}}^{\mathrm{II}\lambda^{\mathrm{II}}}(Q,\alpha) = \lim \varrho_{\mathrm{t}}^{\lambda^{\mathrm{II}}} K_{\mathfrak{B}}(z,\overline{z}), \quad \operatorname{arc}(z_1) \to \alpha$$

(we use the respective abbreviations for upper or lower limits or in the case of the approach $\mathsf{A}^{\mathrm{III}}(\alpha)$).

In the relations corresponding to (2.13) and (2.14) M and m are functions of α that for any fixed α whose absolute value is less than $\frac{\pi}{2}$ are finite and positive, respectively.

In the case of an approach $\mathsf{A}^{\mathrm{IV}}(a_1, a_2)$, ϱ_{n} and ϱ_{t} are replaced by the distance ϱ of the point $P(z_1, z_2)$ lying on the line QP_0 ($P_0 = P_0(a_1, a_2)$) to the point Q:

$$\varrho = \sqrt{|z_1|_{|}^{2} z_2|^2} = t\sqrt{|a_1|^2 + |a_2|^2},$$

and we introduce the abbreviation

$$L_{\mathfrak{B}}^{\mathrm{IV}\lambda^{\mathrm{IV}}}(Q; a_1, a_2) = \lim_{t \to 0} \Big((t\sqrt{|a_1|^2 + |a_2|^2})^{\lambda^{\mathrm{IV}}} K_{\mathfrak{B}}(ta_1, ta_2; ta_1, ta_2) \Big).$$

In the definition of the order, M and m are now functions of P_0 that are finite and positive, respectively, for every P_0 for which the line segment P_0Q (with the exception of finitely many points) lies in the interior of $\mathfrak{B}$.

§ 3

THE REPLACEMENT DOMAIN. Let

$$z_k^* = g_k(z_1, z_2) \quad (k = 1, 2) \tag{3.1}$$

be an analytic map of the domain $\mathfrak{B}$ with the following properties:

(1) In a certain neighborhood $\mathfrak{U}$ of the boundary point $Q(0,0)$, the functions $g_k(z_1, z_2)$ $(k = 1, 2)$ are still defined on the boundary of $\mathfrak{B}$ by their limits, and are continuously differentiable on the intersection of $\mathfrak{U}$ and $\mathfrak{B}$ augmented by the accumulation points.

(2) For the functional determinant of the map we have

$$\lim_{z_1,z_2\to 0} D(z_1, z_2) = \lim_{z_1,z_2\to 0} \frac{\partial(g_1, g_2)}{\partial(z_1, z_2)} = 1. \tag{3.2}$$

A domain $\mathfrak{B}^*$ that is obtained from $\mathfrak{B}$ by such a transformation is called a *replacement domain* of $\mathfrak{B}$ for the boundary point $Q(0,0)$. Near the image point Q^* of Q the domain $\mathfrak{B}^*$ behaves in the same way as we assumed it for the point Q. In the following we will assume that Q and Q^* coincide.

Assume that the coordinates z_1, z_2 for $\mathfrak{B}$ and z_1^*, z_2^* for $\mathfrak{B}^*$ are normal. Then the transformation (3.1) can be written in a neighborhood of $Q(0,0)$ as

$$z_1^* = \tau z_1 + g_{12}(z_1, z_2), \quad z_2^* = \alpha z_1 + \nu z_2 + g_{22}(z_1, z_2), \quad \tau > 0, \tag{3.3}$$

where

$$|\nu| = \frac{1}{\tau}, \quad \lim_{z_1,z_2\to 0} \frac{g_{12}(z_1, z_2)}{\sqrt{|z_1|^2 + |z_2|^2}} = 0, \quad \lim_{z_1,z_2\to 0} \frac{g_{22}(z_1, z_2)}{\sqrt{|z_1|^2 + |z_2|^2}} = 0. \tag{3.4}$$

We call τ the *measure factor* of the transformation.

If $\mathfrak{B}^*$ is the replacement domain of $\mathfrak{B}$ for the point Q, then with regard to the approaches A^{I} or $\mathsf{A}^{\mathrm{II}}(\alpha)$, the following holds for $\mathfrak{B}$:

(a) *The upper and lower orders in Q for $\mathfrak{B}^*$ coincide with those for $\mathfrak{B}$, so that the existence of a certain order in Q holds simultaneously for $\mathfrak{B}$ and $\mathfrak{B}^*$.*

(b) *If a certain order exists, then it is attainable simultaneously for $\mathfrak{B}$ and $\mathfrak{B}^*$ from above or below, respectively.*

(c) *In case of attainability, the limits $\overline{L}_{\mathfrak{B}^*}(Q)$ and $\underline{L}_{\mathfrak{B}^*}(Q)$ satisfy the following relations,*

$$\overline{L}_{\mathfrak{B}^*}(Q) = \tau^\lambda \overline{L}_{\mathfrak{B}}(Q), \quad \underline{L}_{\mathfrak{B}^*}(Q) = \tau^\lambda \underline{L}_{\mathfrak{B}}(Q),$$

so that Q is a simultaneous limit for $\mathfrak{B}$ and $\mathfrak{B}^*$ of the same order, and then it holds for the limit that

$$L_{\mathfrak{B}^*}(Q) = \tau^\lambda L_{\mathfrak{B}}(Q). \tag{3.5}$$

All these claims follow immediately from the transformation formula (1.15) for the kernel function,

$$\mathsf{K}_{\mathfrak{B}^*}(z_1^*, z_2^*; \overline{z}_1^*, \overline{z}_2^*) = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)}{\left|\frac{\partial(g_1, g_2)}{\partial(z_1, z_2)}\right|^2},$$

as well as the limit equations

$$\lim_{z_1, z_2 \to 0} \frac{\varrho_{\mathrm{n}}^*}{\varrho_{\mathrm{n}}} = \tau, \qquad \lim_{z_1, z_2 \to 0} \frac{\varrho_{\mathrm{t}}^*}{\varrho_{\mathrm{t}}} = \tau,$$

and (3.2), if in addition we prove that the approaches A^{I}, $\mathsf{A}^{\mathrm{II}}(\alpha)$ in $\mathfrak{B}$ are transformed via (3.3) into the approaches of the same type in $\mathfrak{B}^*$, and vice versa.

For this proof we assume normal coordinates to be given for both domains $\mathfrak{B}$ and $\mathfrak{B}^*$, and then we can write the transformations in the form (3.3). First, consider the case of the approach A^{I}. From (3.3) it follows that

$$2x_1^* = 2\tau x_1 + g_{12}(z_1, z_2) + \overline{g}_{12}(\overline{z}_1, \overline{z}_2), \tag{3.6}$$

and thus, by (3.4) and

$$\frac{x_1^*}{x_1} = \tau + \frac{g_{12}(z_1, z_2) + \overline{g}_{12}(\overline{z}_1, \overline{z}_2)}{\sqrt{|z_1|^2 + |z_2|^2}} \frac{\sqrt{|z_1|^2 + |z_2|^2}}{2x_1} \tag{3.7}$$

we have

$$\lim_{z_1, z_2 \to 0} \frac{x_1^*}{x_1} = \tau = \lim \frac{\varrho_{\mathrm{n}}^*}{\varrho_{\mathrm{n}}}$$

if the approach happens within a cone

$$\frac{\sqrt{|z_1|^2 + |z_2|^2}}{x_1} \leq M. \tag{3.8}$$

In the same way one proves (3.7) using the inversion formulas

$$z_1 = \frac{1}{\tau} z_1^* + h_{12}(z_1^*, z_2^*), \quad z_2 = \frac{\alpha}{\tau} z_1^* + \frac{1}{\nu} z_2^* + h_{22}(z_1^*, z_2^*), \tag{3.9}$$

if instead of (3.8) the approach is in a cone

$$\frac{\sqrt{|z_1^*|^2 + |z_2^*|^2}}{x_1^*} \leq N \tag{3.10}$$

in $\mathfrak{B}^*$. On the other hand,

$$
\begin{aligned}
\left|\frac{z_1^*}{x_1}\right| &\leq \tau\left|\frac{z_1}{x_1}\right| + \frac{g_{12}(z_1,z_2)}{\sqrt{|z_1|^2+|z_2|^2}}\frac{\sqrt{|z_1|^2+|z_2|^2}}{x_1},\\
\left|\frac{z_2^*}{x_1}\right| &\leq \alpha\left|\frac{z_1}{x_1}\right| + \frac{1}{\tau}\left|\frac{z_2}{x_1}\right| + \frac{g_{22}(z_1,z_2)}{\sqrt{|z_1|^2+|z_2|^2}}\frac{\sqrt{|z_1|^2+|z_2|^2}}{x_1}\\
&\leq \frac{\sqrt{|z_1|^2+|z_2|^2}}{x_1}\left(\sqrt{|\alpha|^2+\frac{1}{\tau^2}} + \frac{|g_{22}(z_1,z_2)|}{\sqrt{|z_1|^2+|z_2|^2}}\right),
\end{aligned}
$$

which implies

$$\frac{\sqrt{|z_1|^2+|z_2|^2}}{x_1} \leq \frac{\sqrt{|z_1|^2+|z_2|^2}}{x_1}\text{const.}, \tag{3.11}$$

and because of

$$\frac{\sqrt{|z_1^*|^2+|z_2^*|^2}}{x_1^*} = \frac{\sqrt{|z_1^*|^2+|z_2^*|^2}}{x_1}\frac{x_1}{x_1^*}$$

it follows, taking into account (3.7), that condition (3.10) is satisfied whenever (3.8) is.

In the same manner, by using the inversion formula (3.9), it follows that condition (3.10) implies condition (3.8), which proves the claimed equivalence of the admissibility conditions at the points Q and Q^*.

The invariance of the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$ follows immediately from

$$\lim_{z_1^*\to 0}\frac{y_1^*}{z_1^*} = \lim_{z_1\to 0}\frac{y_1}{x_1}(=\tan(\alpha)), \tag{3.12}$$

as soon as either of the limits exists.

If we assume the existence and continuity of the second derivative of (2.7), then the equation of the boundary surface can be expressed in normal coordiantes in a neighborhood of $Q(0,0)$:

$$2x_1 = ay_1^2 + 2\mathrm{i}y_1(bz_2 - \overline{b}\overline{z}_2) + cz_2^2 + \overline{c}\overline{z}_2^2 + \sigma|z_2|^2 + \psi(y_1,z_2,\overline{z}_2), \tag{3.13}$$

where

$$\lim_{y_1,z_2\to 0}\frac{\psi(y_1,z_2,\overline{z}_2)}{y_1^2+|z_2|^2} = 0, \qquad \alpha,\sigma \text{ real.}$$

We sharpen the definition of a replacement domain by requiring that the functions g_1 and g_2 in the transformation (3.1) have continuous second derivatives in a neighborhood of Q in the closure of $\mathfrak{B}$. Assuming the coordinates in $\mathfrak{B}$ and its replacement domain $\mathfrak{B}^*$ to be normal, the transformation (3.1) can be written as

$$\begin{aligned} z_1^* &= \tau z_1 + c_{11} z_1^2 + c_{12} z_2^2 + 2b_1 z_1 z_2 + g_{13}(z_1, z_2), \\ z_2^* &= \alpha z_1 + \nu z_2 + c_{21} z_1^2 + c_{22} z_2^2 + 2b_2 z_1 z_2 + g_{23}(z_1, z_2), \end{aligned} \tag{3.14}$$

where

$$\tau > 0, \quad |\nu| = \frac{1}{\tau}, \quad \lim_{z_1, z_2 \to 0} \frac{g_{13}(z_1, z_2)}{|z_1|^2 + |z_2|^2} = 0, \quad \lim_{z_1, z_2 \to 0} \frac{g_{23}(z_1, z_2)}{|z_1|^2 + |z_2|^2} = 0.$$

Via the transformation (3.1) the domain $\mathfrak{B}$ is mapped to $\mathfrak{B}^*$ and the coefficients $\{a, b, c\}$ are mapped to those $\{a^*, b^*, c^*\}$ of $\mathfrak{B}^*$. As we shall see soon, by a suitable choice of coordinates of the transformation (3.14) we can achieve that a^*, b^*, c^* vanish, whereas σ has an invariant meaning, namely, it transforms according to

$$\sigma^* = \tau^3 \sigma, \tag{3.15}$$

or in the case of a general transformation,

$$\sigma^* = \frac{\tau^3 \sigma}{|D_0|^2}, \quad D_0 = \left(\frac{\partial(g_1, g_2)}{\partial(z_1, z_2)} \right)_{\substack{z_1 = 0 \\ z_2 = 0}}. \tag{3.16}$$

If $b = c = 0$ at the point Q for the domain $\mathfrak{B}$, then $\mathfrak{B}$ is called *canonical* at $Q(0,0)$, and a replacement domain $\mathfrak{B}^*$ of $\mathfrak{B}$ for Q that is canonical at Q is called a *canonical replacement domain* for the point Q. For the boundary hypersurface we have the expansion

$$2x_1 = a y_1^2 + \sigma |z_2|^2 + \dots \tag{3.17}$$

Such a canonical replacement domain can be obtained, for example, by applying a transformation

$$z_1^* = z_1 - 2b z_1 z_2 - c z_2^2, \quad z_2^* = z_2, \tag{3.18}$$

whose inversion is

$$z_1 = z_1^* + 2b z_1^* z_2^* + c z_2^{*2} + \dots, \quad z_2 = z_2^*. \tag{3.19}$$

For if we consider that

$$2\mathrm{i} y_1 (b z_2 - \overline{b} \overline{z}_2) = (z_1 - \overline{z}_1)(b z_2 - \overline{b} \overline{z}_2) = 2b z_1 z_2 + 2\overline{b} \overline{z}_1 \overline{z}_2 - (z_1 + \overline{z}_1)(b z_2 + \overline{b} \overline{z}_2),$$

then it follows for the transformed domain that

$$(z_1+\overline{z}_1)(1+bz_2+\overline{b}\overline{z}_2)=ay_1^2+\sigma|z_2|^2+\dots$$

(where we omitted the stars), and hence the expansion (3.17) follows. By (3.15), the sign of σ is invariant under the considered transformations since $\tau>0$. It is identical to minus the Levi expression

$$L(\Phi)=-\begin{vmatrix} 0 & \frac{\partial\Phi}{\partial z_1} & \frac{\partial\Phi}{\partial z_2} \\ \frac{\partial\Phi}{\partial\overline{z}_1} & \frac{\partial^2\Phi}{\partial z_1\partial\overline{z}_1} & \frac{\partial^2\Phi}{\partial z_2\partial\overline{z}_1} \\ \frac{\partial\Phi}{\partial\overline{z}_2} & \frac{\partial^2\Phi}{\partial z_1\partial\overline{z}_2} & \frac{\partial^2\Phi}{\partial z_2\partial\overline{z}_2} \end{vmatrix}.$$

For if we write the equation of the boundary surface (compare (2.7)) as

$$\Phi=z_1+\overline{z}_1-\psi\left(\frac{z_1-\overline{z}_1}{2\mathrm{i}},z_2,\overline{z}_2\right)=0,$$

then

$$\sigma=-[L(\Phi)]_{\substack{z_1=0\\z_2=0}}$$

and the sign of σ is invariant under any analytic transformation. Its meaning is elucidated by the following: Consider any canonical replacement domain for $\mathfrak{B}$ (or only such for which $a=0$), then for the plane $z_1=0$, the following holds:

(1) For $\sigma>0$, in a certain neigborhood of $z_2=0$ and the plane $z_1=0$:

$$2x_1-\psi(y_1,z_2,\overline{z}_2)=-\sigma|z_2|^2-\dots<0,$$

that is, in a certain neighborhood of the origin, the analytic plane $z_1=0$ lies outside the canonical replacement domain.

We will study the case $\sigma>0$, which in a way is the regular case, in §5 and determine σ from the kernel function for very general cases.

(2) $\sigma<0$. Then

$$2x_1-\psi(y_1,z_2,\overline{z}_2)=2x_1-ay_1^2-\sigma|z_2|^2-\dots>0,$$

and thus a certain neigborhood of the point $z_2=0$ of the analytic plane $z_1=0$ lies in the interior of the canonical replacement domain.

Here we can easily show that the kernel function remains bounded under the approaches A^{I} and $\mathsf{A}^{\mathrm{II}}(\alpha)$ to the point Q. We may assume that $a\leq 0$ in the expansion (3.17), for otherwise we can use the transformation

$$z_1'=\frac{z_1}{1+\mu z_1},\quad z_2'=z_2 \tag{3.20}$$

to pass to a replacement domain for which this holds, and since the approaches A^{I} and $\mathsf{A}^{\mathrm{II}}(\alpha)$ behave invariantly under this transformation. If $a \leq 0$, then the domain given by

$$|x_1| \geq 0, \quad |z_1| \leq \delta_1, \quad |z_2| \leq \delta_2 \tag{3.21}$$

for sufficiently small δ_1, δ_2 satisfies

$$2x_1 - \sigma|z_2|^2 - \psi(y_1, z_2, \overline{z}_2) \geq 0.$$

The points

$$z_1 = 0, \quad |z_2| = \delta_2$$

are all contained in the interior of $\mathfrak{B}$ by the above, that is, they have a positive distance κ to the boundary of $\mathfrak{B}$. So, if we consider the points

$$|z_1| \leq \delta_1', \quad |z_2| = \delta_2$$

(where $\delta_1' = \min(\frac{\kappa}{2}, \delta_1)$), then these points have a distance to the boundary of $\mathfrak{B}$ that is greater than $\frac{\kappa}{2}$. At the points whose coordinates satisfy the inequalities (3.21), the following estimate holds for the kernel function

$$\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \leq \frac{32}{\pi^2\kappa^4} \tag{3.22}$$

(compare equation (14) on p. 651 in the work cited in footnote 5) on p. 3). On the other hand, by the maximum property of the kernel function (compare p. 5), the inequality (3.22) also holds for poiints $z_2| \leq \delta_2$ (that also belong to $\mathfrak{B}$). Therefore, on the whole domain we have

$$x_1 > 0, \quad |z_1| \leq \delta_1', \quad |z_2| \leq \delta_2, \tag{3.23}$$

which proves the claim.

(3) In the case $\sigma = 0$ we cannot make such a positive statement on the position of the analytic plane $z_1 = 0$. We will consider two special cases for $\sigma = 0$ in §§6 and 8.

INNER AND OUTER COMPARISON DOMAIN. Domains $\underline{\mathfrak{B}}$ and $\overline{\mathfrak{B}}$ are called *inner* and *outer comparison domains* of the domain $\mathfrak{B}$ for the boundary point Q, respectively, if there exists a replacement domain $\mathfrak{B}^*$ of $\mathfrak{B}$ such that

$$\underline{\mathfrak{B}} \subset \mathfrak{B}^*, \quad \overline{\mathfrak{B}} \supset \mathfrak{B}^*, \tag{3.24}$$

and $\underline{\mathfrak{B}}$ and $\overline{\mathfrak{B}}$ contain the point Q^* (image of Q) and have the same tangent hyperplane at this point as $\mathfrak{B}^*$.

Now, if $\underline{\mathfrak{B}}$ is an inner comparison domain of $\mathfrak{B}$ and $\mathfrak{B}^*$ is the corresponding replacement domain, then by (1.12), the points of $\underline{\mathfrak{B}}$ satisfy

$$\varrho_{\mathrm{n}}^{r} K_{\underline{\mathfrak{B}}}(z;\overline{z}) \geq \varrho_{\mathrm{n}}^{r} K_{\mathfrak{B}^*}(z;\overline{z}). \tag{3.25}$$

So if $\overline{\lambda}(Q;\underline{\mathfrak{B}})$ is the upper order for any innter comparison domain, then the order is $\overline{\lambda}(Q;\mathfrak{B}^*) \leq \overline{\lambda}(Q;\underline{\mathfrak{B}})$, and, since $\overline{\lambda}(Q;\mathfrak{B}) = \overline{\lambda}(Q;\mathfrak{B}^*)$,

$$\overline{\lambda}(Q;\underline{\mathfrak{B}}) \geq \overline{\lambda}(Q;\mathfrak{B}). \tag{3.26}$$

In the same way it follows for the outer comparison domain that

$$\underline{\lambda}(Q;\overline{\mathfrak{B}}) \leq \underline{\lambda}(Q;\mathfrak{B}). \tag{3.27}$$

Therefore, *if it is possible to choose the outer and inner comparison domains for the point Q such that*

$$\underline{\lambda}(Q;\overline{\mathfrak{B}}) = \overline{\lambda}(q;\underline{\mathfrak{B}}),$$

then from $\underline{\lambda}(Q;\overline{\mathfrak{B}}) \leq \underline{\lambda}(Q;\mathfrak{B}) \leq \overline{\lambda}(Q;\mathfrak{B}) \leq \overline{\lambda}(Q;\underline{\mathfrak{B}})$ follows the existence of the order

$$\lambda(Q;\mathfrak{B}) = \underline{\lambda}(Q;\overline{\mathfrak{B}}) = \overline{\lambda}(Q;\underline{\mathfrak{B}}). \tag{3.28}$$

In the case of existence of a certain order $\lambda(Q;\mathfrak{B})$ for becoming infinity of the kernel function in neighborhood of Q it follows from (3.25), (3.24) (compare also p. 13) for the limits $\underline{L}^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q)$, $\overline{L}^{\mathrm{I}\lambda}(Q)$:[11)]

$$\overline{L}^{\mathrm{I}\lambda}_{\underline{\mathfrak{B}}}(Q) \geq L^{\mathrm{I}\lambda}_{\mathfrak{B}_1^*}(Q) = \tau_1^{\lambda} L^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q) \tag{3.29}$$

and

$$\underline{L}^{\mathrm{I}\lambda}_{\overline{\mathfrak{B}}}(Q) \leq L^{\mathrm{I}\lambda}_{\mathfrak{B}_2^*}(Q) = \tau_2^{\lambda} L^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q) \tag{3.30}$$

If it is possible to construct the inner and outer comparison domains for the point Q such that the measure factors of the two transformations coincide, that is, such that

$$\tau_1 = \tau_2 = \tau, \tag{3.31}$$

[11)]Now let $\mathfrak{B}_1^*$, $\mathfrak{B}_2^*$ be two replacment domains of $\mathfrak{B}$ that do not necessarily coincide.

and such that

$$L^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q) = L^{\mathrm{I}\lambda}_{\underline{\mathfrak{B}}}(Q), \tag{3.32}$$

then (3.29) *and* (3.30) *imply that*

$$\tau^{\lambda} L^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q) = L^{\mathrm{I}\lambda}_{\mathfrak{B}}(Q) = L^{\mathrm{I}\lambda}_{\underline{\mathfrak{B}}}(Q) \tag{3.33}$$

exists, that is, the point Q is a limit point of order λ.

For the approaches $\mathsf{A}^{\mathrm{II}}(\alpha)$ and $\mathsf{A}^{\mathrm{III}}(\alpha)$ we can draw the analogous conclusions in a similar manner.

§ 4

For the unit bicylinder $\mathfrak{C}$,

$$|z_1| \leq 1, \quad |z_2| \leq 1 \tag{4.1}$$

the functions

$$\frac{1}{\pi}\sqrt{(m_1+1)(m_2+1)}z_1^{m_1}z_2^{m_2} \quad (m_k = 0, 1, 2, \ldots;\ k = 1, 2)$$

form a complete orthonormal system. We obtain for the kernel function:

$$\begin{aligned} \mathsf{K}_{\mathfrak{C}}(z_1, z_2; \overline{z}_1, \overline{z}_2) &= \frac{1}{\pi^2}\sum_{m_1,m_2}^{\infty}(m_1+1)(m_2+1)|z_1|^{2m_1}|z_2|^{2m_2} \\ &= \frac{1}{\pi^2}\sum_{m}^{\infty}(m+1)|z_1|^{2m}\sum_{m}^{\infty}(m+1)|z_2|^{2m} \\ &= \frac{1}{\pi}\frac{1}{(1-|z_1|^2)^2}\frac{1}{\pi}\frac{1}{(1-|z_2|^2)^2} \\ &= \mathsf{K}_{\mathfrak{C}_1}(z_1;\overline{z}_1)\mathsf{K}_{\mathfrak{C}_2}(z_2;\overline{z}_2). \end{aligned} \tag{4.2}$$

We consider the boundary point $Q(-1, \gamma)$,

$$z_1 = -1, \quad z_2 = \gamma \quad (|\gamma| < 1). \tag{4.3}$$

If we set

$$z_1 = z_1' - 1, \quad z_2 = z_2', \tag{4.4}$$

then after applying the transformation (4.4) we obtain (omitting the primes)

$$K_{\mathfrak{E}_1}(z_1;\bar{z}_1) = \frac{1}{\pi}\frac{1}{((z_1+\bar{z}_1)-|z_1|^2)^2}, \tag{4.5}$$

and in the new coordinates,

$$\varrho_{\mathrm{n}} = \frac{z_1+\bar{z}_1}{2}. \tag{4.6}$$

Therefore,

$$\left(\frac{z_1+\bar{z}_1}{2}\right)^2 \mathsf{K}_{\mathfrak{E}_1}(z_1;\bar{z}_1) = \frac{1}{4\pi}\frac{1}{(1-\frac{|z_1|^2}{z_1+\bar{z}_1})^2}, \tag{4.7}$$

and since

$$\begin{aligned}\frac{|z_1|^2}{z_1+\bar{z}_1} = \left|\frac{((z_1+\bar{z}_1)+(z_1-\bar{z}_1))^2}{4(z_1+\bar{z}_1)}\right| &= \frac{z_1+\bar{z}_1}{4}\left|1+\frac{2(z_1-\bar{z}_1)}{z_1+\bar{z}_1}+\frac{(z_1-\bar{z}_1)^2}{(z_1+\bar{z}_1)^2}\right| \\ &\le \left(\frac{1+M}{2}\right)^2(z_1+\bar{z}_1),\end{aligned}$$

where we assume

$$\left|\frac{z_1-\bar{z}_1}{z_1+\bar{z}_1}\right| < M,$$

it follows that

$$\lim_{z_1\to 0}\left(\frac{z_1+\bar{z}_1}{2}\right)^2 K_{\mathfrak{E}_1}(z_1;\bar{z}_1) = \frac{1}{4\pi}, \tag{4.8}$$

and hence under approach[12] A^{I}

$$L^{\mathrm{I2}}_{\mathfrak{E}}(Q) = \lim_{z_1\to 0, z_2\to\gamma} \varrho_{\mathrm{n}}^2 \mathsf{K}_{\mathfrak{E}}(z_1,z_2;\bar{z}_1,\bar{z}_2) = \frac{1}{4\pi^2}\frac{1}{(1-|\gamma|^2)^2}. \tag{4.9}$$

If $\mathsf{P}(\gamma)$ denotes the *mapping radius*[13] of $\mathfrak{E}$ with respect to γ,

$$\mathsf{P}(\gamma) = (1-|\gamma|^2) = \frac{1}{\sqrt{\pi \mathsf{K}_{\mathfrak{E}_1}(\gamma;\bar{\gamma})}},$$

then

$$L^{\mathrm{I2}}_{\mathfrak{E}}(Q) = \lim_{z_1\to 0, z_2\to\gamma} \varrho_{\mathrm{n}}^2 K_{\mathfrak{E}}(z_1,z_2;\bar{z}_1,\bar{z}_2) = \frac{1}{4\pi^2 \mathsf{P}(\gamma)^2}. \tag{4.10}$$

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[12] Here, the conditions for the approach can be sharpened.

[13] For this definition, compare Bieberbach, *Lehrbuch der Funktionentheorie*, volume II, Springer Verlag (Berlin 1927), p. 322.

The equation (4.10) remains valid if instead of the bicylinder we consider any product domain $\mathfrak{P}$,

$$\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2,$$

where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simply connected domains in the z_1- and z_2-plane, respectively, that are bounded by closed rectifiable Jordan curves. If

$$\zeta_1 = g_1(z_1), \quad \zeta_2 = g_2(z_2)$$

maps $\mathfrak{P}$ to the unit bicylinder $|\zeta_1| \leq 1$, $|\zeta_2| \leq 1$, the by (1.15) the kernel function of $\mathfrak{P}$ is

$$\begin{aligned} \mathsf{K}_{\mathfrak{P}}(z_1, z_2; \bar{z}_1, \bar{z}_2) &= \frac{|g_1'(z_1)|^2}{\pi(1-|g_1(z_1)|^2)^2} \frac{|g_2'(z_2)|^2}{\pi(1-|g_2(z_2)|^2)^2} \\ &= \mathsf{K}_{\mathfrak{A}_1}(z_1; \bar{z}_1)\mathsf{K}_{\mathfrak{A}_2}(z_2; \bar{z}_2). \end{aligned} \tag{4.11}$$

Now let $Q(\gamma_1, \gamma_2)$ be a boundary point of $\mathfrak{P}$, such that γ_1 lies on the boundary of $\mathfrak{A}_1$ and γ_2 lies in the interior of $\mathfrak{A}_2$. For the assumption on the existence of the tangent hyperplane to be satisfied, we assume: if the boundary of $\mathfrak{A}_1$ (parameterized by arclength s) is represented in the form $z_1 = w(s)$, then in some neighborhood of γ_1 the derivative $w'(s)$ exists. Moreover, on this neighborhood $w'(s)$ shall satisfy the Hölder condition $|w'(s+h) - w'(s)| \leq k|h|^\alpha$ for some $0 < \alpha < 1$, $k < \infty$. By a theorem on conformal mappings on the boundary[14)] $g_1(z_1)$ has a continuous derivative in γ_1. But now

$$\mathsf{K}_{\mathfrak{A}_1}(z_1; \bar{z}_1) = \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \bar{\zeta}_1)|g_1'(z_1)|^2,$$

and hence, of σ_{n} denotes the distance in $\mathfrak{E}_1$,

$$\lim_{z_1 \to 0} \varrho_{\mathrm{n}}^2 \mathsf{K}_{\mathfrak{A}_1}(z_1; \bar{z}_1) = \lim_{\zeta_1 \to 0} \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \bar{\zeta}_1)\varrho_{\mathrm{n}}^2 |g_1'(z_1)|^2 = \lim_{\zeta_1 \to 0} \sigma_{\mathrm{n}}^2 \mathsf{K}_{\mathfrak{E}_2}(\zeta_1; \bar{\zeta}_1) = \frac{1}{4\pi},$$

that is,

$$\begin{aligned} L_{\mathfrak{P}}^{\mathrm{II}2}(Q(0,\gamma)) &= \lim_{z_1 \to 0, z_2 \to \gamma} \varrho_{\mathrm{n}}^2 \mathsf{K}_{\mathfrak{P}}(z_1, z_2; \bar{z}_1, \bar{z}_2) = \frac{1}{4\pi} \mathsf{K}_{\mathfrak{A}_2}(z_2; \bar{z}_2) \\ &= \frac{1}{4\pi^2} \frac{1}{\mathsf{P}_{\mathfrak{A}_2}(\gamma)^2}, \end{aligned} \tag{4.12}$$

[14)] Compare Lichtenstein, Enzyklopädie der mathematischen Wissenschaften, and Warschawski, *Über einen Satz von O.D. Kellog*, Nachricthen von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse, 1932, p. 73, where the most recent literature on this subject is listed.

where we set $\gamma_1 = 0$, $\gamma_2 = \gamma$.

For the bicylinder we can use the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$. From

$$\lim_{z_1 \to 0} \frac{|z_1|^2}{((z_1 + \overline{z}_1) - |z_1|^2)^2} = \lim_{z_1 \to 0} \frac{1}{(\frac{z_1 + \overline{z}_1}{|z_1|} - |z_1|)^2} = \frac{1}{4\cos(\alpha)^2},$$

assuming

$$\lim_{z_1 \to 0} \operatorname{arc}(z_1) \to \alpha, \tag{4.13}$$

it follows that

$$L^{\mathrm{II}2}_{\mathfrak{E}}(Q(0,\gamma)) = \lim_{\substack{z_1 \to 0 \\ \operatorname{arc}(z_1) \to \alpha \\ z_2 \to \gamma}} \varrho_{\mathrm{t}}^2 \mathsf{K}_{\mathfrak{E}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{1}{4\pi^2 \cos(\alpha)^2 \mathsf{P}(\gamma)^2}. \tag{4.14}$$

So the limit depends on the angle α. The computation of the last limit is also valid in the situation where instead of a bicylinder we consider a product domain $\mathfrak{P}$ and the boundary of $\mathfrak{A}_1$ at $z_1 = 0$ satisfies the above conditions. The formula changes slightly if the point $z_1 = 0$ is a cusp in which tangents meet at an angle ω.[15) Here, $|\alpha| < \frac{\omega}{2}$ is to be assumed. For if $\zeta_1 = g_1(z_1)$ and $\zeta_2 = g_2(z_2)$ are functions that map the domain $\mathfrak{P}$ to the unit bicylinder, where the point $z_1 = 0$, $z_2 = 0$ is mapped to itself, then

$$|z_1|^2 \mathsf{K}_{\mathfrak{P}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = |z_1|^2 \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \overline{\zeta}_1) |g_1'(z_1)|^2 \mathsf{K}_{\mathfrak{A}_1}(z_2; \overline{z}_2). \tag{4.15}$$

But now

$$\lim_{z_1 \to 0} |z_1|^2 |g_1'(z_1)|^2 \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \overline{\zeta}_1) = \lim_{\zeta_1 \to 0} \frac{|z_1|^2 |g_1'(z_1)|^2}{|\zeta_1|^2} \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \overline{\zeta}_1).$$

By a theorem on conformal mappings on the boundary,

$$\lim_{\zeta_1 \to 0} \frac{|z_1|^2 |g_1'(z_1)|^2}{|\zeta_1|^2} = \left(\frac{\pi}{\omega}\right)^2.$$

On the other hand, due to quasi-conformality[16) the points on the bisector are mapped to the radius and the angle α is mapped to the angle $\frac{\pi\alpha}{\omega}$. Hence

$$\lim_{\zeta_1 \to 0} |\zeta_1|^2 \mathsf{K}_{\mathfrak{E}_1}(\zeta_1; \overline{\zeta}_1) = \frac{1}{4\pi \cos(\frac{\pi\alpha}{\omega})^2}$$

[15) We assume that the cusp is formed by two continuous curve segments. Compare Lichtenstein, *Über die konforme Abbildung ebener analytischer Gebiete mit Ecken*, Journal für die reine und angewandte Mathematik 140 (1911), p. 100, and Warschawski, *Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung*, Mathematische Zeitschrift 35 (1932), p. 321, where notes on the current literature can be found.

[16) Compare Carathéodory, *Elementarer Beweis für den Fundamentalsatz der konformen Abbildungen*, Abhandlungen zum Doktorjubiläum von H.A. Schwarz (Berlin 1914), p. 20, in particular §18-§20.

and therefore

$$L_{\mathfrak{P}}^{\mathrm{II}2}(Q,\alpha) = \frac{1}{4\omega^2\cos(\frac{\pi\alpha}{\omega})^2 \mathsf{P}_{\mathfrak{A}_2}(\gamma)^2}. \tag{4.16}$$

Now consider the hypersphere $\mathfrak{K}$,

$$|z_1|^2 + |z_2|^2 < 1.$$

For the hypersphere $\mathfrak{K}$, the functions

$$C_{n_1,n_2} z_1^{n_1} z_2^{n_2}, \quad |C_{n_1,n_2}|^2 = \frac{1}{\pi^2}\frac{(n_1+n_2+2)!}{n_1!n_2!} \tag{4.17}$$

form a complete orthonormal system[17)], so that the kernel is

$$\begin{aligned}\sum_{n_1,n_2}^{\infty} |C_{n_1,n_2}|^2 |z_1|^{2n_1}|z_2|^{2n_2} &= \frac{1}{\pi^2}\sum_{n_1,n_2}^{\infty}\frac{(n_1+n_2+2)!}{n_1!n_2!}|z_1|^{2n_1}|z_2|^{2n_2} \\ &= \frac{2}{\pi^2(1-(|z_1|^2+|z_2|^2))^3}.\end{aligned} \tag{4.19}$$

If we now apply the transformation

$$z_1' = \frac{z_1}{\sigma}, \quad z_2' = \frac{z_2}{\sigma},$$

then we obtain the kernel function for the hypersphere $\mathfrak{K}_\sigma$, $|z_1|^2 + |z_2|^2 < \frac{1}{\sigma^2}$:

$$\begin{aligned}\mathsf{K}_{\mathfrak{K}_\sigma}(z_1,z_2;\overline{z}_1,\overline{z}_2) &= \frac{2\sigma^4}{\pi^2(1-\sigma^2(|z_1|^2+|z_2|^2))^3} \\ &= \frac{2}{\pi^2\sigma^2(\frac{1}{\sigma^2}-(|z_1|^2+|z_2|^2))^3}.\end{aligned} \tag{4.20}$$

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17) The constants C_{n_1,n_2} are derived from

$$|C_{n_1,n_2}|^2 \int_{\mathfrak{K}} |z_1|^{2n_1}|z_2|^{2n_2}\mathrm{d}\omega = 1. \tag{4.18}$$

We set $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$. Then $\mathrm{d}\omega = r_1 r_2 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \mathrm{d}r_1 \mathrm{d}r_2$, so that

$$\begin{aligned}|C_{n_1,n_2}|^2 4\pi^2 \int_0^1 r_1^{2n_1+1}\mathrm{d}r_1 \int_0^{\sqrt{1-r_1^2}} r_2^{2n_2+1}\mathrm{d}r_2 &= \frac{2\pi^2|C_{n_1,n_2}|^2}{n_2+1}\int_0^1 r_1^{2n_1+1}(1-r_1^2)^{n_2+1}\mathrm{d}r_2 \\ &= \frac{\pi^2|C_{n_1,n_2}|^2}{n_2+1}\int_0^1 u^{n_1}(1-u)^{n_2+1}\mathrm{d}u \\ &= \pi^2|C_{n_1,n_2}|^2\frac{n_1!n_2!}{(n_1+n_2+2)!}.\end{aligned}$$

Now consider the boundary point $z_1 = -\frac{1}{\sigma}\ z_2 = 0$, and change the coordinate system such that via

$$z_1' = z_1 + \frac{1}{\sigma}, \quad z_2' = z_2 \tag{4.21}$$

this boundary point becomes the origin. Now,

$$\sigma^2 \left| z_1 - \frac{1}{\sigma} \right| \left| \overline{z}_1 - \frac{1}{\sigma} \right| = \sigma^2 |z_1|^2 - \sigma |z_1 + \overline{z}_1| + 1.$$

The kernel function of $\mathfrak{K}_\sigma$ is

$$\mathsf{K}_{\mathfrak{K}_\sigma}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{2\sigma}{\pi^2((z_1 + \overline{z}_1) - \sigma|z_1|^2 - \sigma|z_2|^2)^3}. \tag{4.22}$$

The equation for the boundary hypersurface of $\mathfrak{K}_\sigma$ after the transformation (4.21) is

$$-(z_1 + \overline{z}_1) + \sigma|z_1|^2 + \sigma|z_2|^2 = 0 \quad \text{or} \quad (z_1 + \overline{z}_1) = \sigma|z_1|^2 + \sigma|z_2|^2. \tag{4.23}$$

The normal points to the interior, and the expansion (4.23) has the canonical form with $\sigma > 0$. By (4.22),

$$L^{\mathrm{I3}}_{\mathfrak{K}_\sigma}(Q) = \lim_{z_1, z_2 \to 0} \varrho_{\mathrm{n}}^3 K_{\mathfrak{K}_\sigma}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{\sigma}{4\pi^2}. \tag{4.24}$$

As we will see, this result holds under some general conditions for a large class of domains if $\sigma > 0$.

It suggests itself to establish analogous relations to the limit relations (4.10), (4.24) for the minimal function (1.10) and for the invariant (1.17).

For the case of the bicylinder $\mathfrak{C}$, $|z_1 - 1| < 1$, $|z_2| < 1$, and the hypersphere $\mathfrak{K}_\sigma$, $|\frac{1}{\sigma} - z_1|^2 + |z_2| < \frac{1}{\sigma^2}$, we obtain the following expressions for the minimal function:

$$\begin{aligned}
\mathsf{M}_{\mathfrak{C}}(z_1, z_2; \overline{t}_1, \overline{t}_2) &= \frac{(t_1 + \overline{t}_1 - |t_1|^2)^2 (1 - |t_2|^2)^2}{(z_1 + \overline{t}_1 - z_1\overline{t}_1)^2 (1 - z_2\overline{t}_2)^2} \\
\mathsf{M}_{\mathfrak{K}_\sigma}(z_1, z_2; \overline{t}_1, \overline{t}_2) &= \frac{(t_1 + \overline{t}_1 - \sigma|t_1|^2 - \sigma|t_2|^2)^3}{(z_1 + \overline{t}_1 - \sigma z_1\overline{t}_1 - \sigma z_2\overline{t}_2)^3},
\end{aligned} \tag{4.25}$$

from which we obtain the limit formulas

$$\lim_{n\to\infty} n^{2-2p} \mathsf{M}_{\mathfrak{C}} \left(\frac{z_1}{n^p}, z_2; \frac{\overline{t}_1}{n}, \overline{t}_2 \right) = \frac{(t_1 + \overline{t}_1)^2 (1 - |t_2|^2)^2}{z_1^2 (1 - z_2 t_2)^2} \tag{4.26}$$

$$\lim_{n\to\infty} n^{3-3p} \mathsf{M}_{\mathfrak{K}_\sigma} \left(\frac{z_1}{n^p}, \frac{z_2}{n^p}; \frac{\overline{t}_1}{n}, \frac{\overline{t}_2}{n} \right) = \frac{(t_1 + \overline{t}_1)^3}{z_1^3} \tag{4.27}$$

with $0 < p < 1$. Similarly we obtain

$$\boldsymbol{I}_{\mathfrak{G}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{1}{4\pi^2} \quad \text{and} \quad \boldsymbol{I}_{\mathfrak{K}_\sigma}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{2}{9\pi^2}, \tag{4.28}$$

from which

$$\lim_{z_1, z_2 \to 0} \boldsymbol{I}_{\mathfrak{K}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{1}{4\pi^2} \tag{4.29}$$

$$\lim_{z_1, z_2 \to 0} \boldsymbol{I}_{\mathfrak{K}_\sigma}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{2}{9\pi^2}, \tag{4.30}$$

follow.

As will be shown in the following, the minimal function $\mathsf{M}_{\mathfrak{B}}$ and the invariant $\boldsymbol{I}_{\mathfrak{B}}$ of the boundary points considered in §5 and §6 satisfy relations analogous to (4.27), (4.30), and (4.26), (4.29), respectively.

§ 5

THE CASE $\sigma > 0$. POINTS OF THIRD ORDER. For the investigation of points on the boundary with $\sigma > 0$ we assume the approach A^{I} and first prove

Theorem I. *If $\sigma > 0$ at a the point $Q(0,0)$ of $\mathfrak{B}$, then under the approach A^{I},*

$$\overline{\lim_{z_1, z_2 \to 0}} \varrho_{\mathrm{n}}^3 \mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \leq \frac{\sigma}{4\pi^2}. \tag{5.1}$$

PROOF: Let $\mathfrak{K}_1$ denote the hypersphere

$$\left| z_1 - \frac{1}{\sigma_1} \right|^2 + |z_2|^2 < \frac{1}{\sigma_1^2}, \quad \sigma_1 > \sigma. \tag{5.2}$$

We claim that the domain $\mathfrak{K}_{\alpha_1 \beta_1}$, obtained from $\mathfrak{K}_1$ by the transformation

$$z_1' = \frac{z_1}{1 + \alpha_1 z_1}, \quad z_2' = \frac{z_2}{1 + \beta_1 z_1'} \quad (\alpha_1 > 0, \beta_1 \geq 0), \tag{5.3}$$

is a comparison domain of $\mathfrak{B}$ for sufficiently large α_1, β_1. Let $\mathfrak{B}^*$ be a canonical replacment domain of $\mathfrak{B}$, for which the boundary hypersurface is

$$2x_1 = a y_1^2 + \sigma |z_2|^2 + \dots. \tag{5.4}$$

in a neighborhood of Q. We will now show that the domain $\mathfrak{K}_{\alpha_1 \beta_1}$ is contained in $\mathfrak{B}^*$ for sufficiently large $\mathfrak{B}^*$ (and has the same tangent hyperplane as $\mathfrak{B}^*$ and $\mathfrak{B}$ at Q).

If we apply the transformation

$$z_1' = \frac{z_1}{1 + \alpha_1 z_1}, \quad z_2' = z_2 \tag{5.5}$$

to $\mathfrak{K}_1$, then the transformed domain $\mathfrak{K}_{\alpha_1}$ has the representation

$$2x_1 = (\sigma_1 + 2\alpha_1)y_1^2 + \sigma|z_2|^2 + \dots. \tag{5.6}$$

By choosing α_1 large enough such that

$$\sigma_1 + 2\alpha_1 > a,$$

in a sufficiently small neighborhood $\mathfrak{U}_1$ of Q it holds that

$$2x_1 - (\sigma_1 + 2\alpha_1)y_1^2 - \sigma_1|z_2|^2 + \dots \leq 2x_1 - ay_1^2 - \sigma_2|z_2|^2 + \dots, \tag{5.7}$$

that is, in $\mathfrak{U}_1$, every point of $\mathfrak{K}_{\alpha_1}$ also belongs to $\mathfrak{B}^*$. For the intersections $\mathfrak{K}_{\alpha_1}(z_1 = \gamma)$ and $\mathfrak{B}^*(z_1 = \gamma)$ of the domains $\mathfrak{K}_{\alpha_1}$ and $\mathfrak{B}^*$ with the plane $z_1 = \gamma$ it thus holds that

$$\mathfrak{K}_{\alpha_1}(z_1 = \gamma) \subset \mathfrak{B}^*(z_1 = \gamma) \tag{5.8}$$

if γ is sufficiently small, say

$$|\gamma| < l_1(\alpha_1). \tag{5.9}$$

Consider now the intersection $\mathfrak{K}_{\alpha_1}(z_2 = 0)$, that is, the circle

$$\left| z_1 - \frac{1}{\sigma_1 + 2\alpha_1} \right| \leq \frac{1}{\sigma_1 + 2\alpha_1}. \tag{5.10}$$

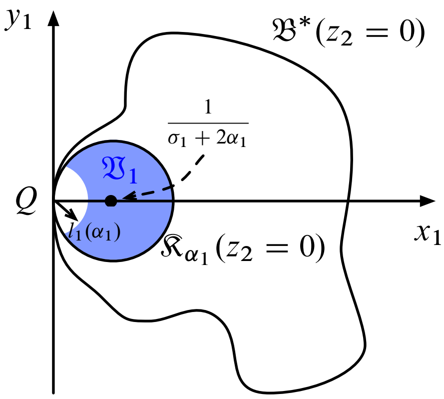
Because of the inequality (5.7), this is contained in the intersection $\mathfrak{B}^*(z_2 = 0)$ for sufficiently small z_1. (If α_1 is chosen sufficiently large, then the circle (5.10) moves into an arbitrarily small neighborhood of $z_1 = 0$, and we can thus arrange that it is completely contained in $\mathfrak{B}^*(z_2 = 0)$.) Once α_1 has been fixed according to the preceding conditions, consider the set $\mathfrak{V}_1$[18)] of values z_1 in the intersection $\mathfrak{K}_{\alpha_1}(z_2 = 0)$ that in addition to (5.10) also satisfy

$$|z_1| > l_1(\alpha_1), \tag{5.11}$$

then those points that are contained in the normal intersection $\mathfrak{B}^*(z_2 = 0)$ in the arbitrarily small neighborhood of Q are inner points of $\mathfrak{B}^*(z_2 = 0)$.

[18)] Shaded in Figure 1.

Figure 1



About every point $(z_1, 0)$ $(z_1 \in \mathfrak{V}_1)$ there is thus a hypersphere with maximal radius $\varrho(z_1) > 0$ whose interior belongs completely to $\boldsymbol{\mathfrak{B}}^*$. As a continuous function in z_1, $\varrho(z_1)$ assumes a positive minimum ϱ^* in the closed $\boldsymbol{\mathfrak{V}}_1$. In particular, every intersection $\mathfrak{B}^*(z_1 = \gamma)$, for γ in $\mathfrak{V}_1$, contains a circle with center $z_2 = 0$ of radius ϱ^*. Considering now the corresponding intersections $\mathfrak{K}_{\alpha_1}(z_1 = \gamma)$ of the domain $\boldsymbol{\mathfrak{K}}_{\alpha_1}$, represented by

$$\left|\frac{z_1}{1 - \alpha_1 z_1} - \frac{1}{\sigma_1}\right|^2_{z_1 = \gamma} + |z_2|^2 \leq \frac{1}{\sigma_1^2}, \tag{5.12}$$

then these are non-empty only if z_1 belongs to the circle (5.10), and are themselves circles with center $z_2 = 0$ and a radius less than $\frac{1}{\sigma_1}$. If we apply the transformation

$$z_1' = z_1, \quad z_2' = \frac{z_2}{1 + \beta_1 z_1} \tag{5.13}$$

to $\boldsymbol{\mathfrak{K}}_{\alpha_1}$, then the intersection $\mathfrak{K}_{\alpha_1}(z_1 = \gamma)$ is mapped to a circle centered at $z_2 = 0$ of radius $\varrho(\gamma)$, for which

$$\varrho(\gamma) \leq \frac{1}{\sigma_1 |1 + \beta_1 \gamma|}$$

holds. Since $|\gamma| \geq l_1(\alpha_1)$ on the set $\mathfrak{V}_1$, β_1 can be chosen in such a way that for all γ in $\mathfrak{V}_1$,

$$\frac{1}{\sigma_1 |1 + \beta_1 \gamma|} < \varrho^* \leq \varrho(\gamma) \tag{5.14}$$

holds. Then, for $|\gamma| \geq l_1(\alpha_1)$,

$$\mathfrak{K}_{\alpha_1 \beta_1}(z_1 = \gamma) \subset \mathfrak{B}^*(z_1 = \gamma). \tag{5.15}$$

On the other hand, the values $|\gamma| < l_1(\alpha_1)$ in the circle (5.10) also satisfy

$$\frac{1}{|1+\beta_1\gamma_1|} < 1,$$

as for these values $\gamma + \overline{\gamma} > 0$. Since therefore

$$\mathfrak{K}_{\alpha_1\beta_1}(z_1 = \gamma) \subset \mathfrak{K}_{\alpha_1}(z_1 = \gamma)$$

for $|\gamma| \leq l_1(\alpha_1)$, in connection with (5.8), the relation (5.15) also holds for $|\gamma| < l_1(\alpha_1)$. Hence

$$\boldsymbol{\mathfrak{K}}_{\alpha_1\beta_1} \subset \boldsymbol{\mathfrak{B}}^*. \tag{5.16}$$

Moreover, both domains have the same tangent hyperplane $z_1 + \overline{z}_1 = 0$ in the common boundary point Q, that is, $\mathfrak{K}_{\alpha_1\beta_1}$ is an inner comparison domain for $\boldsymbol{\mathfrak{B}}$. Now, $\boldsymbol{\mathfrak{K}}_{\alpha_1\beta_1}$ is a replacement domain for $\boldsymbol{\mathfrak{K}}_1$, that is,

$$\begin{aligned} \overline{L}^{\mathrm{I3}}_{\boldsymbol{\mathfrak{B}}}(Q) &= \overline{\lim_{z_1,z_2\to 0}}\, \varrho_{\mathrm{n}}^3 \mathsf{K}_{\boldsymbol{\mathfrak{B}}}(z_1,z_2;\overline{z}_1,\overline{z}_2) \\ &\leq \lim_{z_1,z_2\to 0} \varrho_{\mathrm{n}}^3 \mathsf{K}_{\boldsymbol{\mathfrak{K}}_1}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \frac{\sigma_1}{4\pi^2}, \end{aligned} \tag{5.17}$$

and since this holds for every $\sigma_1 > \sigma$, this completes the proof of (5.1).

Theorem II. *If the point $Q(0,0)$ satisfies*

$$\sigma > 0 \tag{5.18}$$

and if furthermore the sections $\mathfrak{B}^\dagger(z_1 = \gamma)$ of a canonical replacement domain[19] *$\boldsymbol{\mathfrak{B}}^\dagger$ for sufficiently small γ are contained in an arbitrarily given neighborhood of Q, then*

$$L^{\mathrm{I3}}_{\boldsymbol{\mathfrak{B}}}(Q) = \lim_{z_1,z_2\to 0} \varrho_{\mathrm{n}}^3 \mathsf{K}_{\boldsymbol{\mathfrak{B}}}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \frac{\sigma}{4\pi^2}. \tag{5.19}$$

PROOF: Assume the expansion

$$2x_1 = a y_1^2 + \sigma|z_2|^2 + \ldots \tag{3.17}$$

belongs to the canonical replacement domain $\mathfrak{B}^\dagger$. Again, we start with a hypersphere $\boldsymbol{\mathfrak{K}}_2$

$$\left|z_1 - \frac{1}{\sigma_2}\right|^2 + |z_2|^2 < \frac{1}{\sigma_2^2}, \quad \sigma_2 < \sigma, \tag{5.20}$$

[19] Here, $\boldsymbol{\mathfrak{B}}^\dagger$ is once more a replacement domain for $\boldsymbol{\mathfrak{B}}$, but it does not have to coincide with $\boldsymbol{\mathfrak{B}}^*$.

and claim that the domain $\mathfrak{K}_{\alpha_2\beta_2}$ obtained from $\mathfrak{K}_2$ by the transformation

$$z_1' = \frac{z_1}{1-\alpha_2 z_2}, \quad z_2' = z_1(1+\beta_2 z_1') \tag{5.21}$$

for sufficiently large α_2, β_2 is an outer comparison domain for $\mathfrak{B}^\dagger$ (with respect to Q). If we apply the transformation

$$z_1' = \frac{z_1}{1-\alpha_2 z_1}, \quad z_2' = z_2 \tag{5.22}$$

to $\mathfrak{K}_2$, then we obtain the transformed domain $\mathfrak{K}_{\alpha_2}$ with

$$2x_1 = (\sigma_2 - 2\alpha_2)y_1^2 + \sigma_2|z_2|^2 + \dots. \tag{5.23}$$

If we choose α_2 positive and large enough such that

$$\sigma_2 - 2\alpha_2 < a, \tag{5.24}$$

then in a sufficiently small neighborhood $\mathfrak{U}_2$ of Q

$$2x_1 - ay_1^2 - \sigma|z_2|^2 + \dots < 2x_1 - (\sigma_2 - 2\alpha_2)y_1^2 - \sigma_2|z_2|^2 + \dots \tag{5.25}$$

holds, that is, in the neighborhood $\mathfrak{U}_2$, every point of $\mathfrak{B}^\dagger$ is at the same time an inner point of $\mathfrak{K}_{\alpha_2}$. But by assumption there are no other points of $\mathfrak{B}^\dagger$ outside of this neighborhood if γ is sufficiently small, say

$$|\gamma| < l_2(\alpha_2). \tag{5.26}$$

Then

$$\mathfrak{B}^\dagger(z_1 = \gamma) \subset \mathfrak{K}_{\alpha_2}(z_1 = \gamma), \tag{5.27}$$

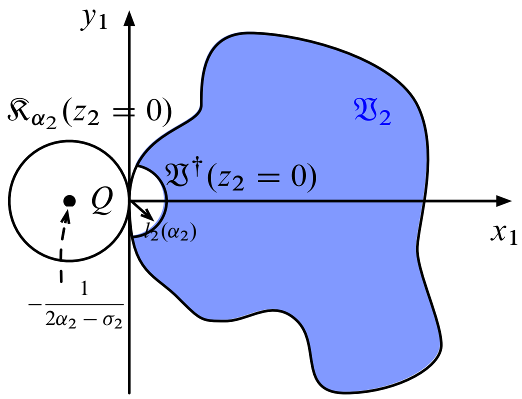
if γ satisfies (5.26). We now consider the section $\mathfrak{K}_{\alpha_2}(z_2 = 0)$. If we assume (as we always can) that

$$\sigma_2 - 2\alpha_2 < 0,$$

then the intersection is formed by the exterior of the circle

$$\left| z_1 - \frac{1}{\sigma_2 - 2\alpha_2} \right| < \frac{1}{2\alpha_2 - \sigma_2}. \tag{5.28}$$

Figure 2



For sufficiently small z_1, this circle lies completely outside of the section $\mathfrak{B}^\dagger(z_2 = 0)$ due to (5.25). If α_2 is chosen sufficiently large, then the circle (5.28) moves into a sufficiently small neighborhood of $z_1 = 0$, and thus we can achieve that it is located completely outside of $\mathfrak{B}^\dagger(z_2 = 0)$.

The set $\mathfrak{V}_2$[20)]of those points of $\boldsymbol{\mathfrak{B}}^\dagger(z_2 = 0)$ for which $|z_1| \geq l_2(\alpha_2)$, is then completely contained in the interior of the intersection $\mathfrak{K}_{\alpha_2}(z_2 = 0)$, that is, they have no points in common with the closed circle (5.28). The sections $\mathfrak{K}_{\alpha_2}(z_1 = \gamma)$, where γ belongs to $\mathfrak{V}_2$, are circles with center $z_2 = 0$, and the radii of these circles have a positive lower bound r. If we apply to $\boldsymbol{\mathfrak{K}}_{\alpha_2}$ the transformation

$$z_1' = z_1, \quad z_2' = z_2(1 + \beta_2 z_1), \tag{5.29}$$

then the section $\mathfrak{K}_{\alpha_2}(z_1 = \gamma)$ is again a circle with center $z_2 = 0$ and of a radius greater or equal to $r|1 + \beta_2\gamma|$. For these values of $|\gamma| > l_2(\alpha_2)$ we can choose β_2 large enough such that

$$r|1 + \beta_2\gamma| > \mathsf{P} \tag{5.30}$$

holds, where P is such that for all sections $\mathfrak{B}^\dagger(z_1 = \gamma)$

$$|z_2| < \mathsf{P}$$

holds. As the domain $\mathfrak{B}^\dagger$ was assumed to be bounded, there always exists a finite such P. If β_2 is chosen such that it satisfies the inequality (5.30), then for $|\gamma| \geq l_2(\alpha_2)$,

$$\mathfrak{B}^\dagger(z_1 = \gamma) \subset \mathfrak{K}_{\alpha_2\beta_2}(z_1 = \gamma). \tag{5.31}$$

[20)] Shaded in Figure 2.

On the other hand,

$$|1+\beta_2\gamma| > 1$$

if $\beta_2 \geq 2\alpha_2 - \sigma_2$, for the inequality $|1+\beta_2\gamma| > 1$, which may also be written as $|\gamma + \frac{1}{\beta_2}| > \frac{1}{\beta_2}$, states that the point γ lies outside the circle with center $-\frac{1}{\beta_2}$ and radius $\frac{1}{\beta_2}$. For $\beta_2 > 2\alpha_2 - \sigma_2$, all points of the section $\mathfrak{K}_{\alpha_2}(z_2 = 0)$ lie outside of this circle by (5.29). Thus for $|\gamma| < l_2(\alpha_2)$,

$$\mathfrak{K}_{\alpha_2\beta_2}(z_1 = \gamma) \supset \mathfrak{K}_{\alpha_2}(z_1 = \gamma), \tag{5.32}$$

so that in connection with (5.31), we obtain that (5.31) is also true for $|\gamma| < l_2(\alpha_2)$. Hence

$$\boldsymbol{\mathfrak{K}}_{\alpha_2\beta_2} \supset \boldsymbol{\mathfrak{B}}^\dagger, \tag{5.33}$$

and furthermore, the two domains have the common boundary point Q and the tangent hyperplane $z_1 + \overline{z}_1 = 0$ at it, that is, $\boldsymbol{\mathfrak{K}}_{\alpha_2\beta_2}$ is an outer comparison domain for $\boldsymbol{\mathfrak{B}}^\dagger$. Now, $\boldsymbol{\mathfrak{K}}_{\alpha_2\beta_2}$ and $\boldsymbol{\mathfrak{B}}^\dagger$ are replacement domains of $\boldsymbol{\mathfrak{K}}_2$ and $\boldsymbol{\mathfrak{B}}$, and thus it holds that

$$\underline{L}^{\mathrm{I3}}_{\boldsymbol{\mathfrak{B}}}(Q) = \varliminf_{z_1,z_2\to 0} \varrho_{\mathrm{n}}^3 \mathsf{K}_{\boldsymbol{\mathfrak{B}}}(z_1,z_2;\overline{z}_1,\overline{z}_2) \geq L^{\mathrm{I3}}_{\boldsymbol{\mathfrak{K}}_2}(Q) = \frac{\sigma_2}{4\pi^2}, \tag{5.34}$$

and as this holds for every $\sigma_2 < \sigma$, we also have

$$\underline{L}^{\mathrm{I3}}_{\boldsymbol{\mathfrak{B}}}(Q) = \varliminf_{z_1,z_2\to 0} \varrho_{\mathrm{n}}^3 \mathsf{K}_{\boldsymbol{\mathfrak{B}}}(z_1,z_2;\overline{z}_1,\overline{z}_2) \geq \frac{\sigma}{4\pi^2}, \tag{5.35}$$

and combined with (5.1), (5.19) follows.

§ 6

In the cases $\sigma > 0$ and $\sigma < 0$, for a canonical replacement domain $\boldsymbol{\mathfrak{B}}^*$, in a neighborhood of the boundary point $Q(0,0)$, the plane $z_1' = 0$ was located either completely outside or completely inside of the domain $\boldsymbol{\mathfrak{B}}^*$, with the exception of Q itself. We assume now that the intersection of the analytic plane $z_1 = 0$ with the boundary of $\boldsymbol{\mathfrak{B}}$ contains, in addition to the point $z_2 = 0$, a whole surface patch $\mathfrak{H}$ that contains the point Q, that is, $z_2 = 0$, in its interior.[21] Therefore

[21] If there exists an analytic surface $z_1 = f(z_2)$ passing through Q and sharing a surface patch with $\boldsymbol{\mathfrak{B}}$, where $f(z_2)$ in $\mathfrak{B}^*$ (the projection of $\boldsymbol{\mathfrak{B}}$ to $z_1 = 0$) is a regular analytic and injective function of z_2, then this case can be reduced to the one considered in this paragraph or in §8 via the transformation $z_1' = z_1 - f(z_2)$, $z_2' = z_2$.

clearly $\sigma = 0$. First, we investigate the case $\sigma = 0$ for this special case. (The assumptions can be slightly generalized by only requiring continuity for the first derivatives of (2.7), where the existence of σ in the sense defined previously is not guaranteed.) We assume the approaches $\mathsf{A}^{\mathrm{II}}(\alpha)$ and $\mathsf{A}^{\mathrm{III}}(\alpha)$ and use normal coordinates throughout.

By the assumption of continuity of the first derivatives of (2.7) it follows from

$$\left(\frac{\partial \psi}{\partial y_1}\right)_{y_1=0} = 0 \tag{6.1}$$

that for

$$|y_1| < \delta \tag{6.2}$$

$$|\psi(y_1, z_2, \overline{z}_2) - \psi(0, z_2, \overline{z}_2)| \leq |y_1| A(y_1, z_2, \overline{z}_2), \quad \lim_{y_1 \to 0} A(y_1, z_2, \overline{z}_2) = 0 \tag{6.3}$$

holds. Form the assumptions made in the beginning, it follows for the z_2-values in $\mathfrak{H}$ that

$$\psi(0, z_2, \overline{z}_2) = 0$$

and hence

$$|\psi(y_1, z_2, \overline{z}_2)| \leq |y_1| A(y_1, z_2, \overline{z}_2). \tag{6.4}$$

If we only consider the points in $\mathfrak{B}$, for which

$$x_1 > 0, \quad |y_1| < N x_1 \tag{6.5}$$

holds, then for these points

$$|\psi(y_1, z_2, \overline{z}_2)| \leq N A(y_1, z_2, \overline{z}_2) x_1.$$

So if we pick a neighborhood $\mathfrak{U}(\varepsilon)$ with

$$|z_2| < \varepsilon, \tag{6.6}$$

such that $A(y_1, z_2, \overline{z}_2) < \frac{2}{N}$ in this neighborhood, which is always possible due to (6.3), then for these z_2,

$$|\psi(y_1, z_2, \overline{z}_2)| < 2x_1, \tag{6.7}$$

that is, the points whose coordinates satisfy the inequalities (6.3), (6.5), (6.6), are interior points of $\mathfrak{B}$.[22] The product domain $\mathfrak{T} = \mathfrak{B} \times \mathfrak{K}$, where

$$\begin{aligned} \mathfrak{B}: &\quad |z_1| \leq \delta, \quad x_1 > 0, \quad |y_1| \leq N x_1, \\ \mathfrak{K}: &\quad |z_2| \leq \varepsilon \end{aligned}$$

is thus contained in $\mathfrak{B}$ and has the point $Q(0,0)$ as a boundary point. So, given an approach $\mathsf{A}^{\mathrm{II}}(\alpha)$ in $\mathfrak{B}$, it is enough to choose

$$N > \tan(\alpha)$$

to ensure this is an approach in $\mathfrak{T}$ in the sense of the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$ defined in § 3. Because of $\mathfrak{T} \subset \mathfrak{B}$ and (1.5) we thus have

$$\overline{\lim_{z_1,z_2\to 0}} |z_1|^2 \mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2) \leq \lim_{z_1,z_2\to 0} |z_1|^2 \mathsf{K}_{\mathfrak{T}}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \frac{1}{4\omega^2\varepsilon^2\cos(\frac{\pi\alpha}{\omega})^2}, \tag{6.8}$$

where $\omega = 2\arctan(N)$. So the following holds:

Theorem III. *If in a boundary point $Q(0,0)$ of $\mathfrak{B}$ or of the canonical replacement domain $\mathfrak{B}^*$, the analytic plane lying in the tangent hyperplane (at Q) of $\mathfrak{B}$ or $\mathfrak{B}^*$, respectively, has a common surface segment with the boundary of $\mathfrak{B}$ or $\mathfrak{B}^*$, respectively, that contains Q as an inner point, then for the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$,*

$$\overline{L}^{\mathrm{II}2}_{\mathfrak{B}}(Q,\alpha) = \overline{\lim_{z_1,z_2\to 0,\mathrm{arc}(z_1)\to\alpha}} \varrho_t^2 \mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2)$$

for every $|\alpha| < \frac{\pi}{2}$.

Under stricter assumptions we prove:

Theorem IV. *Assume the domain $\mathfrak{B}$ or one of its replacement domains $\mathfrak{B}^*$ satisfies the conditions of Theorem III. Moreover, the analytic hyperplane lying in the tangent hyperplane at Q of $\mathfrak{B}$ or $\mathfrak{B}^*$, respectively, is assumed to have no common points with the interior of $\mathfrak{B}$ or $\mathfrak{B}^*$, respectively, and the points of the domain obtained from $\mathfrak{B}$ or $\mathfrak{B}^*$ by the transformation*

$$z_1' = \sqrt[n]{z_1}, \quad z_2' = z_2 \tag{6.9}$$

shall satisfy

$$z_1' + \overline{z}_1' \geq 0$$

[22] The interior of $\mathfrak{B}$ is given by (6.7) in a neighborhood of Q, cf. (2.8).

if n is sufficiently large. Then in the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$ for any α, the point Q is of order $\lambda = 2$.

In view of Theorem III, we only need to prove that for every approach $\mathsf{A}^{\mathrm{II}}(\alpha)$

$$\varlimsup_{z_1,z_2\to 0} \varrho_t^2 \mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2) > 0 \quad (\varrho_t = |z_1|). \tag{6.10}$$

Let $\mathfrak{B}'$ denote the domain obtained from $\mathfrak{B}$ or $\mathfrak{B}^*$ by the transformation (6.9). For the kernel function of the transformed domain, we obtain by (1.15)

$$\mathsf{K}_{\mathfrak{B}'}(z_1'z_2';\overline{z}_1',\overline{z}_2') = n^2|z_1'|^{2n-2}\mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2)$$

and hence also

$$|z_1'|^2\mathsf{K}_{\mathfrak{B}'}(z_1'z_2';\overline{z}_1',\overline{z}_2') = n^2|z_1|^2\mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2). \tag{6.11}$$

The transformation (6.9) maps the x_1-axis to itself, and under the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$, $\operatorname{arc}(z_1) \to \alpha$, $|\alpha| < \pi$, corresponds to the approach

$$\operatorname{arc}(z_1') \to \frac{\alpha}{n}. \tag{6.12}$$

So if $|z_1'|\mathsf{K}_{\mathfrak{B}}(z_1'z_2';\overline{z}_1',\overline{z}_2')$ remains above a certain bound under the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$, then the same will hold for the kernel of $\mathfrak{B}$ if the approach is in the sense of $\mathsf{A}^{\mathrm{II}}(\alpha)$. If now $\mathfrak{P}$ denotes a simply connected domain of the z_2-plane containing the projection of all sections $\mathfrak{B}'(z_1 = \gamma)$ in its interior (hence in particular $\mathfrak{H} = \mathfrak{B}'(z_1 = 0)$ and the point $z_2 = 0$), then $\mathfrak{B}'$ lies in the product domain

$$\mathfrak{R} = \mathfrak{E} \times \mathfrak{P},$$

where $\mathfrak{E}$ is the half-plane $z_1' + \overline{z}_1' > 0$, $z_2 = 0$. Now let any approach $\mathsf{A}^{\mathrm{II}}(\alpha)$ for the point Q with respect to $\mathfrak{B}$ be given. This corresponds to an approach in $\mathfrak{B}'$ with limit (6.12) which also represents an approach $\mathsf{A}^{\mathrm{II}}(\frac{\alpha}{n})$ for the product domain. Hence

$$|z_1|^2\mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \frac{1}{n^2}|z_1'|^2\mathsf{K}_{\mathfrak{B}'}(z_1',z_2';\overline{z}_1',\overline{z}_2') \geq |z_1'|^2\mathsf{K}_{\mathfrak{R}}(z_1,z_2;\overline{z}_1,\overline{z}_2) > 0, \tag{6.13}$$

which proves Theorem IV.

After having provided a class of boundary points with order $\lambda(Q;\mathfrak{B}) = 2$, in the remainder of this paragraph and in §8 we will encounter important special types of boundary points Q of this sort, for which we prove the existence of the limits

$L^{\mathrm{II}2}_{\mathfrak{B}}(Q,\alpha)$ and $L^{\mathrm{III}2}_{\mathfrak{B}}(Q,\alpha)$. To study the first type of these boundary points, we need an auxiliary function $f_\nu(z)$. This is the function mapping the half-plane $z+\overline{z}>0$ to the triangle $O_1A_1B_1$ in the w-plane, where $\overline{O_1A_1}=\overline{O_1B_1}=1$ and O_1, A_1, B_1 correspond to the points $z=0$, $z=-\mathrm{i}$, $z=\infty$. Here, let $\overline{OO_1}$ be the positive u-axis, and let the points O and O_1 have the coordinates 0 and 1. The angles OO_1B_1 and OO_1A_1 are both equal to $\frac{\pi}{2\nu}$. Then

$$f_\nu(z)=1-t_\nu(\mathrm{i}z), \tag{6.14}$$

where $t_\nu(z)$ is a function defined and studied in a previous article.[23)]

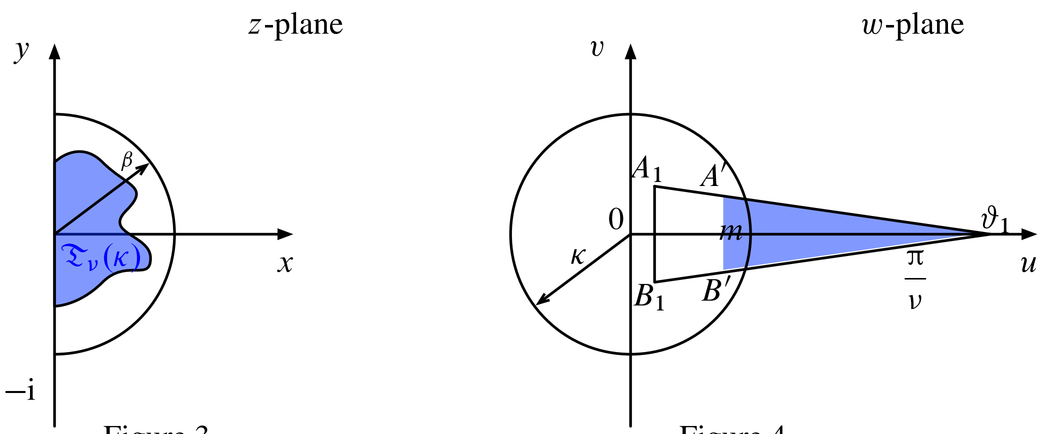


Figure 3 Figure 4

Lemma I. *Let $0<\kappa<1$. In the domain $\mathfrak{T}_\nu(\kappa)$ of the right z-halfplane given by*

$$|f_\nu(z)|>\kappa, \tag{6.15}$$

the following inequalities hold for sufficiently large ν:

$$|f_\nu(z)<(1-k(\nu)|z|^{\frac{1}{\nu}}, \qquad \lim_{\nu\to\infty}k(\nu)=1, \tag{6.16}$$

$$|\theta_\nu(z)|=|\operatorname{arc}(f_\nu(z))|<M_1|z|^{\frac{1}{\nu}}\sin\left(\frac{\pi}{2\nu}\right), \qquad 0<M_1<\frac{2}{\kappa}. \tag{6.17}$$

[23)]Compare the article *Über ausgezeichnete Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen*, Mathematische Annalen 104 (1931), p. 611, §2. In the following we will use the estimate (13) for $t_{\nu(z)}$ derived there.

By equation (13) in the previously mentioned article, it holds for $|z| < \beta < 1$ that

$$f_\nu(z) = 1 - z^{\frac{1}{\nu}} + z^{\frac{1}{\nu}} g_\nu(z) = 1 - |z|^{\frac{1}{\nu}} \mathrm{e}^{\mathrm{i}\frac{\varphi}{\nu}}(1 - g_\nu(z)), \quad \varphi = \operatorname{arc}(z), \tag{6.18}$$
$$g_\nu(z) = \mathrm{O}\left(\frac{1}{\nu}\right).$$

Now consider the semicircle $z + \overline{z} > 0$, $|z| < \beta$. Then the image $A'mB'$ (compare figure 4) of the boundary segment $|z| = \beta$ will come arbitrarily close to zero for sufficiently large ν, that is, it lies within the circle

$$|w| < \kappa. \tag{6.19}$$

Due to the uniqueness of the map it thus follows that the points, for which the inequality (6.15) holds, lies in the interior of the circle $|z| < \beta$.[24] But for these points, the representation (6.18) applies, which immediately yields (6.16). Furthermore, for z in $\mathfrak{T}_\nu(\kappa)$, and $M_2 \leq 1 + \varepsilon_\nu$ with $\lim_{\nu\to\infty} \varepsilon_\nu = 0$,

$$|\operatorname{Im}(f_\nu(z))| \leq |1 - f_{\nu(z)}| \sin\left(\frac{\pi}{2\nu}\right) < M_2 |z|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right), \tag{6.20}$$

hence for sufficiently large ν, $|\operatorname{Im}(f_\nu(z))| < \frac{\kappa}{2}$, and thus

$$|\operatorname{Re}(f_\nu(z))| > |f_\nu(z)| - |\operatorname{Im}(f_\nu(z))| > \frac{\kappa}{2}. \tag{6.21}$$

From (6.20) and (6.21) it follows that

$$|\theta_\nu(z)| < |\tan(\theta_\nu(z)| \leq \frac{2M_2}{\kappa} |z|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right) = M_1 |z|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right), \qquad M_1 = \frac{2M_2}{\kappa}, \tag{6.22}$$

which proves the claim (6.17). The points in the right z-half-plane lying outside of $\mathfrak{T}_\nu(\kappa)$ satisfy

$$|f_\nu(z)| \leq \kappa. \tag{6.23}$$

We now wish to describe a category of boundary points Q of which we will show that they are limit points of second order. For the point $Q(0, 0)$ we assume that the section $\mathfrak{H} = \mathfrak{B}(z_1 = 0)$ (belonging to the boundary of $\mathbf{\mathfrak{B}}$) is a starshaped domain relative to $z_2 = 0$,[25] that is, the equation defining the boundary curve is

[24] In the article mentioned in the previous footnote, our β is denoted by λ.
[25] Compare the footnote on p. 31.

$R = h(\theta)$, and moreover we assume that the function $h(\theta)$ satisfies the following Lipschitz condition[26)]

$$|h(\theta) - h(\theta')| \leq N_1|\theta - \theta'|, \qquad N_1 < \infty. \tag{6.24}$$

For the sections $\mathfrak{B}(z_1 = \gamma)$ we assume that, for $|\gamma| < \delta$,

$$\frac{\mathfrak{B}_1(z_1 = 0)}{m(|\gamma|)} \subset \mathfrak{B}(z_1 = \gamma) \subset m(|\gamma|)\mathfrak{B}(z_1 = 0), \tag{6.25}$$

where δ is a given arbitrarily small positive quantity, and

$$m(|\gamma|) = 1 + N_2|\gamma|^{\frac{1}{\tau}}, \qquad 0 < \tau < \infty, N_2 < \infty.$$

(Here, by $m(|\gamma|)\mathfrak{B}(z_1 = 0)$ we mean the domain obtained from $\mathfrak{B}(z_1 = 0)$ via the transformation $z_2' = m(|\gamma|)z_2$.) From (6.24) it follows that for every z_2 in the section $\mathfrak{B}(z_1 = \gamma)$, the inequality

$$\frac{h(\mathrm{arc}(z_2))}{1 + N_2|\gamma|^{\frac{1}{\tau}}} \leq |z_2| \leq (1 + N_2|\gamma|^{\frac{1}{\tau}})h(\mathrm{arc}(z_2)) \tag{6.26}$$

holds. Finally, we assume that $z_1 + \bar{z}_1 \geq 0$ for the domain $\mathfrak{B}$.

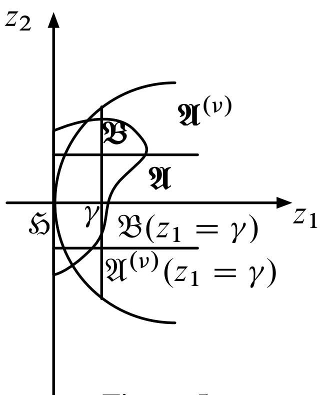


Figure 5

We construct a domain $\mathfrak{A}^{(\nu)}$ in the following way. Let $\mathfrak{E}$ denote the half-plane $z_1 + \bar{z}_1 \geq 0$, and consider the product domain

$$\mathfrak{A} = \mathfrak{E} \times \mathfrak{H}.$$

[26)]The inequality (6.24) can be replaced by $|h(\theta) - h(\theta')| \leq N_1|\theta - \theta'|^{\mu}$ ($\mu > 0$). Then the following arguments have to modified insignificantly.

The domain $\mathfrak{A}^{(\nu)}$ is obtained from $\mathfrak{A}$ via the transformation

$$z_1' = z_1, \quad z_2' = \frac{z_2}{f_\nu(z_1)}. \tag{6.27}$$

If $h_{\min}$ is a lower bound for the $h(\theta)$ (that is, if $0 \le h_{\min} \le h(\theta)$ for all $0 \le \theta \le 2\pi$) and $Z_2^{\max}$ is an upper bound for $|z_2|$ of all coordinates in the domain $\mathfrak{B}$, then κ (compare Lemma I) shall be chosen such that

$$0 < \kappa < \frac{h_{\min}}{Z_2^{\max}} \le 1. \tag{6.28}$$

After fixing κ, we choose ν large enough such that the inqualities (6.16), (6.17), (6.23) hold, and moreover such that

$$\nu > 2\tau, \quad k(\nu) \ge \frac{1}{2}. \tag{6.29}$$

If β denotes the radius of a circle centered at $z_1 = 0$ whose interior contains $\mathfrak{T}_\nu(\kappa)$, then finally we assume ν large enough such that[27]

$$N_2(\beta(\kappa,\nu))^{\frac{1}{\tau}-\frac{1}{\nu}} < \left(\frac{1}{2} - \frac{3}{2}\frac{N_1 M_1}{h_{\min}} \sin\left(\frac{\pi}{2\nu}\right)\right), \qquad M_1 \le \frac{2}{\kappa}, \tag{6.30}$$

$$\beta(\kappa,\nu) < \delta. \tag{6.31}$$

First, we consider the sections $\mathfrak{A}^{(\nu)}(z_1 = \gamma)$ for γ in $\mathfrak{T}_\nu(\kappa)$, where $\mathfrak{T}_\nu(\kappa)$ is the domain introduced on p. 36. The section $\mathfrak{A}(z_1 = \gamma)$ yields the starshaped domain $\mathfrak{H}$ for all points of $\mathfrak{E}$. Hence every section $\mathfrak{A}^{(\nu)}(z_1 = \gamma)$ is again a starshaped domain, whose boundary is given by the equation

$$R = h(\gamma,\theta) = \frac{h(\theta + \vartheta_\nu(\gamma))}{|f_\nu(\gamma)|}, \qquad \operatorname{arc}(f_\nu(\gamma)) = \vartheta_\nu(\gamma). \tag{6.32}$$

Now, our chosen ν satisfies by (6.29) and (6.16)

$$|f_\nu(\gamma)| \le 1 - \frac{1}{2}|\gamma|^{\frac{1}{\nu}}, \tag{6.33}$$

and by (6.24) and (6.17)

$$\begin{aligned} h(\theta + \vartheta_\nu(\gamma)) &\ge h(\theta) - N_1|\vartheta_\nu(\gamma)| \ge h(\theta) - N_1 M_1 |\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right) \\ &\ge h(\theta)\left(1 - \frac{N_1 M_1}{h_{\min}}|\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right) = h(\theta)\left(1 - N_3|\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right), \end{aligned} \tag{6.34}$$

[27] That it is possible to choose β such that (6.30) and (6.31) are satisfied follows from the fact that for fixed κ and sufficiently large ν the domain $\mathfrak{T}_\nu(\kappa)$ lies in the interior of an arbitrarily large circle centered at $z = 0$ (compare the proof of Lemma I).

with $N_3 = \frac{N_1 M_1}{h_{\min}}$, and thus

$$h(\gamma, \theta) \geq h(\theta)\left(1 - N_3|\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right)\left(1 + \frac{1}{2}|\gamma|^{\frac{1}{\nu}}\right). \tag{6.35}$$

On the other hand, by (6.26), the z_2-coordinate of a point in $\mathfrak{B}(z_1 = \gamma)$ satisfies

$$|z_2| \leq h(\theta)(1 + N_2|\gamma|^{\frac{1}{\tau}}), \qquad \theta = \mathrm{arc}(z_2). \tag{6.36}$$

Since now, by (6.30), for $|\gamma| \leq \beta$,

$$1 + (N_2|\gamma|^{\frac{1}{\tau}-\frac{1}{\nu}})|\gamma|^{\frac{1}{\nu}} < 1 + |\gamma|^{\frac{1}{\nu}}\left(\frac{1}{2} - N_3 \sin\left(\frac{\pi}{2\nu}\right) - \frac{1}{2}N_3|\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right) \tag{6.37}$$

it follows (for γ in $\mathfrak{T}_\nu(\kappa)$ z_2 in $\mathfrak{B}(z_1 = \gamma)$),

$$|z_2| < h(\gamma, \mathrm{arc}(z_2)), \tag{6.38}$$

that is,

$$\mathfrak{A}^{(\nu)}(z_1 = \gamma) \supset \mathfrak{B}(z_1 = \gamma). \tag{6.38a}$$

For the γ-values outside of $\mathfrak{T}_\nu(\kappa)$, by (6.23),

$$|f_\nu(\gamma)| \leq \kappa. \tag{6.39}$$

If $\mathfrak{H}_{\min}$ denotes a circle of radius $h_{\min}$ centered at the origin, and $\mathfrak{Z}^{\max}$ denotes a circle of radius $Z_2^{\max}$ with the same center, then by (6.38) and (6.39), for all γ outside of $\mathfrak{T}_\nu(\kappa)$,

$$\mathfrak{A}^{(\nu)}(z_1 = \gamma) \supset \frac{1}{\kappa}\mathfrak{H}_{\min} \supset \frac{Z_2^{\max}}{h_{\min}}\mathfrak{H}_{\min} = \mathfrak{Z}^{\max} \supset \mathfrak{B}(z_1 = \gamma), \tag{6.40}$$

which, together with (6.38a), implies:

$$\boldsymbol{\mathfrak{A}}^{(\nu)} \supset \boldsymbol{\mathfrak{B}}. \tag{6.41}$$

Now, $\boldsymbol{\mathfrak{A}}^{(\nu)}$ is obtained from $\boldsymbol{\mathfrak{A}}$ via the transformation (6.27), and hence

$$\lim_{z_1, z_2 \to 0} \frac{\partial(z_1', z_2')}{\partial(z_1, z_2)} = \lim_{z_1 \to 0} \frac{1}{f_\nu(z_1)} = 1. \tag{6.42}$$

It thus follows from (1.15) and (6.42) that

$$L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{A}}^{(\nu)}}(Q) = L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{A}}}(Q) = \frac{1}{4\pi^2 \cos(\alpha)^2 \mathsf{P}^2}, \tag{6.43}$$

where P is the mapping radius of $\mathfrak{H} = \boldsymbol{\mathfrak{B}}(z_1 = 0)$ with respect to $z_2 = 0$, and hence by (6.41),

$$L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{A}}}(Q) \geq \frac{1}{4\pi^2 \cos(\alpha)^2 \mathsf{P}^2}. \tag{6.44}$$

We now construct the inner comparison domain. Let $\mathfrak{D}$ denote the intersection of $\mathfrak{T}_\nu(\kappa)$ and $\mathfrak{B}(z_2 = 0)$.[28)] Consider the product domain

$$\boldsymbol{\mathfrak{D}} = \mathfrak{D} \times \mathfrak{H}$$

and apply to it the transformation

$$z_1' = z_1, \quad z_2' = z_2 f_\nu(z_1). \tag{6.45}$$

Then the transformed domain $\boldsymbol{\mathfrak{D}}^{(\nu)}$ is

$$\mathfrak{D}^{(\nu)}(z_1 = \gamma) = f_\nu(\gamma)\mathfrak{D}(z_1 = \gamma) = f_\nu(\gamma)\mathfrak{H}. \tag{6.46}$$

$\mathfrak{D}^{(\nu)}(z_1 = \gamma)$ is again starshaped. The equations of the boundary curve is

$$R = h^*(\gamma, \theta) = h(\theta - \vartheta_\nu(\gamma))|f_\nu(\gamma)|, \tag{6.46a}$$

and thus the sections $\mathfrak{B}(z_1 = \gamma)$ satisfy

$$h^*(\gamma, \theta) \leq h(\theta) \left(1 + N_3 |\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right) \left(1 - \frac{1}{2}|\gamma|^{\frac{1}{\nu}}\right).$$

On the other hand, by (6.25) for z_2 in $\mathfrak{B}(z_1 = \gamma)$,

$$|z_2| > h(\theta)(1 - N_2 |\gamma|^{\frac{1}{\tau}}), \qquad \theta = \operatorname{arc}(z_2),$$

and hence, if ν is sufficiently large according to the earlier conditions ((6.16), (6.17), etc.), then

$$1 - N_2|\gamma|^{\frac{1}{\tau}} \geq 1 - |\gamma|^{\frac{1}{\tau}} \left(\frac{1}{2} - N_3 \sin\left(\frac{\pi}{2\nu}\right) + \frac{1}{2} N_3 |\gamma|^{\frac{1}{\nu}} \sin\left(\frac{\pi}{2\nu}\right)\right).$$

This implies

$$\mathfrak{D}^{(\nu)}(z_1 = \gamma) \subset \mathfrak{B}(z_1 = \gamma) \quad \text{and} \quad \boldsymbol{\mathfrak{D}}^{(\nu)} \subset \boldsymbol{\mathfrak{B}}, \tag{6.47}$$

and hence

$$L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{D}}^{(\nu)}}(Q, \alpha) \geq L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{B}}}(Q, \alpha).$$

[28)] Or the connected component containing the boundary point $z_2 = 0$ of this intersection.

Now, by (6.42),

$$L^{\mathrm{II}2}_{\mathfrak{D}^{(\nu)}}(Q,\alpha) = L^{\mathrm{II}2}_{\mathfrak{D}}(Q,\alpha) = \frac{1}{4\pi^2\cos(\alpha)^2\mathsf{P}^2}, \tag{6.48}$$

so that the comparison of (6.44) and (6.48) yields

$$L^{\mathrm{II}2}_{\mathfrak{B}}(Q,\alpha) = \frac{1}{4\pi^2\cos(\alpha)^2\mathsf{P}^2}. \tag{6.49}$$

§ 7

In the following we study the case that the analytic tangent plane $z_1 = 0$ has a common surface segment with the boundary of $\mathfrak{B}$ under different assumptions on $\mathfrak{B}$ than in §6. The current paragraph serves to introduce the tools for this endeavour by introducing and studying a certain class of special domains.

In this paragraph, by a *sector* $\mathcal{S}$ we mean a circular sector OAB in the z_1-plane whose vertex O lies in the point $z_1 = 0$. It is completely determined by the angles ϑ_1 and ϑ_2 ($> \vartheta_1$) enclosed by its radii with a fixed direction, say the positive x_1-axis, and by the radius $\varrho = \overline{OA} = \overline{OB}$ of the circle.

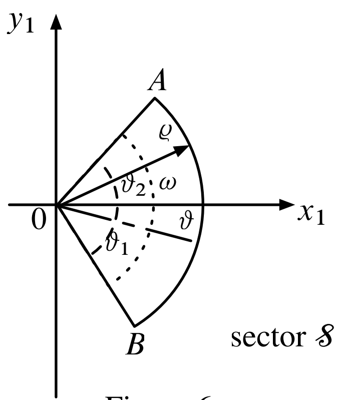


Figure 6

We also write $\mathcal{S}(\vartheta_1, \vartheta_2, \varrho)$ to emphasize the dependence of $\mathcal{S}$ on ϑ_1, ϑ_2, ϱ. In the following, we study domains $\mathfrak{S}$ with the following properties:

1. The sections $\mathfrak{S}(z_2 = \gamma)$ are non-empty only for those values γ belonging to a domain $\mathfrak{H}$ in the z_2-plane, and then they are sectors $\mathcal{S}$.

2. For the determining values ϑ_1, ϑ_2, ϱ of the section $\mathfrak{S}(z_2 = \gamma)$, which are now functions of γ, it holds that

$$\vartheta_2(\gamma) - \vartheta_1(\gamma) = \omega = \text{const.} > 0, \qquad \varrho = \text{const.} \tag{7.1}$$

3. $\vartheta(\gamma) = \frac{\vartheta_2(\gamma)+\vartheta_1(\gamma)}{2}$.

4. There exists a sector $\mathcal{S}(\vartheta_1^{(0)}, \vartheta_2^{(0)}, \varrho)$, independent of γ, that has only the point $z_1 = 0$ in common with all $\mathfrak{S}(z_2 = \gamma) = \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \varrho)$.[29)]

5. There exists β (with $0 < \beta < \pi$) such that

$$-\pi + \beta \leq \vartheta_1(\gamma)\frac{\pi}{\omega}, \quad \vartheta_2(\gamma)\frac{\pi}{\omega} \leq \pi - \beta. \tag{7.2}$$

Common to all sectors $\mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \varrho)$ is a fixed sector $\mathcal{S}(\vartheta_1^{(1)}, \vartheta_2^{(2)}, \varrho)$ with $\vartheta_2^{(1)} - \vartheta_1^{(1)} > 0$.

For the positive x_1-axis we choose a ray contained in the interior of $\mathcal{S}(\vartheta_1^{(1)}, \vartheta_2^{(2)}, \varrho)$. Then

$$\vartheta_{1\sup} < 0 < \vartheta_{2\inf}, \tag{7.3}$$

where $\vartheta_{1\sup}$ is the upper bound of $\vartheta_1(\gamma)$ and $\vartheta_{2\inf}$ is the lower bound of $\vartheta_2(\gamma)$.

In the following, we will occasionally write $\mathcal{T}(\theta, \omega, \varrho)$ instead of $\mathcal{S}(\theta, \omega, \varrho)$. Via any of the transformations

$$\widetilde{z}_1 = \mu z_1, \quad \widetilde{z}_2 = z_2, \qquad (\mu > 0), \tag{7.4}$$

or

$$\widetilde{z}_1 = \mathrm{e}^{\mathrm{i}\theta} z_1, \quad \widetilde{z}_2 = z_2, \qquad (\theta \text{ real}), \tag{7.5}$$

or

$$\widetilde{z}_1 = z_1^{\beta}, \quad \widetilde{z}_2 = z_2, \qquad (\mu > 0), \tag{7.6}$$

the domain $\mathfrak{S}$ is mapped to a domain $\widetilde{\mathfrak{S}}$ with the same properties,[30)] and for the determining parameters, we then obtain

$$\widetilde{\vartheta}_2(\gamma) = \vartheta_2(\gamma), \quad \widetilde{\vartheta}_1(\gamma) = \vartheta_1(\gamma), \quad \widetilde{\varrho} = \mu\varrho, \quad \widetilde{\omega} = \omega, \quad \widetilde{\vartheta}(\gamma) = \vartheta(\gamma), \tag{7.7}$$

[29)] Possibly also the point $z_1 = \infty$.

[30)] In order not to lose property 4 under the transformation (7.6), it might be necessary to consider $\widetilde{\mathfrak{S}}$ in a (multiply covered) Riemannian space.

or

$$\widetilde{\vartheta}_2(\gamma) = \vartheta_2(\gamma)+\theta, \quad \widetilde{\vartheta}_1(\gamma) = \vartheta_1(\gamma)+\theta, \quad \widetilde{\varrho} = \varrho, \quad \widetilde{\omega} = \omega, \quad \widetilde{\vartheta}(\gamma) = \vartheta(\gamma)+\theta, \tag{7.8}$$

or

$$\widetilde{\vartheta}_2(\gamma) = \beta\vartheta_2(\gamma), \quad \widetilde{\vartheta}_1(\gamma) = \beta\vartheta_1(\gamma), \quad \widetilde{\varrho} = \varrho^{\beta}, \quad \widetilde{\omega} = \omega^{\beta}, \quad \widetilde{\vartheta}(\gamma) = \beta\vartheta(\gamma). \tag{7.9}$$

For domains $\mathfrak{S}$, we investigate the behavior of the kernel function in a neighborhood of the boundary point $Q(0, a_2)$, where a_2 is an inner point of $\mathfrak{H}$. In general, the tangent hyperplane does not exist at the point $Q(0, a_2)$ (with the exception of $\omega = \pi$). However, in analogy to the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$, we can introduce corresponding notions of approach and order. Under an *approach* $\mathsf{A}^{\mathrm{II}}(\alpha)$ we understand here

$$\begin{gathered} z_1 \to 0, \quad z_2 \to a_2, \\ \operatorname{arc}(z_1) \to \alpha, \quad |\alpha - \vartheta(a_2)| < \frac{\omega}{2}. \end{gathered} \tag{7.10}$$

For this approach, we study

$$|z_1|^2 \mathsf{K}_{\mathfrak{S}}(z_1, z_2; \overline{z}_1, \overline{z}_2), \tag{7.10a}$$

and in particular will prove the existence of

$$L^{\mathrm{II}2}_{\mathfrak{S}}(Q(0, a_2), \alpha) = \lim_{t\to 0, \varphi\to\alpha, z_2\to a_2} t^2 \mathsf{K}_{\mathfrak{S}}(t e^{i\varphi}, z_2; t e^{-i\varphi}, \overline{z}_2), \tag{7.11}$$

where $t e^{i\varphi}$ $(t < \varrho)$ is an inner point of the section $\mathfrak{S}(z_2 = a_2)$, and t is real and positive.

Under the transformations (7.4), (7.5), (7.6), the kernel functions become

$$|z_1|^2 \mathsf{K}_{\widetilde{\mathfrak{S}}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \left|\frac{z_1}{\mu}\right|^2 \mathsf{K}_{\mathfrak{S}}\left(\frac{z_1}{\mu}, z_2; \frac{\overline{z}_1}{\mu}, \overline{z}_2\right), \tag{7.12}$$

or

$$|z_1|^2 \mathsf{K}_{\widetilde{\mathfrak{S}}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = |z_1|^2 \mathsf{K}_{\mathfrak{S}}\left(\frac{z_1}{e^{i\theta}}, z_2; \frac{\overline{z}_1}{e^{-i\theta}}, \overline{z}_2\right), \tag{7.13}$$

or

$$|z_1|^2 \mathsf{K}_{\widetilde{\mathfrak{S}}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{1}{\beta^2} |z_1^{\frac{1}{\beta}}|^2 \mathsf{K}_{\mathfrak{S}}\left(z_1^{\frac{1}{\beta}}, z_2; \overline{z}_1^{\frac{1}{\beta}}, \overline{z}_2\right). \tag{7.14}$$

If we replace the parameter t for the domain $\mathfrak{S}$ by μt, then the expressions (7.10a) and (7.12) become identical in the two domains. On the other hand, for identical

$\vartheta(\gamma)$ and ω, the domain $\widetilde{\mathfrak{S}}$ has radius $\widetilde{\varrho} = \mu\varrho$. For our investigation, the value of ϱ is thus irrelevant, and the limit (7.11) is also independent of ϱ.

A particularly important class of domains $\mathfrak{S}$ is given by the domains $\mathfrak{S}^\dagger$, for which $\varrho = \infty$. The transformation (7.4) maps a domain $\mathfrak{S}^\dagger$ to itself. So if we set $z = t\mathrm{e}^{\mathrm{i}\varphi}$, then by (7.12),

$$|t|^2 \mathsf{K}_{\mathfrak{S}^\dagger}(t\mathrm{e}^{\mathrm{i}\varphi}, z_2; t\mathrm{e}^{-\mathrm{i}\varphi}, \overline{z}_2) \tag{7.15}$$

is a function, independent of t. If $\mathfrak{H}_m$ is a simply connected domain contained in the interior of $\mathfrak{H}$, then $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$ shall denote the following variable domain of (z_2, α): z_2 is a point of $\mathfrak{H}_m$ and the corresponding α is a point in the interval

$$\vartheta_1(z_2) + \varepsilon \leq \alpha \leq \vartheta_2(z_2) - \varepsilon. \tag{7.16}$$

For values (z_2, α) from $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$, the expression (7.15) is, for fixed $t = t_0 > 0$, a uniformly continuous function in α and z_2 (compare §1, p. 8). Now, since (7.15) is a function not depending on t, the following holds:

Lemma II. *For (a_2, α) in $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$, the limit*

$$L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}(Q(0, a_2), \alpha) = \lim_{t\to 0, \varphi\to\alpha, z_2\to a_2} |t^2| \mathsf{K}_{\mathfrak{S}^\dagger}(t\mathrm{e}^{\mathrm{i}\varphi}, z_2; \mathrm{e}^{-\mathrm{i}\varphi}, \overline{z}_2) \tag{7.17}$$

exists and is a uniformly continuous function on $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$ in the variables a_2 and α.

Lemma III. *Suppose the sequence of domains*[31)]

$$\mathfrak{S}^{\dagger(m)} = \sum \mathfrak{S}^{\dagger(m)}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{S}(\vartheta_1^{(m)}(\gamma), \vartheta_2^{(m)}(\gamma), \infty), \qquad m = 1, 2, 3, \ldots$$

converges uniformly on $\mathfrak{H}$ to the domain

$$\mathfrak{S}^\dagger = \sum \mathfrak{S}^\dagger(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \infty), \tag{7.17a}$$

in the sense that for each γ,

$$\vartheta^{(m)}(\gamma) \to \vartheta(\gamma), \quad \text{and} \quad \omega^{(m)} \to \omega. \tag{7.18}$$

Then:

$$\lim_{m\to\infty} L^{\mathrm{II}2}_{\mathfrak{S}^{\dagger(m)}}(Q(0, a_2), \alpha) = L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}(Q(0, a_2), \alpha). \tag{7.19}$$

31) For the notation $\sum_{\gamma}^{\mathfrak{H}}$, see Hausdorff, *Mengenlehre*, Berlin 1927, §1.

For the proof, apply the transformation

$$\widetilde{z}_1 = z_1^{1-\tau}, \quad \widetilde{z}_2 = z_2, \qquad \tau > 0, \tag{7.20}$$

to $\mathfrak{S}^\dagger$. The thus obtained domain $\widetilde{\mathfrak{S}}^\dagger$ is again a $\mathfrak{S}^\dagger$-domain with angles

$$\widetilde{\vartheta}_1(\gamma) = (1-\tau)\vartheta_1(\gamma), \quad \widetilde{\vartheta}_2(\gamma) = (1-\tau)\vartheta_2(\gamma).$$

Since

$$\vartheta_1(\gamma) < 0, \quad \vartheta_2(\gamma) > 0, \tag{7.21}$$

we have

$$\widetilde{\vartheta}_1(\gamma) = (1-\tau)\vartheta_1(\gamma) > \vartheta_1(\gamma), \quad \widetilde{\vartheta}_2(\gamma) = (1-\tau)\vartheta_2(\gamma) < \vartheta_2(\gamma), \tag{7.22}$$

and for sufficiently large m, due to (7.21), (7.3) and (7.18),

$$\widetilde{\vartheta}_1(\gamma) > \vartheta_1^{(m)}(\gamma), \quad \widetilde{\vartheta}_2(\gamma) < \vartheta_2^{(m)}(\gamma). \tag{7.23}$$

Hence, for sufficiently large m,

$$\mathcal{S}(\widetilde{\vartheta}_1(\gamma), \widetilde{\vartheta}_2(\gamma), \infty) \subset \mathcal{S}(\widetilde{\vartheta}_1^{(m)}(\gamma), \widetilde{\vartheta}_2^{(m)}(\gamma), \infty), \tag{7.24}$$

that is,

$$\widetilde{\mathfrak{S}}^\dagger \subset \mathfrak{S}^{\dagger(m)}.$$

From this, we obtain under the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$

$$\lim_{|z_1|\to 0, \mathrm{arc}(z_1)\to\alpha, z_2\to a_2} |z_1|^2 \mathsf{K}_{\mathfrak{S}^{\dagger(m)}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \le \lim_{|z_1|\to 0, \mathrm{arc}(z_1)\to\alpha, z_2\to a_2} |z_1|^2 \mathsf{K}_{\widetilde{\mathfrak{S}}^\dagger}(z_1, z_2; \overline{z}_1, \overline{z}_2),$$

hence also

$$\overline{\lim_{m\to\infty}} L^{\mathrm{II}2}_{\mathfrak{S}^{\dagger(m)}}(Q(0,a_2),\alpha) \le L^{\mathrm{II}2}_{\widetilde{\mathfrak{S}}^\dagger}(Q(0,a_2),\alpha).$$

Now, by (7.14),

$$L^{\mathrm{II}2}_{\widetilde{\mathfrak{S}}^\dagger}(Q(0,a_2),\alpha) = \frac{1}{(1-\tau)^2} L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}\left(Q(0,a_2), \frac{\alpha}{1-\tau}\right). \tag{7.25}$$

As τ can be chosen arbitrarily close to 0 and $L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}(Q(0,a_2),\alpha)$ is a continuous function in α, it follows that

$$\overline{\lim}\, L^{\mathrm{II}2}_{\mathfrak{S}^{\dagger(m)}}(Q(0,a_2),\alpha) \le L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}(Q(0,a_2),\alpha) \tag{7.26}$$

On the other hand, if we apply the transformation

$$\approxeq\!\!\!z_1 = z_1^{1+\tau}, \quad \approxeq\!\!\!z_2 = z_2, \tag{7.27}$$

then we obtain in the same manner that for sufficiently large m,

$$\overset{\approx}{\widetilde{\mathfrak{S}}} \supset \mathfrak{S}^{\dagger(m)}, \tag{7.28}$$

and from this

$$\underline{\lim}\, L^{\mathrm{II}2}_{\mathfrak{S}^{\dagger(m)}}(Q(0,a_2),\alpha) \geq L^{\mathrm{II}2}_{\overset{\approx}{\widetilde{\mathfrak{S}}}^{\dagger}}(Q(0,a_2),\alpha). \tag{7.29}$$

Now (7.26) and (7.29) imply (7.19), which proves the lemma.

By a $\mathcal{J}$-*circle* we mean a circle with center $\frac{R\mathrm{e}^{\mathrm{i}\vartheta(\gamma)}}{2\cos(\vartheta(\gamma))}$ and radius $\frac{R}{2\cos(\vartheta(\gamma))}$ located in the plane $z_2 = \gamma$. (The points $z_1 = 0$ and $z_1 = R$ have all circle peripheries in common.) We will also write $\mathcal{J}(\vartheta(\gamma), R)$ to emphasize the dependence of $\mathcal{J}$ on $\vartheta(\gamma)$ and R. In doing so, we assume that $\vartheta(\gamma)$ satisfies all conditions on $\frac{\vartheta_1(\gamma)+\vartheta_2(\gamma)}{2}$ stated on p. 42. By a $\mathfrak{J}$-*domain* we mean a domain

$$\mathfrak{J} = \sum \mathfrak{J}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{J}(\vartheta(\gamma), R), \qquad R = \text{const.}$$

Via the transformation

$$z_1^{\dagger} = \frac{z_1}{1 - \frac{z_1}{R}}, \quad z_2^{\dagger} = z_2, \tag{7.30}$$

the $\mathfrak{J}$-domain is mapped to the domain $\mathfrak{S}^{\dagger} = \sum \mathfrak{S}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{T}(\vartheta(\gamma), \pi, \infty)$, where the boundary point $z_1 = R$ is mapped to the point $z_1^{\dagger} = \infty$. As the transformation (7.30) is regular in a neighborhood of $z_1 = 0$, $z_2 = a_2$, and $\lim \frac{\partial(z_1^{\dagger}, z_2^{\dagger})}{\partial(z_1, z_2)} = 1$, by (3.5) and (3.12) we have,

$$L^{\mathrm{II}2}_{\mathfrak{J}}(Q(0,a_2),\alpha) = L^{\mathrm{II}2}_{\mathfrak{S}^{\dagger}}(Q(0,a_2),\alpha), \tag{7.31}$$

and by Lemma II, (7.31) is a uniformly continuous function on $\mathfrak{h}_m(\vartheta_1+\varepsilon, \vartheta_2-\varepsilon)$ in the variables a_2 and α.

The domain $\mathfrak{S} = \sum \mathfrak{S}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \varrho) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{T}(\vartheta(\gamma), \omega, \varrho)$ under the transformation

$$\widetilde{z}_1 = z_1^{\frac{\pi}{\omega}}, \quad \widetilde{z}_2 = z_2, \tag{7.32}$$

is mapped to the domain

$$\widetilde{\mathfrak{S}} = \sum \widetilde{\mathfrak{S}}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{T}(\widetilde{\vartheta}(\gamma), \pi, \widetilde{\varrho}), \quad \widetilde{\vartheta}(\gamma) = \frac{\pi}{\omega}, \widetilde{\varrho} = \varrho^{\frac{\pi}{\omega}}. \tag{7.33}$$

From now on we denote by $\mathfrak{S}_0^\dagger$ and $\mathfrak{J}_0$ a certain $\mathfrak{S}^\dagger$- or $\mathfrak{J}$-domain, namely, with angles

$$\widetilde{\vartheta}_1(\gamma) = \frac{\pi}{\omega}\vartheta_1(\gamma), \quad \widetilde{\vartheta}_2(\gamma) = \frac{\pi}{\omega}\vartheta_2(\gamma), \tag{7.34}$$

and with $R = \widetilde{\varrho}\sin(\beta)$ (see (7.2) and (7.1) for the meanings of β and $\widetilde{\varrho}$, respectively). Then

$$\mathfrak{J}_0 \subset \widetilde{\mathfrak{S}} \subset \mathfrak{S}_0^\dagger \tag{7.35}$$

and therefore

$$\mathsf{K}_{\mathfrak{J}_0}(z_1, z_2; \overline{z}_1, \overline{z}_2) \geq \mathsf{K}_{\widetilde{\mathfrak{S}}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \geq \mathsf{K}_{\mathfrak{S}_0^\dagger}(z_1, z_2; \overline{z}_1, \overline{z}_2). \tag{7.36}$$

From (7.31) and (7.36) it follows that

$$L^{\text{II2}}_{\widetilde{\mathfrak{S}}}(Q(0, a_2), \alpha) \tag{7.37}$$

exists and equals (7.31). Hence (7.37) is a uniformly continuous function on $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$ in the variables α and a_2. If we introduce the polar coordinates $z_1 = T\mathrm{e}^{\mathrm{i}\Phi}$ and $\widetilde{z}_1 = t\mathrm{e}^{\mathrm{i}\varphi}$ for the quantities in (7.32), then by (7.14) and (7.32)

$$\left(\frac{\pi}{\omega}\right)^2 t^2 \mathsf{K}_{\widetilde{\mathfrak{S}}}(t\mathrm{e}^{\mathrm{i}\varphi}, \widetilde{z}_2; t\mathrm{e}^{-\mathrm{i}\varphi}, \overline{\widetilde{z}}_2) = T^2 \mathsf{K}_{\mathfrak{S}}(T\mathrm{e}^{\mathrm{i}\Phi}, z_2; T\mathrm{e}^{-\mathrm{i}\Phi}, \overline{z}_2), \tag{7.38}$$

$$t = T^{\frac{\pi}{\omega}}, \quad \varphi = \frac{\pi}{\omega}\Phi, \quad \widetilde{z}_2 = z_2.$$

Since $t \to 0$ implies $T \to 0$ and the transformation (7.32) corresponds to the transformation $\varphi = \frac{\pi}{\omega}\Phi, \widetilde{z}_2 = z_2$, (7.38) implies the following theorem.

Theorem V. *$L^{\text{II2}}_{\mathfrak{S}}(Q(0, a_2), \alpha)$ exists for values (a_2, α) in $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$, and is a uniformly continuous function on $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$ in a_2 and α.*

From Theorem V and Lemma III now follows and important generalization of the latter, namely:

Corollary I. *Suppose the sequence of domains*

$$\mathfrak{S}^{(m)} = \sum \mathfrak{S}^{(m)}(z_2 = \gamma) = \sum_{\gamma}^{\mathfrak{H}} \mathcal{S}(\vartheta_1^{(m)}(\gamma), \vartheta_2^{(m)}(\gamma), \varrho^{(m)}), \quad 0 < \varrho_0 \leq \varrho^{(m)} \leq \varrho_1 < \infty,$$

converges to the domain $\mathfrak{S} = \sum \mathfrak{S}(z_2 = \gamma) = \sum_\gamma^{\mathfrak{H}} \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \varrho)$ *in the sense that* $\vartheta^{(m)}(\gamma) \to \vartheta(\gamma)$ *and* $\omega^{(m)} \to \omega$ *uniformly on* $\mathfrak{H}$ *for each* γ. *Then*

$$\lim_{m\to\infty} L^{\mathrm{II}2}_{\mathfrak{S}^{(m)}}(Q(0, a_2), \alpha) = L^{\mathrm{II}2}_{\mathfrak{S}}(Q(0, a_2), \alpha).$$

Remark. Let $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \mathfrak{H}_3 \subset \ldots$ denote an ascending sequence of domains converging to a kernel domain $\mathfrak{H}$. Let $\mathfrak{S}^\dagger$ be the domain given by (7.17a) and $\mathfrak{S}^\dagger_m = \sum_\gamma^{\mathfrak{H}_m} \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \infty)$. Then: $\lim_{m\to\infty} L^{\mathrm{II}2}_{\mathfrak{S}^\dagger_m}(Q(0, a_2), \alpha) = L^{\mathrm{II}2}_{\mathfrak{S}^\dagger}(Q(0, a_2), \alpha)$, uniformaly on $\mathfrak{h}_m(\vartheta_1 + \varepsilon, \vartheta_2 - \varepsilon)$ in a_2 and α. For, if $f_m(z_2)$ (with $f_m(a_2) = a_2$, $f'_m(a_2) > 0$) denotes the function that maps $\mathfrak{H}_m$ to $\mathfrak{H}$, and $r_m(z_2)$ the corresponding inverse function, then the transformation $\widetilde{z}_1 = z_1$, $\widetilde{z}_2 = f_m(z_2)$ maps the domain $\mathfrak{S}^\dagger_m$ to $\widetilde{\mathfrak{S}}^\dagger_m = \sum \mathcal{S}(\vartheta_1^{(m)}(\gamma), \vartheta_2^{(m)}(\gamma), \infty)$, where $\vartheta_k^{(m)}(\gamma) = \vartheta_k(r_m(\gamma))$. By (1.15) it holds that $L^{\mathrm{II}2}_{\widetilde{\mathfrak{S}}^\dagger_m}(Q, \alpha) = |r'_m(a_2)|^2 L^{\mathrm{II}2}_{\mathfrak{S}^\dagger_m}(Q, \alpha)$. By a well-known theorem from complex analysis, $\lim_{m\to\infty} f_m(z_2) = z_2)$ uniformly on $\mathfrak{H}$. Together with Lemma III, this yields the desired limit. An analogous limit relation can be obtained for domains $\mathfrak{S}$ by (7.31), (7.37) etc.

§ 8

Let z_1, z_2 be normal coordinates for the point Q of $\mathfrak{B}$, and assume again that the plane $z_1 = 0$ has a common surface segment $\mathfrak{H}$ with the boundary of $\mathfrak{B}$ with inner point $z_2 = 0$. In this paragraph, we make the following assumptions:

1. $\mathfrak{H} = \mathfrak{B}(z_1 = 0)$ is a starshaped domain for which the boundary curve satisfies
$$R = h(\theta), \tag{8.1}$$
and $h(\theta)$ satisfies the Lipschitz condition
$$|h(\theta) - h(\theta')| \le A|\theta - \theta'|. \tag{8.2}$$

2. We assume that all z_1-coordinates satisfy $|z_1| \le 1$ (we can easily achieve this by a simple transformation), and assume that the sections $\mathfrak{B}(z_1 = 0)$ satisfy
$$\mathfrak{B}(z_1 = 0) \subset \frac{1}{1 - \gamma^{\frac{1}{\nu}}} \mathfrak{H}, \qquad 0 < \nu < \infty. \tag{8.3}$$

3. For each non-empty section $\mathfrak{B}(z_2 = \gamma)$ we assume that there is a line $\mathfrak{g}(\gamma)$ passing through the point $z_1 = 0$, for which the angle $\Psi(\gamma)$ with the positive x_1-axis satisfies

$$|\Psi(\gamma) - \Psi(\gamma')| \leq C|\gamma - \gamma'|^{\mu}, \qquad \mu > 0, \tag{8.4}$$

and

$$\Psi(\gamma) + C|\gamma|^{\mu}|z_1|^{\frac{\mu}{\nu}} < \operatorname{arc}(z_1) < \pi + \Psi(\gamma) - C|\gamma|^{\mu}|z_1|^{\frac{\mu}{\nu}}, \tag{8.5}$$

where z_1 is any point in the section $\mathfrak{B}(z_2 = \gamma)$.

It follows in particular from (8.5) that the section $\mathfrak{B}(z_2 = \gamma)$ and its boundary, with the exception of the point $z_1 = 0$, lie on one side of the line $\mathfrak{g}(\gamma)$. For those values of γ belonging to $\mathfrak{H}$, by 4., the line $\mathfrak{g}(\gamma)$ is the tangent of the boundary of $\mathfrak{B}(z_2 = \gamma)$ in the point $z_1 = 0$ (compare figure 9).

4. We make the assumption on the sections $\mathfrak{B}(z_2 = \gamma)$ that for every closed domain $\mathfrak{H}'$ contained in the interior of $\mathfrak{H}$, there is a positive number κ (independent of γ) such that for γ in $\mathfrak{H}'$, every section $\mathfrak{B}(z_2 = \gamma)$ contains a circle of radius κ that touches the boundary of $\mathfrak{B}(z_2 = \gamma)$ in the point $z_1 = 0$.[32)]

5. For $\Psi(\gamma)$, the following inequality holds:

$$-\pi + \beta \leq \Psi(\gamma) \leq -\beta, \qquad \beta > 0. \tag{8.6}$$

(So there is a fixed sector $\mathcal{S}$ (independent of γ) that has only the point $z_1 = 0$ in common with $\mathfrak{B}(z_2 = \gamma)$.)

Under these conditions we prove:

Theorem VI. *For every domain* $\mathfrak{B}$ *of the given type, there exists the limit*

$$L^{\mathrm{II}2}_{\mathfrak{B}}(Q, \alpha) = \lim_{z_1, z_2 \to 0, \operatorname{arc}(z_1) \to \alpha} |z_1|^2 \mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \tag{8.7}$$

under the approach $\mathsf{A}^{\mathrm{II}}(\alpha)$.

Proof: Apply the transformation

$$\widetilde{z}_1 = z_1, \quad \widetilde{z}_2 = (1 - z_1^{\frac{1}{\nu}})z_2, \tag{8.8}$$

to the domain $\mathfrak{B}$ and prove for the transformed domain $\widetilde{\mathfrak{B}}$:

[32)]Instead of 4., we could make the weaker assumption that for each $\mathfrak{H}'$ there exists a sequence $\mathcal{T}(\Psi(\gamma) + \frac{\pi}{2}, \omega_m, \varrho_m)$, $m = 1, 2, 3, \ldots$, with $\varrho_m > 0$, $\lim_{m \to \infty} \omega_m = \pi$, such that for each γ in $\mathfrak{H}'$ it holds that $\mathcal{T}(\Psi(\gamma) + \frac{\pi}{2}, \omega_m, \varrho_m) \subset \mathfrak{B}(z_2 = \gamma)$. Compare p. 53.

1. We have
$$\widetilde{\mathfrak{B}}(z_1 = 0) \supset \widetilde{\mathfrak{B}}(z_1 = \gamma). \tag{8.9}$$

2. If γ is a point in $\mathfrak{H}$, then the line $\mathfrak{g}(\gamma)$ touches the boundary of $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ in $z_1 = 0$, and the section $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ lies on one side of this tangent.[33)]

Part 1 follows immediately from assumption 2 above.

For part 2: If
$$2x_1 = \psi(y_1, z_2, \overline{z}_2) \tag{2.7}$$
is the equation of the boundary surface, then from the assumptions 4 and 3 on ψ it follows that
$$\frac{\psi(y_1, z_2, \overline{z}_2)}{2y_1} = \chi(y_1, z_2, \overline{z}_2) \tag{8.9a}$$
is a continuous function in the variables y_1, z_2, $\overline{z}_2$. Now, for every γ in $\mathfrak{H}$,
$$\tan(\Psi(\gamma)) = \lim_{y_1 \to 0} \frac{2y_1}{\psi(y_1, z_2, \overline{z}_2)} = \frac{1}{\chi(0, \gamma, \overline{\gamma})},$$
and for the tan of the angle enclosed by the tangent and the boundary curve of $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ (given by $2\widetilde{x}_1 = \psi(\widetilde{y}_1, \frac{\widetilde{z_2}}{1-\widetilde{z}_1^{\frac{1}{\nu}}}, \frac{\widetilde{\overline{z}_2}}{1-\widetilde{\overline{z}}_1^{\frac{1}{\nu}}}))$ we obtain, again using (8.9a),
$$\lim_{\widetilde{y}_1 \to 0} \frac{2\widetilde{y}_1}{2\widetilde{x}_1} = \lim_{x_1, y_1 \to 0} \frac{1}{\chi\left(y_1, \frac{\gamma}{1-z_1^{\frac{1}{\nu}}}, \frac{\overline{\gamma}}{1-\overline{z}_1^{\frac{1}{\nu}}}\right)} = \frac{1}{\chi(0, \gamma, \overline{\gamma})}, \tag{8.10}$$
so that the line $\mathfrak{g}(\gamma)$ is the tangent at the point $z_1 = 0$ to the boundary of the section $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ of the transformed domain, which proves the first part of 2.

Let $z_1 = r_1^0 e^{i\varphi_0}$, $z_2 = \gamma$ be a point of $\mathfrak{B}$ and $z_1 = r_1^0 e^{i\varphi_0}$, $z_2 = \widetilde{\gamma}$ the corresponding point in $\widetilde{\mathfrak{B}}$. Now we have to show that $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ lies on one side of $\mathfrak{g}(\gamma)$. In combination with $\gamma - \widetilde{\gamma} = \gamma z_1^{\frac{1}{\nu}}$ it follows from (8.4) that
$$|\Psi(\gamma) - \Psi(\widetilde{\gamma})| \leq C|\gamma|^{\mu}|z_1|^{\frac{\mu}{\nu}}. \tag{8.11}$$

We will now show that for the angle χ enclosed by $\mathfrak{g}(\widetilde{\gamma})$ and the line connecting the point $z_1 = 0$ with the point z_1
$$0 \leq \chi \leq \pi \tag{8.12}$$

33) In the following, γ can denote a point in the z_1- as well as in the z_2-plane.

holds, which implies the second part of 2. Namely,

$$\chi = \varphi_0 - \Psi(\widetilde{\gamma}) = \varphi_0 - \Psi(\gamma) + \Psi(\gamma) - \Psi(\widetilde{\gamma}), \quad \varphi_0 = \operatorname{arc}(z_1),$$

and hence

$$|\varphi_0 - \Psi(\gamma)| - |\Psi(\gamma) - \Psi(\widetilde{\gamma})| < \chi < |\varphi_0 - \Psi(\gamma)| + \pi + |\Psi(\gamma) - \Psi(\widetilde{\gamma})|,$$

and the inequalities (8.11) and (8.5) yield (8.12).

On the other hand, by (1.15), we have the relation

$$|\widetilde{z}_1|^2 \mathsf{K}_{\widetilde{\mathfrak{B}}}(\widetilde{z}_1, \widetilde{z}_2; \overline{\widetilde{z}}_1, \overline{\widetilde{z}_2}) = |z_1|^2 \mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) : |1 - z_1^{\frac{1}{\nu}}|^2, \tag{8.13}$$

and it is therefore enough to prove the statemen of Theorem VI for the domain $\widetilde{\mathfrak{B}}$. Now let $\mathfrak{H}_m$, $m = 1, 2, 3 \ldots$, be a sequence of closed domains contained in $\mathfrak{H}$, such that $\mathfrak{H}_m$ is contained in the interior of $\mathfrak{H}_{m+1}$, and the $\mathfrak{H}_m$ fully exhaust the domain $\mathfrak{H}$. Let $\widetilde{\mathfrak{B}}^{(m)}$ denote the domain

$$\widetilde{\mathfrak{B}}^{(m)} = \sum_{\gamma}^{\mathfrak{H}_m} \widetilde{\mathfrak{B}}(z_2 = \gamma). \tag{8.14}$$

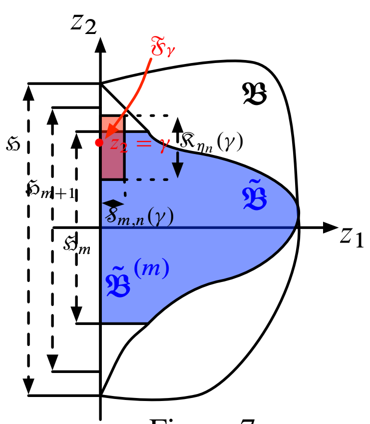


Figure 7

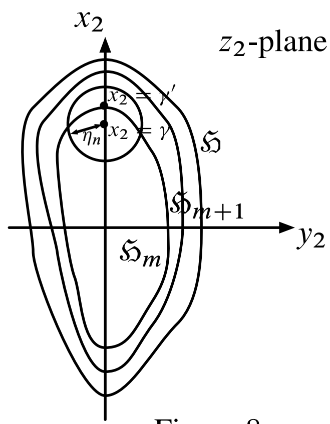


Figure 8

For the domain $\widetilde{\mathfrak{B}}^{(m)}$, we construct the following:

I. A domain

$$\mathfrak{S}^{(m)} = \sum_{\gamma}^{\mathfrak{H}_m} \mathfrak{S}^{(m)}(z_2 = \gamma) \supset \widetilde{\mathfrak{B}}^{(m)} \tag{8.15}$$

of the type studied in §7, p. 42, where the sector $\mathfrak{S}^{(m)}(z_2 = \gamma) = \mathcal{S}(\vartheta_1(\gamma), \vartheta_2(\gamma), \varrho)$ is a semicircle whose diameter lies on the line $\mathfrak{g}(\gamma)$ and whose radius ϱ is large enough such that, for γ in $\mathfrak{H}_m$,

$$\mathfrak{S}^{(m)}(z_2 = \gamma) \supset \widetilde{\mathfrak{B}}(z_2 = \gamma). \tag{8.16}$$

As $\widetilde{\mathfrak{B}}$ is bounded and $\widetilde{\mathfrak{B}}(z_2 = \gamma)$ lies, by assumption 3, on one side of the line $\mathfrak{g}(\gamma)$, this construction is clearly possible. From assumption 5 it follows that properties 4 and 5 of §7 are satisfied.

II. A sequence of domains

$$\boldsymbol{\mathfrak{S}}^{(m,n)} = \sum_{\gamma}^{\mathfrak{H}_m} \mathfrak{S}^{(m,n)}(z_2 = \gamma), \tag{8.17}$$

such that for every m

$$\boldsymbol{\mathfrak{S}}^{(m,n)} \subset \widetilde{\boldsymbol{\mathfrak{B}}}^{(m)} \tag{8.18}$$

holds, and the angles $\omega_{m,n}$ of $\boldsymbol{\mathfrak{S}}^{(m,n)}$ satisfy

$$\lim_{n\to\infty} \omega_{m,n} = \pi (= \omega_m), \tag{8.19}$$

uniformly in γ for fixed m on $\mathfrak{H}_m$.

Let ε_n be positive and $\lim_{n\to\infty} \varepsilon_n = 0$. As the function $\Psi(\gamma)$ is uniformly continuous, there exists η_n such that in each point γ of $\mathfrak{H}_m$ there is a circle $\mathfrak{K}_{\eta_n}(\gamma)$ with radius η_n and center γ, such that in the interior of $\mathfrak{K}_{\eta_n}(\gamma)$:

$$\Psi(\gamma) - \frac{\varepsilon_n}{2} < \Psi(z_2) < \Psi(\gamma) + \frac{\varepsilon_n}{2}, \tag{8.20}$$

for z_2 in $\mathfrak{K}_{\eta_n}(\gamma)$.

Furthermore, we may assume η_n small enough such that the circle $\mathfrak{K}_{\eta_n}(\gamma)$ lies in $\mathfrak{H}_{m+1}$ for each γ in $\mathfrak{H}_m$. By assumption, for each γ in $\mathfrak{H}_{m+1}$, hence a fortiori in $\mathfrak{K}_{\eta_n}(\gamma)$, there is a circle of radius κ_{m+1} that is contained in $\mathfrak{B}(z_2 = \gamma')$ (for γ' in $\mathfrak{K}_{\eta_n}(\gamma) \subset \mathfrak{H}_{m+1}$) and touches the line $\mathfrak{g}(\gamma')$. Then $\mathfrak{B}(z_2 = \gamma)$ contains a sector with angle $\omega_n = \pi - \varepsilon_n$ and radius $\tau_{m,n} = 2\kappa_{m+1} \cos(\frac{\omega_n}{2})$, where the angles $\vartheta_1^{(n)}(\gamma')$ and $\vartheta_2^{(n)}(\gamma')$ of the sector satisfy

$$\vartheta_2^{(n)}(\gamma') - \vartheta_1^{(n)}(\gamma') = \omega_n, \quad \frac{\vartheta_1^{(n)}(\gamma') + \vartheta_2^{(n)}(\gamma')}{2} = \Psi(\gamma') + \frac{\pi}{2}, \tag{8.21}$$

for γ' in $\mathfrak{K}_{\eta_n}(\gamma)$. By (8.20), we thus obtain for z_2 in $\mathfrak{K}_{\eta_n}(\gamma)$:

$$\begin{aligned} \vartheta_1^{(n)}(z_2) &= \Psi(z_2) + \frac{\varepsilon_n}{2} < \Psi(\gamma) + \varepsilon_n, \\ \vartheta_2^{(n)}(z_2) &= \Psi(z_2) + \pi - \frac{\varepsilon_n}{2} > \Psi(\gamma) - \varepsilon_n + \pi, \end{aligned} \tag{8.22}$$

hence for z_2 in $\mathfrak{K}_{\eta_n}(\gamma)$

$$\vartheta_1^{(n)}(z_2) < \vartheta_1^{*(n)}(\gamma) < \vartheta_2^{*(n)}(\gamma) < \vartheta_2^{(n)}(z_2), \tag{8.23}$$

where we put

$$\begin{aligned} \vartheta_1^{*(n)}(\gamma) &= \vartheta_1^{(n)}(\gamma) + \frac{\varepsilon_n}{2} = \Psi(\gamma) + \varepsilon_n, \\ \vartheta_2^{*(n)}(\gamma) &= \vartheta_2^{(n)}(\gamma) - \frac{\varepsilon_n}{2} = \Psi(\gamma) + \pi - \varepsilon_n, \\ &(\vartheta_2^{*(n)}(\gamma) - \vartheta_1^{*(n)}(\gamma) = \pi - 2\varepsilon_n). \end{aligned} \tag{8.24}$$

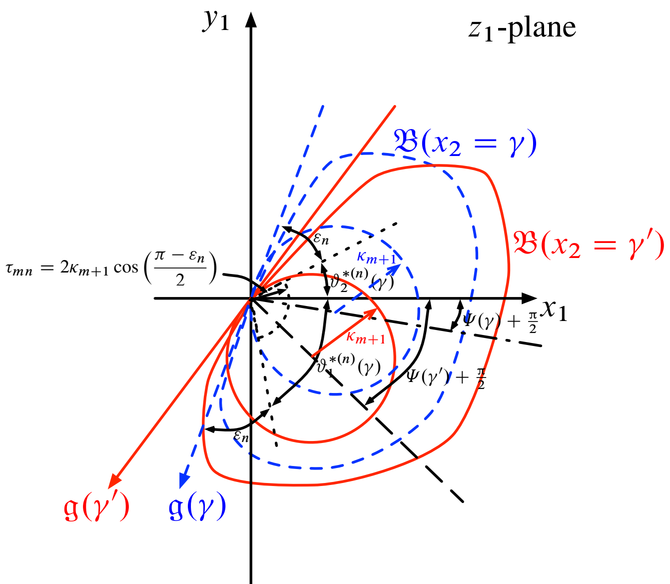


Figure 9

If $\mathcal{S}_{m,n}(\gamma)$ denotes the sector with angles $\vartheta_1^{*(n)}(\gamma)$ and $\vartheta_2^{*(n)}(\gamma)$ with radius

$$\tau_{m,n} = 2\kappa_{m+1} \cos\left(\frac{\pi - \varepsilon_n}{2}\right),$$

then for γ' in $\mathfrak{K}_{\eta_n}(\gamma)$ (with γ in $\mathfrak{H}_m$), $\mathcal{S}_{m,n}$ is contained in $\mathfrak{B}(z_2 = \gamma)$.[34)] Hence the product domain

$$\mathbf{\mathfrak{S}}_{m,n}(\gamma) = \mathfrak{K}_{\eta_n}(\gamma) \times \mathcal{S}_{m,n}(\gamma) \tag{8.25}$$

is contained in $\mathbf{\mathfrak{B}}^{(m+1)}$ (compare figure 7).

If we now return to the domain $\widetilde{\mathbf{\mathfrak{B}}}^{(m)}$ and consider the sections $\widetilde{\mathfrak{B}}(z_2 = \gamma)$, then these correspond to the sections $\mathfrak{B}(z_2(1-z_1^{\frac{1}{\nu}}) = \gamma)$ of the domain $\mathbf{\mathfrak{B}}$ with the analytic surface $\mathfrak{F}_\gamma$ given by $z_2(1-z_1^{\frac{1}{\nu}}) = \gamma$. The intersection of $\mathfrak{F}_\gamma$ with the product domain $\mathbf{\mathfrak{S}}_{m,n}(\gamma)$ is a surface segment in $\mathfrak{F}_\gamma$ that is bounded by a closed curve. This boundary curve $\mathfrak{f}_\gamma$ is the intersection of the surface $\mathfrak{F}_\gamma$ with $\mathbf{R}[\mathbf{\mathfrak{S}}_{m,n}(\gamma)]$ (where $\mathbf{R}[\ldots]$ denotes the boundary). Then $\mathfrak{f}_\gamma$ is composed of the intersections of $\mathfrak{F}_\gamma$ with $\mathfrak{K}_{\eta_n}(\gamma)\times\mathbf{R}[\mathcal{S}_{m,n}(\gamma)]$ and with $(\mathcal{S}_{m,n}(\gamma)+\mathbf{R}[\mathcal{S}_{m,n}(\gamma)])\times\mathbf{R}[\mathfrak{K}_{\eta_n}(\gamma)]$. The z_1-coordinates of the first part of $\mathfrak{f}_\gamma$ either lie on the two rays $re^{i\vartheta_1^{*(n)}(\gamma)}$, $re^{i\vartheta_2^{*(n)}(\gamma)}$ $(0 \le r < \infty)$ or on a circle with radius $\tau_{m,n}$. For the z_2-coordinates of the remaining points of $\mathfrak{f}_\gamma$, (8.26) below holds. We will now show that the the z_1-coordinate of the points on $\mathfrak{f}_\gamma$ satisfy the inequality $|z_1| \ge \varrho_{m,n} > 0$ unless they lie on the rays mentioned above. Namely, consider the part of the boundary of $\mathbf{\mathfrak{S}}_{m,n}$ whose z_2-coordinate is given by

$$|z_2 - \gamma| = \eta_n. \tag{8.26}$$

Then $|z_1|$ has a positive minimum for the points on the curve segement $\mathfrak{c}$ obtained by intersecting (8.26) with $\mathfrak{F}_\gamma$.[35)] If we now apply to

$$\mathfrak{S}_{m,n}(z_2(1-z_1^{\frac{1}{\nu}}) = \gamma) \subset \mathfrak{B}(z_2(1-z_1^{\frac{1}{\nu}}) = \gamma)$$

the transformation (8.8), we obtain

$$\widetilde{\mathfrak{S}}_{m,n}(z_2 = \gamma) \subset \widetilde{\mathfrak{B}}(z_2 = \gamma).$$

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34) If, instead of assumption 4, we make the assumption stated in the footnote on p. 50, then κ_{m+1} in the last equation has to be replaced by ϱ_m.

35) Namely, if $z_2 = \gamma + \eta_n e^{i\varphi}$, the points in the intersection satisfy

$$|z_1| = \left|1 - \frac{\gamma}{\eta_n e^{i\varphi} + \gamma}\right|^\nu = \left|\frac{\eta_n^\nu e^{i\nu\varphi}}{(\eta_n e^{i\varphi} + \gamma)^\nu}\right|,$$

and the boundedness of γ implies the existence of a positive lower bound for $|z_1|$. So if we choose

$$\varrho_{m,n} = \min\left(2\kappa_{m+1}\cos\left(\frac{\pi - \varepsilon_n}{2}\right), \left|\frac{\eta_n^\nu e^{i\nu\varphi}}{(\eta_n e^{i\varphi} + \gamma)^\nu}\right|\right),$$

then the z_1-coordinates of the curve $\mathfrak{c}$ satisfy the inequality $|z_1| \ge \varrho_{m,n}$.

The section $\widetilde{\mathfrak{B}}_{m,n}(z_2 = \gamma)$, now in the plane $z_2 = \gamma$, is bounded by the two rays $z_1 = r\mathrm{e}^{\mathrm{i}\vartheta_1^{*(n)}(\gamma)}$ and $z_1 = r\mathrm{e}^{\mathrm{i}\vartheta_2^{*(n)}(\gamma)}$ and a curve segment $\mathfrak{c}$ that satisfies $|z_1| \geq \varrho_{m,n}$. Hence $\widetilde{\mathfrak{S}}_{m,n}(z_2 = \gamma)$ contains a sector $\mathcal{S}(\vartheta_1^{*(n)}(\gamma), \vartheta_2^{*(n)}(\gamma), \varrho_{m,n})$ with

$$\lim_{n\to\infty} \omega_{m,n} = \lim_{n\to\infty} (\vartheta_2^{*(n)}(\gamma) - \vartheta_1^{*(n)}(\gamma)) = \lim_{n\to\infty} \pi - 2\varepsilon_n = \pi. \tag{8.27}$$

So if we define the domain

$$\boldsymbol{\mathfrak{S}}^{(m,n)} = \sum_{\gamma}^{\mathfrak{H}_m} \mathfrak{S}^{(m,n)}(z_2 = \gamma) \tag{8.28}$$

by

$$\mathfrak{S}^{(m,n)}(z_2 = \gamma) = \mathcal{S}(\vartheta_1^{*(n)}(\gamma), \vartheta_2^{*(n)}(\gamma), \varrho_{m,n}), \tag{8.29}$$

then

$$\boldsymbol{\mathfrak{S}}^{(m,n)} \subset \widetilde{\boldsymbol{\mathfrak{B}}}^{(m)} \subset \boldsymbol{\mathfrak{S}}^{(m)}, \tag{8.30}$$

and moreover, due to (8.21), (8.22), (8.24),

$$\lim_{n\to\infty} \vartheta_1^{*(n)}(\gamma) = \Psi(\gamma), \quad \lim_{n\to\infty} \vartheta_2^{*(n)}(\gamma) = \Psi(\gamma) + \pi. \tag{8.31}$$

Hence by (8.30),

$$|z_1|^2 \mathsf{K}_{\boldsymbol{\mathfrak{S}}^{(m)}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \leq |z_1|^2 \mathsf{K}_{\widetilde{\boldsymbol{\mathfrak{B}}}^{(m)}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \leq |z_1|^2 \mathsf{K}_{\boldsymbol{\mathfrak{S}}^{(m,n)}}(z_1, z_2; \overline{z}_1, \overline{z}_2) \tag{8.32}$$

and therefore

$$L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{S}}^{(m)}}(Q, \alpha) \leq \underline{L}^{\mathrm{II}2}_{\widetilde{\boldsymbol{\mathfrak{B}}}^{(m)}}(Q, \alpha) \leq \overline{L}^{\mathrm{II}2}_{\widetilde{\boldsymbol{\mathfrak{B}}}^{(m)}}(Q, \alpha) \leq \overline{\lim_{n\to\infty}} L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{S}}^{(m,n)}}(Q, \alpha), \tag{8.33}$$

and since by the corollary of Theorem V

$$\overline{\lim_{n\to\infty}} L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{S}}^{(m,n)}}(Q, \alpha) = L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{S}}^{(m)}}(Q, \alpha) \tag{8.34}$$

holds, $L^{\mathrm{II}2}_{\widetilde{\boldsymbol{\mathfrak{B}}}^{(m)}}(Q, \alpha)$ exists and is

$$L^{\mathrm{II}2}_{\widetilde{\boldsymbol{\mathfrak{B}}}^{(m)}}(Q, \alpha) = L^{\mathrm{II}2}_{\boldsymbol{\mathfrak{S}}^{(m)}}(Q, \alpha). \tag{8.35}$$

Part II

The totality of domains that can be mapped to one another by pairs of functions in two complex variables $z_1^* = z_1^*(z_1, z_2)$, $z_2^* = z_2^*(z_1, z_2)$, are called a *class of equivalent domains*.

If we restrict ourselves to classes to which belongs at least one finite and simple domain, and consider all regular functions $h(z_1, z_2)$ on such a domain $\mathfrak{B}$ for which

$$\int_{\mathfrak{B}} |h(z_1, z_2)|^2 \mathrm{d}\omega \le 1 \qquad (\mathrm{d}\omega = \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2)$$

holds, then the squares of the absolute values of the h attain a maximum in every point $\{t_1, t_2\}$ of $\mathfrak{B}$ that defines a positive and analytic[1)] function $\mathsf{K}_{\mathfrak{B}}(t_1, t_2; \bar{t}_1, \bar{t}_2)$ in $\mathfrak{B}$. For every point $\{t_1, t_2\}$ there exists one and (up to a factor of absolute value 1) only one function for which this maximum is attained. Dividing this function for the sake of normalization by $\sqrt{\mathsf{K}_{\mathfrak{B}}(t_1, t_2; \bar{t}_1, \bar{t}_2)}$, the thus obtained function $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ assumes the value 1 in the point $\{t_1, t_2\}$ and yields the minimal values for the integral $\int_{\mathfrak{B}} |h|^2 \mathrm{d}\omega$ when compared to all regular functions $h(z_1, z_2)$ on $\mathfrak{B}$ that satisfy $|h(t_1, t_2)| = 1$. We call $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2) = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)}{\mathsf{K}_{\mathfrak{B}}(t_1, t_2; \bar{t}_1, \bar{t}_2)}$ the *minimal function* of the domain $\mathfrak{B}$ with the base point $\{t_1, t_2\}$.[2)] The Hermitian differential form

$$\mathrm{d}s^2 = \sum_{m,n}^{2} \mathsf{T}_{m\bar{n}} \mathrm{d}z_m \mathrm{d}\bar{z}_n, \quad \mathsf{T}_{m\bar{n}} = \frac{\partial^2 \log \mathsf{K}_{\mathfrak{B}}}{\partial z_m \partial \bar{z}_n} \tag{1.16*}$$

is invariant under transformations by pairs of functions in two complex variables.[3)]

[1)] By an analytic function $p(z_1, z_2; t_1, t_2)$ we mean a function in the for variables z_1, z_2, t_1, t_2 that can be expanded into a convergent series in a neighborhood of every regular point in a sufficiently small polycylinder $|z_k| \le \delta_k$, $|t_k| \le \delta_k$.

[2)] Compare *Über unendliche Hermitesche Formen, die zu einem Bereiche gehören, nebst Anwendungen auf Fragen der Abbildung durch Funktionen von zwei komplexen Veränderlichen*, Mathematische Zeitschrift 29 (1929), p. 641 to 677, in particular §1, in the following cited as article H. In the present work, we denote the minimal functions of $\mathfrak{B}$ with base point $\{t_1, t_2\}$ by $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ rather than by $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)$ as we did in earlier works. For the kernel function, on the other hand, we keep the old notation $\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)$, as it is an *antianalytic* function in t_1, t_2.

[3)] Compare *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande. I*, Journal für die reine und angewandte Mathematik 169 (1933), p. 1 to 42, in particular §1, in the following cited as K; the formulas taken from it will be indicated by a star. See also *Über eine in der Theorie der Funktionen von zwei komplexen Veränderlichen auftretende unitäre Geometrie*, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen 36 (1933), p. 307, and *Sur quelques propriétés des transformations par un couple de fonctions de deux variables complexes*, Rendiconti Accademia Nazionale dei Lincei (6) 19 (1934), p. 474 to 478.

The metric given by (1.16*) is positive definite[4)] and has the property that in the corresponding Riemannian space the invariant $\boldsymbol{I}$ (given by (1.17*)), where the following relations between the contracted curvature tensor, the fundamental tensor and the covariant derivatives of $\boldsymbol{I}$ holds,

$$\mathsf{T}_{m\overline{n}} + \sum_{p=1}^{2} R^{\cdots p}_{\overline{n}mp} = \frac{\partial^2 \log(\boldsymbol{I})}{\partial z_m \partial \overline{z}_n}. \tag{1.18*}$$

The Riemannian space defined by the metric (1.16*) within a class of equivalent domains shall be called the *primal space* of the class.

In the case of one complex variable and finite simply connected domains $\mathfrak{B}^2$, $\mathsf{M}_{\mathfrak{B}^2}(z;t)$ is the derivative of (appropriately normalized) circle mapping that maps the point $\{t\}$ to the center of the circle. Here, the differential form

$$\mathrm{d}s^2 = \frac{\partial^2 \log(\mathsf{K}_{\mathfrak{B}^2}(z,\overline{z}))}{\partial z \partial \overline{z}} |\mathrm{d}z|^2$$

is transformed to the Poincaré metric of the unit circle.

Let it be remarked that the introduction of the primal space leads to an interesting corollary: Every domain $\mathfrak{B}$ of the class can be obtained by introducing the (Cartesian) coordinates z_1, z_2 for $\mathfrak{B}$ in the primal space. This fact allows us to use the results of differential geometry in the theory of mappings given by pairs of functions in two complex variables. In particular, it follows that when mapping the domains onto each other, certain quantities appear, namely invariants (scalars), integral invariants (densities), tensors etc., that transform in a very particular way when mapping from $\mathfrak{B}$ to another domain $\mathfrak{B}^*$ in the class (in our sense, mapping from $\mathfrak{B}$ to $\mathfrak{B}^*$ means that in the primal space we switch from the coordinates z_1, z_2 to the new coordinates z_1^*, z_2^*).

One of the important problems in the theory of functions is the study of the analytic functions in two variables defined on a class of domains. If we use the

[4)] In §1 of K it was shown that the determinant N of the differential form (1.16*) is positive everywhere on $\mathfrak{B}$, which only implies that the form is definite. But similar to the proof of positivity of N, we can conclude that

$$\mathsf{T}_{1\overline{1}} = \frac{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\overline{1}\overline{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\overline{1}\overline{0}} \end{vmatrix}}{\mathsf{K}^2} = \frac{1}{2\mathsf{K}^2} \sum_{\kappa} \sum_{\mu} \begin{Vmatrix} \varphi_\kappa(z_1,z_2) \frac{\partial \varphi_\kappa(z_1,z_2)}{\partial z_1} \\ \varphi_\mu(z_1,z_2) \frac{\partial \varphi_\mu(z_1,z_2)}{\partial z_2} \end{Vmatrix} > 0, \quad \mathsf{K}_{mn\overline{p}\overline{q}} \equiv \frac{\mathrm{d}^{m+n+p+q} \mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2)}{\mathrm{d}z_1^m \mathrm{d}z_2^n \mathrm{d}\overline{z}_1^p \mathrm{d}\overline{z}_2^q}$$

and $\mathsf{T}_{2\overline{2}} > 0$, from which the positivity of the form follows.

give metric for this investigation, mainly questions on the primal spaces and the characterization of the domains belonging to a primal domain arise.

The present work makes certain contributions to this problems by making some statements on the situation at the boundary. It based mainly on the results in K. As was shown there, under certain natural assumptions on the structure of the domain, the kernel functions tends to infinity when approaching the boundary. However, the order of becoming infinite can be different at different points on the boundary. Accordingly, in K the boundary points of zeroth, second, third and fourth order were introduced. In each class of these boundary points we can identify a subclass of points with a simple structure, which are called limit points. To conveniently formulate the function theoretic properties of a limit point Q of second or third order, it is helpful to introduce the so-called *normal coordinates* relative to Q. By the assumption on Q, the boundary hypersurface of $\mathfrak{B}$ in Q has a tangent hyperplane $\mathfrak{e}^3$. The normal coordinates relative to Q are obtained by making Q the origin and choosing the analytic plane $\mathfrak{C}^2$ lying in $\mathfrak{e}^3$ and passing through Q and the plane orthogonal to $\mathfrak{C}^2$ as z_1- and z_2-coordinates, respectively.[5)] For certain types of approaches to Q discussed in K, there exists, as shown in K,

$$\lim_{\{z_1,z_2\}\to Q}(z_1+\overline{z}_1)^m \mathsf{K}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2), \quad m=2,3.$$

The method to prove this limit relation, which will be further developed in the present work, shall here be illustrated by considering the corresponding situation for the case of one complex variable. Let $\mathfrak{B}^2$ be a convex domain, Q one of its boundary points, where the boundary curve in Q has positive curvature. Now, the normal coordinates with respect to Q shall be those coordinates for which Q is the origin and the inner normal to Q becomes the positive x-axis. Let $\mathfrak{J}^2$ and $\mathfrak{A}^2$ be two circles touching the y-axis in Q, where $\mathfrak{J}^2$ contains $\mathfrak{B}^2$ (inner comparison domain). The kernel function of a circle $\mathfrak{K}^2$ or radius r touching the y-axis in Q has (in normal coordinates relative to Q) the value $\frac{1}{\pi\left(z+\overline{z}+\frac{|z|^2}{r}\right)^2}$. If the approach takes place in a the domain $\mathfrak{W}^2_\alpha$ of angles $\frac{2|z|}{z_1+\overline{z}_1}<\frac{1}{\cos(\alpha)}$, $|\alpha|<\frac{\pi}{2}$, then it follows that $\lim_{z\to Q}(z+\overline{z})^2\mathsf{K}_{\mathfrak{K}^2}(z,\overline{z})=\frac{1}{\pi}$ (the limit is thus independent of the radius of the circle). Since on the other hand by (1.12*), $\mathsf{K}_{\mathfrak{A}^2}(z,\overline{z})\leq\mathsf{K}_{\mathfrak{B}^2}(z,\overline{z})\leq\mathsf{K}_{\mathfrak{J}^2}(z,\overline{z})$ holds, it follows that also $\lim_{z\to Q}(z+\overline{z})^2\mathsf{K}_{\mathfrak{B}^2}(z,\overline{z})=\frac{1}{\pi}$.

[5)]In §1, the structure of the limit points of second and third order will be studied more closely, and the normal coordinates for both types of boundary points will be described in more detail.

For manifolds of dimension less than 4, the upper index indicates the dimension of the manifold.

In §1 of the present work, we investigate the behavior of the metric given by (1.16*) at the aforementioned limit points. It will be shown that for limit points of order m (for a certain approach described later in the text and using the normal coordinates relative to Q) the relations hold:

$$\lim_{\{z_1,z_2\}\to Q}(z_1+\overline{z}_1)^2\mathrm{d}s^2 = m|\mathrm{d}z_1|^2, \qquad \lim_{\{z_1,z_2\}\to Q} \boldsymbol{I}_{\mathfrak{B}}(z_1,z_2;\overline{z}_1,\overline{z}_2) = \frac{m-1}{m^2\pi^2}, \qquad m = 2,3.$$

Following the investigation of the kernel function, it seems reasonable to study families of functions whose square means satisfy certain inequalities, and in particular the behavior of these functions in the neighborhood of a limit point of second and third order. In §2 we consider a limit point Q of third order and a sequence of points $\mathfrak{P}^0$ in the interior of $\mathfrak{B}$ with $\lim_{\nu\to\infty}\{t_1^{(\nu)}, t_2^{(\nu)}\} = Q$.

To every point $\{t_1,t_2\}$ in $\mathfrak{P}^0$ we associate a regular function $f(z_1,z_2;t_1,t_2)$ in two variables z_1, z_2 on $\mathfrak{B}$ that assumes the value 1 at $\{t_1,t_2\}$ (that is, $f(t_1,t_2;t_1,t_2) = 1$), while $\int_{\mathfrak{B}}|f|^2\mathrm{d}\omega$ satisfies the inquality (in normal coordinates relative to Q)

$$\int_{\mathfrak{B}}|f(z_1,z_2;t_1,t_2)|^2\mathrm{d}\omega_z \;\le\; \frac{\pi^2}{2\sigma}(t_1+\overline{t}_1)^3\big(1+C(t_1+\overline{t}_1)^r\big)$$

where σ is a quantity determined by the structure of the boundary point, and $C < \infty$, $r > 0$ are constants independent of the position of the point $\{t_1,t_2\}$. If the approach of the pair of points $\{z_1,z_2;t_1,t_2\}$ is towards $\{0,0;0,0\}$ in such a way that there is always some relation between the x_1-coordinate of $\{z_1,z_2\}$ and that of the associated point $\{t_1,t_2\}$, then

$$\lim_{\{z_1,z_2;t_1,t_2\}\to\{0,0;0,0\}} \frac{z_1^3}{(t_1+\overline{t}_1)^3} f(z_1,z_2;t_1,t_2) = 1. \tag{4.3}$$

The minimal function $M_{\mathfrak{B}}(z_1,z_2;t_1,t_2)$ satisfies both of the given conditions, and in particular the limit relation (4.3).

Analogous formulas are derived in §3 for the approach of limit points of second order.

I am indebted to Herr Erwin Klein for the help in preparing this workd.

§ 1

A further development of the methods employed in K allows it, as we shall see in the next paragraph, to derive limit relations for several characteristic quantities of Hermitian metric.

An important tool are the so-called *Jacobian reductions* $\boldsymbol{J}_{\mathfrak{B}}(X_{00},\dots,X_{mn}) = \boldsymbol{J}_{\mathfrak{B}}(X)$ of certain Hermitian forms,[6)] which were already introduced in an earlier work (for different purposes).[7)] The precise definition of the $\boldsymbol{J}_{\mathfrak{B}}(X)$ will be given on p. 62.

In a Jacobian reduction, the coefficients of the X_{pq} appear as the aforementioned quantities of interest. On the other hand, in analogy to (1.12*), the reductions $\boldsymbol{J}_{\mathfrak{B}}(X)$ and $\boldsymbol{J}_{\mathfrak{B}^*}(X)$ of two domains $\mathfrak{B}^* \subset \mathfrak{B}$ satisfy for all values X_{pq}: $\boldsymbol{J}_{\mathfrak{B}}(X) \leq \boldsymbol{J}_{\mathfrak{B}^*}(X)$.

This fact allows us, as will be shown in the following, to employ the methods from K to derive the aforementioned limit relations.

For the following computations it is helpful to introduce symbols for certain matrices, namely,

$$[X]^{mn} = \begin{pmatrix} X_{00} \\ X_{10} \\ \vdots \\ X_{mn} \end{pmatrix}, \quad [D]^{mn} = \begin{pmatrix} \mathsf{K} & \mathsf{K}_{00\overline{10}} & \mathsf{K}_{00\overline{01}} & \cdots & \mathsf{K}_{00\overline{mn}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\overline{10}} & \mathsf{K}_{10\overline{01}} & \cdots & \mathsf{K}_{10\overline{mn}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{K}_{mn00} & \mathsf{K}_{mn\overline{10}} & \mathsf{K}_{mn\overline{01}} & \cdots & \mathsf{K}_{mn\overline{mn}} \end{pmatrix},$$

$$[\overline{X}]'^{mn} = (\overline{X}_{00}, \overline{X}_{10}, \dots, \overline{X}_{mn}),$$

$$\begin{aligned} &[\mathsf{K}(z_1,z_2;\bar t_1,\bar t_2)]^{mn} \\ &= \big(\mathsf{K}(z_1,z_2;\bar t_1,\bar t_2), \mathsf{K}_{00\overline{10}}(z_1,z_2;\bar t_1,\bar t_2), \mathsf{K}_{00\overline{01}}(z_1,z_2;\bar t_1,\bar t_2), \dots, \mathsf{K}_{00\overline{mn}}(z_1,z_2;\bar t_1,\bar t_2)\big) \end{aligned}$$

where

$$\mathsf{K} = \mathsf{K}_{\mathfrak{B}}(t_1,t_2;\bar t_1,\bar t_2), \quad \mathsf{K}_{pr\overline{qu}} = \frac{d^{p+r+q+u}\mathsf{K}_{\mathfrak{B}}(t_1,t_2;\bar t_1,\bar t_2)}{dt_1^p\, dt_2^r\, d\bar t_1^q\, d\bar t_2^y},$$

$$\mathsf{K}_{00\overline{pr}}(z_1,z_2;\bar t_1,\bar t_2) = \frac{d^{p+r}\mathsf{K}_{\mathfrak{B}}(z_1,z_2;\bar t_1,\bar t_2)}{d\bar t_1^p\, d\bar t_2^r}.$$

6) As we work with functions in two variables, we will mainly use double indices. In the following, we will use the order

$$(00), (10), (01), (20), (11), (02), \dots$$

The term preceding the (mn)-th term in this order will be denoted by $(m_\nu n_\nu)$, and $\sum_{(pq)=(00)}^{(mn)}$ means a summation over all terms up to the (mn)-th term in the given sequence.

7) Compare H §2 as well as *Über Hermitesche Formen, die zu einem Bereich gehören*, Sitzungsberichte der Berliner Mathematischen Gesellschaft 26 (1927), p. 178.

The lower indices attached to the matrix symbols indicate the missing columns in the matrix.

The following problem leads to the Jacobian reductions: *Determine the minimum of the integral*

$$\int_{\mathfrak{B}} |h(z_1, z_2)|^2 \mathrm{d}\omega, \quad \mathrm{d}\omega = \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \tag{1.1}$$

among all regular and square-integrable functions on $\mathfrak{B}$ *that satisfy the conditions*

$$\begin{gathered} h(t_1, t_2) = X_{00}, \quad h_{10}(t_1, t_2) = X_{10}, \quad h_{10}(t_1, t_2) = X_{01}, \\ \ldots, \quad h_{mn}(t_1, t_2) = X_{mn}, \end{gathered} \tag{1.2}$$

where

$$h_{mn}(t_1, t_2) = \left[\frac{\mathrm{d}^{m+n} h(z_1, z_2)}{\mathrm{d}z_1^m \mathrm{d}z_2^n}\right]_{\substack{z_1 = t_1 \\ z_2 = t_2}}$$

and the X_{pq} *are given constants. The desired minimum takes the value*

$$-\frac{\begin{vmatrix} 0 & [\overline{X}]'^{mn} \\ [X]^{mn} & [D]^{mn} \end{vmatrix}}{|[D]^{mn}|}. \tag{1.3}$$

The Hermitian form[8] (1.3) *can also be written as*[9]

$$\sum_{(kl)=(00)}^{(mn)} \frac{|[D]^{k_\nu l_\nu}|}{|[D]^{kl}|} |\boldsymbol{l}_{\mathfrak{B}}^{kl}(X)|^2, \quad \boldsymbol{l}_{\mathfrak{B}}^{kl}(X) = (-1)^{[(kl)+1]} \frac{|[X]^{kl}[D]_{kl}^{kl}|}{|[D]^{k_\nu l_\nu}|}.^{10)} \tag{1.4}$$

We call (1.4) the *Jacobian reduction*

$$\boldsymbol{J}_{\mathfrak{B}}(X_{00}, X_{10}, \ldots, X_{mn}).$$

The point $\{t_1, t_2\}$, assumed to be an inner point of $\mathfrak{B}$, is called the *base point* of the Jacobian reduction $\boldsymbol{J}_{\mathfrak{B}}(X)$.

PROOF: Let $\varphi^{(\nu)}(z_1, z_2)$, $\nu = 1, 2, 3, \ldots$ a complete orthogonal system of functions on $\mathfrak{B}$. Then any square-integrable regular function on $\mathfrak{B}$ can be written in the form

$$h(z_1, z_2) = \sum_{\nu=1}^{\infty} A_\nu \varphi^{(\nu)}(z_1, z_2), \quad A_\nu \equiv \int_{\mathfrak{B}} h \overline{\varphi}^{(\nu)} \mathrm{d}\omega, \tag{1.5}$$

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8) Now consider the X_{pq} as variables.

9) See H, in particular p. 671 to 674.

10) $(-1)^{[(pq)+1]}$ means 1 if the index pair (pq) has an odd, and -1 if it has an even number in the order given in footnote 6).

where

$$\int_{\mathfrak{B}} |h(z_1, z_2)|^2 \mathrm{d}\omega = \sum_{\nu=1}^{\infty} |A_\nu|^2 \tag{1.6}$$

holds (the completeness property). Conversely, for every set of values A_ν with finite $\sum_{\nu=1}^{\infty} |A_\nu|^2$, there exists a regular function on $\mathfrak{B}$ with the representation (1.5),[11] our task is reduced to computing the minimum of the infinite Hermitian form $H(A) = \sum_{\nu=1}^{\infty} |A_\nu|^2$ under the condition

$$\sum_{\nu=1}^{\infty} A_\nu \varphi_{kl}^{(\nu)} = X_{kl}, \quad (kl) = (00), (10), (01), \ldots, (mn), \tag{1.7}$$

where

$$\varphi_{kl}^{(\nu)} = \left[\frac{\mathrm{d}^{k+l}\varphi^{(\nu)}(z_{1,2})}{\mathrm{d}z_1^k \mathrm{d}z_2^l}\right]_{\substack{z_1=t_1\\ z_2=t_2}} .$$

To obtain this minimum, we differentiate the expression

$$\sum_{\nu=1}^{\infty} |A_\nu|^2 - \sum_{(kl)=(00)}^{(mn)} \mu^{kl}\Big(\sum_{\nu=1}^{\infty} A_\nu \varphi_{kl}^{(\nu)} - X_{kl}\Big) - \sum_{(kl)=(00)}^{(mn)} \overline{\mu}^{kl}\Big(\sum_{\nu=1}^{\infty} \overline{A}_\nu \overline{\varphi}_{kl}^{(\nu)} - \overline{X}_{kl}\Big) \tag{1.8}$$

and obtain

$$A_\nu = \sum_{(kl)=(00)}^{(mn)} \overline{\mu}^{kl} \overline{\varphi}_{kl}^{(\nu)}. \tag{1.9}$$

If we now substitute the thus obtained expressions for A_ν in (1.7) and exchange the order of summation (which is possible due to the absolute convergence of the sequence (1.7) and of $\sum_{\nu=1}^{\infty} |\varphi^(\nu)_{kl}|^2)$[12], then we obtain

$$\sum_{(kl)=(00)}^{(mn)} \overline{\mu}^{kl} \mathsf{K}_{rq\overline{kl}} = X_{rq}, \quad (rq) = (00), (10), (01), \ldots, (mn), \tag{1.10}$$

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[11]Compare *Zwei Sätze aus dem Ideenkreis des Schwarzschen Lemmas über die Funktionen von zwei komplexen Veränderlichen*, Mathematische Annalen 109 (1934), p. 324, and Hammerstein, *Über die Approximation von Funktionen zweier komplexer Veränderlicher durch Polynome*, Sitzungsberichte der preußischen Akademie der Wissenschaften (mathematisch-physikalische Klasse) 1933, p. 259, in particular Hilfssatz I.

[12]The convergence of $\sum_{\nu=1}^{\infty} |\varphi_{kl}^{(\nu)}|^2$, $(kl) > (00)$, is proved in the same manner as the finitness of $\sum_{\nu=1}^{\infty} |\varphi^{(\nu)}|^2$. To find a bound for $\sum_{\nu=1}^{\infty} |\varphi_{kl}^{(\nu)}|^2$, the minimum of $\int_{\mathfrak{B}} |f|^2 \mathrm{d}\omega$ under the condition $\left[\frac{\mathrm{d}^{k+l} f}{\mathrm{d}z_1^k \mathrm{d}z_2^l}\right]_{\substack{z_1=t_1\\ z_2=t_2}}$ needs to be found.

from which

$$\overline{\mu}^{kl} = (-1)^{[(kl)+1]}\frac{|[X]^{mn}[D]^{mn}_{kl}|}{|[D]^{mn}|} \tag{1.11}$$

follows. The desired minimum is

$$\sum_{\nu=1}^{\infty}|A_\nu|^2 = \sum_{\nu=1}^{\infty}\overline{A}_\nu\Big(\sum_{(kl)=(00)}^{(mn)}\overline{\mu}^{kl}\overline{\varphi}^{(\nu)}_{kl}\Big) = \sum_{(kl)=(00)}^{(mn)}\overline{\mu}\overline{X}_{kl} = \sum_{(kl)=(00)}^{(mn)}(-1)^{[(kl)+1]}X_{kl}\frac{|[X]^{mn}[D]^{mn}_{kl}|}{|[D]^{mn}|} \tag{1.12}$$

which implies (1.4). Interchanging the order of summation is now possible, since both $\sum_{\nu=1}^{\infty}|A_\nu|^2$ and $\sum_{\nu=1}^{\infty}|A_\nu\overline{\varphi}^{(\nu)}_{kl}|$, $(kl) = (00), (10), \ldots, (mn)$, exist. We can now show that (1.3) can also be written in the form (1.4). Like any other Hermitian form, (1.3) can be brought into the form

$$\boldsymbol{J}_{\mathfrak{B}}(X_{00}, X_{10}, \ldots, X_{mn}) = \sum_{(kl)=(00)}^{(mn)}|\boldsymbol{L}^{kl}_{\mathfrak{B}}(X)|^2, \tag{1.13}$$

where

$$\boldsymbol{L}^{kl}_{\mathfrak{B}}(X) \equiv \sqrt{\lambda^{kl}_{\mathfrak{B}}}(A^{kl00}_{\mathfrak{B}}X_{00} + A^{kl10}_{\mathfrak{B}}X_{10} + \ldots + A^{klk_\text{v}l_\text{v}}_{\mathfrak{B}}X_{k_\text{v}l_\text{v}} + X_{kl})$$

If we now fix the first (m_v, n_v)[13] variables $X_{00}, X_{10}, \ldots, X_{m_\text{v}n_\text{v}}$ and let X_{mn} vary, then this expression will have a minimum, namely $\boldsymbol{J}_{\mathfrak{B}}(X_{00}, X_{10}, \ldots, X_{m_\text{v}n_\text{v}})$. On the other hand, the same minimal value is obtained if we substitute

$$X_{mn} = -\sum_{(kl)=(00)}^{(m_\text{v}n_\text{v})}A^{mnkl}_{\mathfrak{B}}X_{kl}$$

in (1.13). Then

$$\sum_{(kl)=(00)}^{(m_\text{v}n_\text{v})}|\boldsymbol{L}^{kl}_{\mathfrak{B}}(X)|^2 = \boldsymbol{J}_{\mathfrak{B}}(X_{00}, \ldots, X_{m_\text{v}n_\text{v}})$$

or

$$\boldsymbol{J}_{\mathfrak{B}}(X_{00}, \ldots, X_{mn}) = \boldsymbol{J}_{\mathfrak{B}}(X_{00}, \ldots, X_{m_\text{v}n_\text{v}}) + |\boldsymbol{L}^{mn}_{\mathfrak{B}}(X_{00}, \ldots, X_{mn})|^2. \tag{1.14}$$

But for $\boldsymbol{L}^{mn}_{\mathfrak{B}}(X_{00}, \ldots, X_{mn})$ we obtain

$$\boldsymbol{L}^{mn}_{\mathfrak{B}}(X_{00}, X_{10}, \ldots, X_{mn}) = (-1)^{[(mn)+1]}\frac{|[X]^{mn}[D]^{mn}_{mn}|}{\sqrt{|[D]^{mn}[D]^{m_\text{v}n_\text{v}}|}} \tag{1.15}$$

[13] See footnote 6) for the meaning of (m_v, n_v).

By successively determining the respective last linear expression in $\boldsymbol{L}_{\mathfrak{B}}^{kl}$ in the forms $\boldsymbol{J}_{\mathfrak{B}}^{kl}(X)$, $(kl) = (mn), (m_\mathrm{v} n_\mathrm{v}), \dots, (01), (10), (00)$, we obtain (1.3).

Remark. For every base point $\{t_1, t_2\}$ there exists one and (up to a factor of absolute value 1) only one solution function

$$f(z_1, z_2) = -\frac{\begin{vmatrix} 0 & [\mathsf{K}(z_1, z_2; \bar{t}_1, \bar{t}_2)]'^{mn} \\ [X]^{mn} & [D]^{mn} \end{vmatrix}}{|[D]^{mn}|} \tag{1.16}$$

for the minimum problem on p. 62.[14)]

By (1.5) and (1.9),

$$f(z_1, z_2) = \sum_{\nu=1}^{\infty} A_\nu \varphi^{(\nu)}(z_1, z_2) = \sum_{\nu=1}^{\infty} \overline{\mu}^{kl} \sum_{(kl)=(00)}^{mn)} \varphi^{(\nu)}(z_1, z_2) \overline{\varphi}_{kl}^{(\nu)} = \sum_{\nu=1}^{\infty} \overline{\mu}^{kl} \mathsf{K}_{00\overline{kl}}(z_1, z_2; \bar{t}_1, \bar{t}_2).$$

(Exchanging the order of summation is allowed, as every sequence $\sum_{\nu=1}^{\infty} |\varphi^{(\nu)}(z_1, z_2) \overline{\varphi}_{kl}^{(\nu)}(\bar{t}_1, \bar{t}_2)|$ converges uniformly on every subdomain of $\mathfrak{B}$ contained in the interior of $\mathfrak{B}$.) Taking into account (1.11), this implies (1.16).

In the following arguments, we shall only use the following Jacobian reductions[15)]

$$\begin{aligned} \boldsymbol{J}_{\mathfrak{B}}(X_{00}, X_{10}, X_{01}) &= \frac{1}{\mathsf{K}} |X_{00}|^2 + \frac{\mathsf{K}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}} \left| -\frac{\mathsf{K}_{1000}}{\mathsf{K}} X_{00} + X_{10} \right|^2 \\ &\quad + \frac{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} & \mathsf{K}_{00\bar{0}\bar{1}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} & \mathsf{K}_{10\bar{0}\bar{1}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{1}\bar{0}} & \mathsf{K}_{01\bar{0}\bar{1}} \end{vmatrix}} \left| \frac{\begin{vmatrix} \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{1}\bar{0}} \end{vmatrix}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}} X_{00} - \frac{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{1}\bar{0}} \end{vmatrix}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}} X_{10} + X_{01} \right|^2, \end{aligned} \tag{1.17}$$

their parts and the Jacobian reductions under the conditions $h(t_1, t_2) = X_{00}$, $h_{01}(t_1, t_2) = X_{01}$:

$$\boldsymbol{H}_{\mathfrak{B}}(X_{00}, X_{01}) = \frac{1}{\mathsf{K}} |X_{00}|^2 + \frac{\mathsf{K}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{0}\bar{1}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{0}\bar{1}} \end{vmatrix}} \left| -\frac{\mathsf{K}_{0100}}{\mathsf{K}} X_{00} + X_{01} \right|^2. \tag{1.18}$$

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14) We will not use this result in the present work.

15) Establishing the formulas for $\boldsymbol{J}_{\mathfrak{B}}(X)$ in full generality was done because the method employed here allows to derive analogous limit relations for further important quantities of the metric.

For the hypersphere $\mathfrak{K}$ given by $|z_1 - \frac{1}{\sigma}|^2 + |z_2|^2 < \frac{1}{\sigma^2}$, we have by (4.22*):

$$\begin{aligned} \boldsymbol{J}_{\mathfrak{K}}(X_{00}, X_{10}, X_{01}) = {} & \frac{\pi^2 E^3}{2\sigma}|X_{00}|^2 + \frac{\pi^2 E^5}{6\sigma^3\left(\frac{1}{\sigma^2} - |t_2|^2\right)} \left|\frac{3 - 3\sigma \bar{t}_1}{E} X_{00} + X_{10}\right|^2 \\ & + \frac{\pi^2 E^4(1-\sigma^2|t_2|^2)}{6\sigma^2} \left| - \frac{3\bar{t}_2}{\frac{1}{\sigma^2} - |t_2|^2} X_{00} + \frac{\left(\frac{1}{\sigma} - t_1\right)\bar{t}_2}{\frac{1}{\sigma^2} - |t_2|^2} X_{10} + X_{01}\right|^2, \end{aligned} \tag{1.19}$$

$$\boldsymbol{H}_{\mathfrak{K}}(X_{00}, X_{10}) = \frac{\pi^2 E^3}{2\sigma}|X_{00}|^2 + \frac{\pi^2 E^5}{6\sigma^2(t_1 + \bar{t}_1 - \sigma|t_1|^2)} \left| - \frac{3\bar{t}_2 \sigma}{E} X_{00} + X_{10}\right|^2, \tag{1.20}$$

where $E = t_1 + \bar{t}_1 - \sigma|t_1|^2 - \sigma|t_2|^2$.

For a bicylinder $\mathfrak{E}$ given by $|\varrho_1 - z_1| < \varrho_1$, $|z_2| < \varrho_2$ we obtain by (4.11*):

$$\begin{aligned} & \boldsymbol{J}_{\mathfrak{E}}(X_{00}, X_{10}, X_{01}) \\ & = \pi^2 E_1^2 E_2^2 |X_{00}|^2 + \frac{\pi^2}{2} E_1^4 E_2^2 \left| \frac{2\left(1 - \frac{\bar{t}_1}{\varrho_1}\right)}{E_1} X_{00} + X_{10}\right|^2 + \frac{\pi^2}{2} E_1^2 E_2^4 \left| - \frac{2\bar{t}_2}{\varrho_2 E_2} X_{00} + X_{10}\right|^2, \end{aligned} \tag{1.21}$$

$$\boldsymbol{H}_{\mathfrak{E}}(X_{00}, X_{01}) = \pi^2 E_1^2 E_2^4 |X_{00}|^2 + \frac{\pi^2}{2} E_1^2 E_2^4 \left| - \frac{2\bar{t}_2}{\varrho_2 E_2} X_{00} + X_{10}\right|^2, \tag{1.22}$$

where $E_1 = t_1 + \bar{t}_1 - \frac{|t_1|^2}{\varrho_1}$, $E_2 = \varrho_2 - \frac{|t_2|^2}{\varrho_2}$.

From the fact that $\boldsymbol{J}_{\mathfrak{B}}(X)$ is the value of the minimum for the problem posed on p. 62, a property follows that is important for further studies of the Jacobian reductions:

Lemma I. *If $\mathfrak{B} \subset \mathfrak{B}^*$, then the Jacobian reductions $\boldsymbol{J}_{\mathfrak{B}}(X)$ and $\boldsymbol{J}_{\mathfrak{B}^*}(X)$ with the same base point $\{t_1, t_2\} \in \mathfrak{B} \subset \mathfrak{B}^*$ satisfy for any set of values X_{pq}:*

$$\boldsymbol{J}_{\mathfrak{B}}(X_{00}, X_{10}, \ldots, X_{mn}) \leq \boldsymbol{J}_{\mathfrak{B}^*}(X_{00}, X_{10}, \ldots, X_{mn}). \tag{1.23}$$

§ 2

We now turn to establishing the anounced equations for the limit points of third order and first recall a few results from K. As before we assume that the interior of $\mathfrak{B}$ in a sufficiently small neighborhood $\mathfrak{U}$ of the boundary point Q is given by (2.1*)[16)]

$$\Phi(z_1, z_2; \overline{z}_1, \overline{z}_2) > 0, \tag{2.1}$$

and that the boundary $\mathfrak{b}^3$ of $\mathfrak{B}$ is given by

$$\Phi(z_1, z_2; \overline{z}_1, \overline{z}_2) = 0, \tag{2.2}$$

and that the tangent hyperplane $\boldsymbol{e}^3$ at the point Q exists. As mentioned in the preface (p. 59), the plane $\mathfrak{C}^2$ containing Q and lying in $\boldsymbol{e}^3$ is chosen to as $z_1 = 0$, and its orthogonal plane as $z_2 = 0$ (the latter is determined up to a rotation about $z_1 = 0$, as it is only required that it contains Q).

Finally, we can choose the positive direction of the x_1-axis such that the inner normal at the point Q corresponds to values $x_1 > 0$, $y_1 = x_2 = y_2 = 0$ (compare K §2). For the following, we use these normal coordinates relative to Q from §2 of K. Then (2.2) is written as

$$2x_1 - \psi(y_1, x_2, \overline{z}_2) = 0, \tag{2.3}$$

where

$$\left(\frac{\partial \psi}{\partial y_1}\right)_{\substack{y_1=0\\ z_2=0}} = 0, \quad \left(\frac{\partial \psi}{\partial z_2}\right)_{\substack{y_1=0\\ z_2=0}} = 0, \quad \left(\frac{\partial \psi}{\partial \overline{z}_2}\right)_{\substack{y_1=0\\ z_2=0}} = 0.$$

By slightly strengthening the assumptions made on ψ in §3 of K,[17)] we assume that ψ is three times smoothly differentiable in a neighborhood of the point Q. Instead of (2.3) we can then write

$$\Phi \equiv 2x_1 - \left(ay_1^2 + 2\mathrm{i}y_1(bz_2 - \overline{b}\overline{z}_2) + cz_2^2 + \overline{c}\overline{z}_2^2 + \sigma|z_2|^2 + \psi_3(y_1, z_2, \overline{z}_2)\right) = 0, \tag{2.4}$$

where $\lim_{y_1 \to 0, z_1 \to 0} \frac{\psi_3(y_1, z_2, \overline{z}_2)}{y_1^2 + |z_2|^2} = 0$, and a, σ are real. $\psi_3(y_1, z_2, \overline{z}_2)$ is a function in $y_1, z_2, \overline{z}_2$ that is three times differentiable in a neighborhood of the coordinate origin Q. Assume that there exists an analytic map[18)] of the domain $\mathfrak{B}$ of the form (3.19*)

$$z_1 = z_1^* + 2bz_1^* z_2^* + cz_2^* + \dots, \quad z_2 = z_2^*, \tag{2.5}$$

[16)]For simplicity, equations that already appeared in K are indicated by a *.

[17)]This strengthening will only be used in §§3 to 5.

[18)]By an analytic map $z_k^* = g_k(z_1, z_2)$, $k = 1, 2$, of a domain $\mathfrak{B}$, we will mean a bijective map of $\mathfrak{B}$, where the $g_k(z_1, z_2)$ are regular and uniformly bounded functions in two complex variables on $\mathfrak{B}$. The function $z_1(z_1^*, z_2^*)$ in (2.5) is of course also regular on $\mathfrak{B}$.

and so the equation for $\mathbf{e}^3$ assumes the form (3.17*)

$$2x_1^* = ay_1^{*2} + \sigma|z_2^*|^2 + \psi_3^*(y_1^*, z_2^*, \overline{z}_2^*) \tag{2.6}$$

in a neighborhood of Q.[19)]

A domain $\mathfrak{B}^*$ obtained from $\mathfrak{B}$ via (2.5) for which the boundary hypersurface in a neighborhood of Q is of the form (2.6) is called the *canonical replacement domain* of $\mathfrak{B}$ for the point Q.

As shown in K, §5, *if $\sigma > 0$ and there exists a canonical replacement domain $\mathfrak{B}^*$ of $\mathfrak{B}$ whose sections $\mathfrak{B}^*(z_1 = \gamma)$ for sufficiently small γ lie in an arbitrarily small neighborhood of Q, then Q is a limit point of third order*, which shall be denoted by Q_3 in the following.

By an *approach* A^{I} we mean convergence $\{z_1, z_2\} \to Q_3$ such that the point $\{z_1, z_2\}$ from $\mathfrak{B}$ remains within a cone $\mathfrak{W}_\alpha$, where $\mathfrak{W}_\alpha$ denotes the totality of all rays through the boundary point Q_3 whose angle to the inner normal is less than α. Using the point Q_3 for normal coordinates, the cone $\mathfrak{W}_\alpha$ is characterized by the inequality

$$\frac{\sqrt{|z_1|^2 + |z_2|^2}}{x_1} < \frac{1}{\cos(\alpha)}, \quad |\alpha| < \frac{\pi}{2}. \tag{2.7}$$

Finally, let $\mathfrak{W}_{c\alpha}$ denote the subset of $\mathfrak{W}_\alpha$ that in addition to (2.7) also satisfies

$$0 < x_1 < c. \tag{2.8}$$

Functions that converge uniformly to 0 upon an approach A^{I} to Q_3 (where the path of the approach is in $\mathfrak{W}_{c\alpha}$) will be denoted by the symbol Ω.

Theorem I. *Under an approach A^{I}, the following limit relations hold at a limit point Q_3:*[20)]

$$\lim(z_1 + \overline{z}_1)^4 \frac{\mathrm{d}\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)}{\mathrm{d}z_1} = -\frac{6\sigma}{\pi^2}, \quad \lim(z_1 + \overline{z}_1)^4 \frac{\mathrm{d}\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2)}{\mathrm{d}z_2} = 0, \tag{2.9}$$

$$\lim(z_1 + \overline{z}_1)^2 \mathrm{d}s^2 = \lim(z_1 + \overline{z}_1)^2 \sum_{m,n=1}^{2} \mathsf{T}_{m\overline{n}} \mathrm{d}z_m \mathrm{d}\overline{z}_n = 3|\mathrm{d}z_1|^2, \tag{2.10}$$

$$\lim \boldsymbol{I}_{\mathfrak{B}}(z_1, z_2; \overline{z}_1, \overline{z}_2) = \frac{2}{9\pi^2}. \tag{2.11}$$

19)We emphasize that for the proof of Theorem I in §5 of K only the uniqueness of (2.5) in a sufficiently small neighborhood of Q is required, and therefore $\mathfrak{b}^{*3}$ can always be brought into the form (5.4*) in the neighborhood of Q.

20)As agreed upon before, we will use normal coordinates for Q_3 in the following arguments.

PROOF: By the arguments in §3 of K, a sequence converging to Q_3 in the sense of A^{I} is mapped to a sequence of the same type under a transformation (2.5). Moreover, by (3.7*), under the approach A^{I},

$$z_1^* + \overline{z}_1^* = (z_1 + \overline{z}_1)(1 + \Omega).$$

It is therefore sufficient to prove that the claimed limit relations hold for the canonical replacement domain $\mathfrak{B}^*$.[21)]

A domain $\mathfrak{J}$ (or $\mathfrak{A}$) contained in $\mathfrak{B}^*$ (or containing $\mathfrak{B}^*$) and containing the boundary point Q_3 and in it the same tangent hyperplane as $\mathfrak{B}^*$ is called the *inner* (or *outer*) *comparison domain.*

As shown in §5 of K, such comparison domains $\mathfrak{J}$ or $\mathfrak{A}$ are obtained from the hyperspheres

$$\left|z_1 - \frac{1}{\sigma_k}\right|^2 + |z_2|^2 < \frac{1}{\sigma_k^2}, \quad k = 1, 2 \tag{2.12}$$

via the transformations (5.3*)

$$z_1' = \frac{z_1}{1 + \alpha_1 z_1}, \quad z_2' = z_2 \left(\frac{1 + \alpha_1 z_1}{1 + (\alpha_1 + \beta_1) z_1}\right) \tag{2.13}$$

or (5.21*)[22)]

$$z_1' = \frac{z_1}{1 - \alpha_2 z_1}, \quad z_2' = z_2 \left(\frac{1 + (\beta_2 - \alpha_2) z_1}{1 - \alpha_2 z_1}\right), \tag{2.14}$$

where σ_k, $k = 1, 2$, are two arbitrary positive quantities satisfying $\sigma_2 < \sigma < \sigma_1$, and α_k, β_k, $k = 1, 2$, are certain contants given in K.[23)]

We proceed with the computation of the coefficients in the Jacobian reductions $\boldsymbol{J}(X_{00}, X_{10}, X_{01})$ and $\boldsymbol{H}(X_{00}, X_{01})$ of $\mathfrak{J}$ and $\mathfrak{A}$, which themselves are functions of the base point. Since we are primarily interested in the behavior of the functions under the approach A^{I}, we will use the symbol Ω introduced on p. 68. The

21)In the following, we omit the $*$ in z_1^*, z_2^*.

22)In (5.3*) it should be $z_2' = \frac{z_2}{1+\beta_1 z_1'}$ rather than $z_2' = \frac{z_2}{1+\beta_1 z_1}$, and in (5.21*) it should be $z_2' = z_2(1 + \beta_2 z_1')$ rather than $z_2' = z_2(1 + \beta_2 z_1)$.

23)In §5 of K, we tacitly assumed that $\mathfrak{B}^*(z_2 = 0)$ is the projection of $\mathfrak{B}^*$ to the z_2-plane (that is, the set of all z_2-coordinates of $\mathfrak{B}^*$). If this is not the case, we can use this projection rather than $\mathfrak{B}^*(z_2 = 0)$ in K for the construction of the outer comparison domain. Since now in a sufficiently small neighborhood of the point Q_3 the boundary $\mathfrak{b}^{*3}$ is given by (2.6), it follows that the boundary curve of the projection of $\mathfrak{B}^*$ to $z_2 = 0$ is given by $2x_1 = ay_1^2 + \sigma|z_2|^2 + \psi_3(y_1, z_2, \overline{z}_2)$. The arguments of §5 in K may thus be repeated in the general case without difficulty.

functions appearing here still depend on σ, and will use the symbol Ω only if the function also converges uniformly in σ to 0 on the interval for σ.

For short, we write:[24)]

$$
\begin{aligned}
A^{\mathfrak{J}} &= (1-\alpha_1 z_1)(1+\beta_1 z_1) = 1+\Omega,\\
A^{\mathfrak{A}} &= (1+\alpha_2 z_1)(1+\beta_2 z_1)^2 = 1+\Omega,\\
E^{\mathfrak{J}} &= z_1+\overline{z}_1-(2\alpha_1+\sigma_1)|z_1|^2-\sigma_1|z_2(1+(\beta_1-\alpha_1)z_1-\alpha_1\beta_1 z_1^2)|^2\\
E^{\mathfrak{A}} &= z_1+\overline{z}_1-(2\alpha_2-\sigma_2)|z_1|^2-\sigma_2|z_2|^2+\beta_2(z_1+\overline{z}_1)^2+|z_1|^2(z_1+\overline{z}_1)(\beta_2^2+\beta_2(2\alpha_2-\sigma_2))\\
&\quad -\alpha_2\sigma_2(z_1+\overline{z}_1)|z_2|^2+\beta_2^2(2\alpha_2-\sigma_2)|z_1|^4-\sigma_2\alpha_2^2|z_1z_2|^2 = (z_1+\overline{z}_1)(1+\Omega).
\end{aligned}
\tag{2.15}
$$

For the derivatives of these quantities, we obtain:[25)]

$$
\begin{gathered}
A_{10}^{\mathfrak{J}} \equiv \frac{\mathrm{d}A^{\mathfrak{J}}}{\mathrm{d}z_1} = (\beta_1-\alpha_1)-\Omega, \quad A_{10}^{\mathfrak{A}} = (\alpha_2+2\beta_2)+\Omega,\\
E_{1000} \equiv \frac{\mathrm{d}E}{\mathrm{d}z_1} = 1+\Omega, \quad E_{0100} \equiv \frac{\mathrm{d}E}{\mathrm{d}z_2} = -\sigma_k\overline{z}_2(1+\Omega),\\
E_{10\overline{1}\overline{0}}^{\mathfrak{J}} = -(2\alpha_1+\sigma_1)(1+\Omega), \quad E_{10\overline{1}\overline{0}}^{\mathfrak{A}} = (2\alpha_2-\sigma_2+\beta_2)(1+\Omega), \quad E_{01\overline{1}\overline{0}}^{\mathfrak{J}} = -\sigma_1 z_2(1+\Omega),\\
E_{01\overline{1}\overline{0}}^{\mathfrak{A}} = 0, \quad E_{01\overline{0}\overline{1}} = -\sigma_k(1+\Omega).
\end{gathered}
$$

Since by (4.22*) and (1.15*)[26)]

$$
\mathsf{K} = \frac{2\sigma_k|A|^2}{\pi^2E^3} = \frac{2\sigma_k}{\pi^2(z_1+\overline{z}_1)^3}(1+\Omega), \tag{2.16}
$$

it follows that

$$
\begin{aligned}
&\mathsf{K}_{1000} = \frac{2\sigma_k}{\pi^2}\left(\frac{A_{10}\overline{A}}{E^3}-\frac{3A\overline{A}E_{1000}}{E^4}\right) = -\frac{6\sigma_k(1+\Omega)}{\pi^2(z_1\overline{z}_1)^4}, \quad \mathsf{K}_{0100} = \frac{6\sigma_k^2 z_2(1+\Omega)}{\pi^2(z_1\overline{z}_1)^4},\\
&\mathsf{K}_{10\overline{1}\overline{0}} = \frac{24\sigma_k}{\pi^4(z_1+\overline{z}_1)^5}(1+\Omega), \quad \mathsf{K}_{01\overline{1}\overline{0}} = -\frac{24\sigma_k^2\overline{z}_2}{\pi^2(z_1+\overline{z}_1)^5}(1+\Omega),\\
&\mathsf{K}_{01\overline{0}\overline{1}} = \frac{6\sigma_k^2}{\pi^2(z_1+\overline{z}_1)^4}(1+\Omega)
\end{aligned}
\tag{2.17}
$$

---

24)To facilitate the eventual handling of the main formulas, the intermediate results of our computations are compiled in (2.15) to (2.22).

25)If the indices $\mathfrak{J}$ or $\mathfrak{A}$ are missing for the $\mathsf{K}_{mn\overline{p}\overline{q}}$, $A, \ldots$, then the respective formular holds for both $\mathfrak{J}$ and $\mathfrak{A}$.

26)In those cases in which it is clear from the context which domain the kernel function refers to, the name of the domain in the index of K is omitted.

and therefore

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix} = \frac{12\sigma_k^2(1+\Omega)}{\pi^4(z_1+\bar{z}_1)^8}, \tag{2.18}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{000\bar{1}} \\ \mathsf{K}_{0100} & \mathsf{K}_{010\bar{1}} \end{vmatrix} = \frac{12\sigma_k^3(1+\Omega)}{\pi^4(z_1+\bar{z}_1)^7}, \tag{2.19}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{1}\bar{0}} \end{vmatrix} = \frac{12\sigma_k^3\bar{z}_2(1+\Omega)}{\pi^4(z_1+\bar{z}_1)^8}, \tag{2.20}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} & \mathsf{K}_{000\bar{1}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} & \mathsf{K}_{100\bar{1}} \\ \mathsf{K}_{0100} & \mathsf{K}_{01\bar{1}\bar{0}} & \mathsf{K}_{010\bar{1}} \end{vmatrix} = \frac{72\sigma_k^4(1+\Omega)}{\pi^6(z_1+\bar{z}_1)^{12}}, \tag{2.21}$$

To obtain the limit relations for $\mathsf{T}_{1\bar{1}} = \frac{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}}{\mathsf{K}^2}$, we employ the Jacobian reduction $\boldsymbol{J}(0, X_{10})$. By Lemma I,

$$\boldsymbol{J}_{\mathfrak{F}}(0, X_{10}) \leq \boldsymbol{J}_{\mathfrak{B}^*}(0, X_{10}) \leq \boldsymbol{J}_{\mathfrak{A}}(0, X_{10}). \tag{2.22}$$

If λ^{10} denotes the expression[27)]

$$\frac{\mathsf{K}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}},$$

then it follows from (2.22) that

$$\lambda^{10}_{\mathfrak{F}} \leq \lambda^{10}_{\mathfrak{B}^*} \leq \lambda^{10}_{\mathfrak{A}}, \tag{2.23}$$

from which we obtain by (2.18) and (2.16)

$$\frac{\frac{2\sigma_1(1+\Omega)}{\pi^2(z_1+\bar{z}_1)^3}}{\frac{12\sigma_1^2(1+\Omega)}{\pi^4(z_1+\bar{z}_1)^8}} \leq \lambda^{10}_{\mathfrak{B}^*} \leq \frac{\frac{2\sigma_2(1+\Omega)}{\pi^2(z_1+\bar{z}_1)^3}}{\frac{12\sigma_2^2(1+\Omega)}{\pi^4(z_1+\bar{z}_1)^8}}$$

and

$$\frac{\pi^2}{6\sigma_1} \leq \lim \frac{\lambda^{10}_{\mathfrak{B}^*}}{(z_1+\bar{z}_1)^5} \leq \frac{\pi^2}{6\sigma_2}. \tag{2.24}$$

As $\sigma_1 - \sigma_2$ may become arbitrarily small,

$$\lim \frac{\lambda^{10}_{\mathfrak{B}^*}}{(z_1+\bar{z}_1)^5} = \frac{\pi^2}{6\sigma}. \tag{2.25}$$

27) Compare (1.13*).

Finally, by (2.16),

$$\lim(z_1+\bar{z}_1)^2 \mathsf{T}_{1\bar{1}}^{\mathfrak{B}^*} = \lim(z_1+\bar{z}_1)^2 \frac{1}{\mathsf{K}_{\mathfrak{B}^*}\lambda_{\mathfrak{B}^*}^{10}} = 3. \tag{2.26}$$

Using $\boldsymbol{H}(0, X_{01})$ and $\boldsymbol{J}(0, 0, X_{01})$ in a similar way, we obtain:

$$\lim \frac{\mu_{\mathfrak{B}^*}^{01}}{(z_1+\bar{z}_1)^4} = \lim \frac{1}{(z_1+\bar{z}_1)^4} \frac{\mathsf{K}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{0100} \\ \mathsf{K}_{00\bar{0}\bar{1}} & \mathsf{K}_{01\bar{0}\bar{1}} \end{vmatrix}} = \frac{\pi^2}{6\sigma^2}, \tag{2.27}$$

$$\lim \frac{\lambda_{\mathfrak{B}^*}^{01}}{(z_1+\bar{z}_1)^4} = \lim \frac{1}{(z_1+\bar{z}_1)^4} \frac{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{00\bar{1}\bar{0}} \\ \mathsf{K}_{1000} & \mathsf{K}_{10\bar{1}\bar{0}} \end{vmatrix}}{\begin{vmatrix} \mathsf{K} & \mathsf{K}_{1000} & \mathsf{K}_{0100} \\ \mathsf{K}_{00\bar{1}\bar{0}} & \mathsf{K}_{10\bar{1}\bar{0}} & \mathsf{K}_{01\bar{1}\bar{0}} \\ \mathsf{K}_{00\bar{0}\bar{1}} & \mathsf{K}_{10\bar{0}\bar{1}} & \mathsf{K}_{01\bar{0}\bar{1}} \end{vmatrix}} = \frac{\pi^2}{6\sigma}, \tag{2.28}$$

which imply

$$\lim(z_1+\bar{z}_1) \mathsf{T}_{2\bar{2}}^{\mathfrak{B}^*} = \lim \frac{z_1+\bar{z}_1}{\mathsf{K}_{\mathfrak{B}^*}\mu_{\mathfrak{B}^*}^{01}} = 2\sigma \tag{2.29}$$

and by ((1.17)*)

$$\lim \boldsymbol{I}_{\mathfrak{B}^*}(z_1, z_2; \bar{z}_1, \bar{z}_2) = \lim \mathsf{K}_{\mathfrak{B}^*}^3 \lambda_{\mathfrak{B}^*}^{10} \lambda_{\mathfrak{B}^*}^{01} = \frac{2}{9\pi^2}, \tag{2.30}$$

which proves (2.11).
Using $\boldsymbol{J}(\frac{X_{00}}{\sqrt{(z_1+\bar{z}_1)^3}}, \frac{X_{10}}{\sqrt{(z_1+\bar{z}_1)^5}})$ leads to a limit relation for $\frac{\mathrm{d}\mathsf{K}_{\mathfrak{B}^*}}{\mathrm{d}z_1}$. From

$$\begin{aligned} \boldsymbol{J}_{\mathfrak{Z}}\left(\frac{X_{00}}{\sqrt{(z_1+\bar{z}_1)^3}}, \frac{X_{10}}{\sqrt{(z_1+\bar{z}_1)^5}}\right) &\leq \boldsymbol{J}\left(\frac{X_{00}}{\sqrt{(z_1+\bar{z}_1)^3}}, \frac{X_{10}}{\sqrt{(z_1+\bar{z}_1)^5}}\right) \\ &\leq \boldsymbol{J}_{\mathfrak{A}}\left(\frac{X_{00}}{\sqrt{(z_1+\bar{z}_1)^3}}, \frac{X_{10}}{\sqrt{(z_1+\bar{z}_1)^5}}\right) \end{aligned} \tag{2.31}$$

together with (2.16), (2.25) and (2.17) it follows that

$$\begin{aligned} \frac{\pi^2}{2\sigma_1}|X_{00}|^2 + \frac{\pi^2}{6\sigma_1}|3X_{00} + X_{10}|^2 &\leq \frac{\pi^2}{2\sigma}|X_{00}|^2 + \frac{\pi^2}{6\sigma}\left| -\lim(z_1+\bar{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}X_{00} + X_{10}\right|^2 \\ &\leq \frac{\pi^2}{2\sigma_2}|X_{00}|^2 + \frac{\pi^2}{6\sigma_2}|3X_{00} + X_{10}|^2, \end{aligned}$$

from which we obtain

$$-\lim(z_1+\overline{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}=3,\qquad \text{that is, }\lim(z_1+\overline{z}_1)^4\mathsf{K}_{1000}=-\frac{6\sigma}{\pi^2},\tag{2.32}$$

and taking into account (2.16), we obtain the first relation in (2.9). Moreover, from

$$\boldsymbol{H}_{\mathfrak{B}^*}(X_{00},0)\le \boldsymbol{H}_{\mathfrak{A}}(X_{00},0)\tag{2.33}$$

and (2.16), (2.27), (2.17) it follows that

$$\begin{aligned}&\frac{\pi^2}{2\sigma}(z_1+\overline{z}_1)^3(1+\Omega)+\frac{\pi^2}{6\sigma^2}(z_1+\overline{z}_1)^2\left|-(z_1+\overline{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}\right|^2\\&\le\frac{\pi^2}{2\sigma_2}(z_1+\overline{z}_1)^3(1+\Omega)+\frac{\pi^2}{6\sigma_2^2}(z_1+\overline{z}_1)^4\left|-\frac{3\sigma_2\overline{z}_2}{z_1+\overline{z}_1}\right|^2.\end{aligned}$$

By dividing this inequality by $(z_1+\overline{z}_1)^2$, we obtain

$$\lim(z_1+\overline{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}=0\tag{2.34}$$

and by taking into account (2.16), the second relation in (2.9).

Now, it follows from

$$\boldsymbol{J}_{\mathfrak{B}^*}(0,X_{10},0)\le \boldsymbol{J}_{\mathfrak{A}}(0,X_{10},0)\tag{2.35}$$

and (2.25), (2.28), (2.20), (2.18) that

$$\begin{aligned}&\frac{\pi^2}{6\sigma}(z_1+\overline{z}_1)^5+\frac{\pi^2}{6\sigma^2}(z_1+\overline{z}_1)^4\frac{\left\|\begin{matrix}\mathsf{K}&\mathsf{K}_{00\overline{1}\overline{0}}\\\mathsf{K}_{0100}&\mathsf{K}_{01\overline{1}\overline{0}}\end{matrix}\right\|}{\left\|\begin{matrix}\mathsf{K}&\mathsf{K}_{00\overline{1}\overline{0}}\\\mathsf{K}_{1000}&\mathsf{K}_{10\overline{1}\overline{0}}\end{matrix}\right\|}\\&\le\frac{\pi^2}{6\sigma_2}(z_1+\overline{z}_1)^5+\frac{\pi^2}{6\sigma_2^2}(z_1+\overline{z}_1)^4|\sigma_2z_2|^2,\end{aligned}\tag{2.36}$$

that is (compare (2.18))

$$\lim(z_1+\overline{z}_1)^8\begin{vmatrix}\mathsf{K}&\mathsf{K}_{00\overline{1}\overline{0}}\\\mathsf{K}_{0100}&\mathsf{K}_{01\overline{1}\overline{0}}\end{vmatrix}=0,$$

from which we finally obtain

$$\lim(z_1+\overline{z}_1)^2\mathsf{T}^{\mathfrak{B}^*}_{1\overline{2}}=\lim(z_1+\overline{z}_1)^2\frac{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{00\overline{1}\overline{0}}\\\mathsf{K}_{0100}&\mathsf{K}_{01\overline{1}\overline{0}}\end{vmatrix}}{\mathsf{K}^2}=0.\tag{2.37}$$

Now (2.10) follows from (2.26), (2.37) and (2.29).

We now consider the points of second order. The boundary manifold $\mathfrak{b}^3$ in a neighborhood of Q shall again be given by (2.3), where we now only assume ψ to be differentiable once (in all variables). Moreover, in a neighborhood of Q, $\mathfrak{b}^3$ shall have common surface segment $\mathfrak{H}^2$ with an analytic surface $z_1 - g(z_2) = 0$. Here, $g_2(z)$ with $g'(0) > 0$ is a unique and regular function in z_2 on $\mathfrak{B}^2$ (the projection of $\mathfrak{b}^3$ onto the z_2-plane) and Q an interior point of $\mathfrak{H}^2$ in $z_1 - g(z_2)$.[28)]

Via the transformation $z_1' = z_1 - g(z_2)$, $z_2' = z_2$, we can achieve that the surface segment $\mathfrak{H}^2$ lies in the plane $z_1' = 0$. We will always use these coordinates z_1', z_2' when investigating points of second order.[29)] Under certain natural assumptions on $\mathfrak{B}$ (see K, §6)[30)], the order of approaching infinity of the kernel function when approaching these boundary points is $\lambda(Q, \mathfrak{B}) = 2$. In the second part of §6 in K, we identified a special class of these boundary points and showed that they are limit points. The assumptions (slightly changed compared to §6) on the structure of the boundary points (denoted by Q_2) are:

1. The section $\mathfrak{H}^2 = \mathfrak{b}^3(z_2 = 0)$ is a starshaped domain with respect to $z_2 = 0$. If $R = h(\theta)$ is the equation of a boundary curve, then we assume that $h(\theta)$ is continuously differentiable. That is,

$$|h(\theta) \geq a > 0, \quad |h'(\theta)| \leq B < \infty, \quad 0 \leq \theta \leq 2\pi,$$

   where a and B are suitable constants.

2. The section $\mathfrak{B}(z_1 = 0)$ contains a circle $\mathfrak{T}^2$ of radius $\frac{1}{\varrho} > 0$ that touches the y_1-axis in the point $z_1 = 0$.

3. For the sections $\mathfrak{B}(z_1 = \gamma)$ we assume that

$$\frac{\mathfrak{B}(z_1 = 0)}{m(|\gamma|)} \subset \mathfrak{B}(z_1 = \gamma), \quad \text{for } |\gamma| \leq \delta,\ \gamma \in \mathfrak{T}^2, \tag{2.38}$$

$$m(|\gamma|)\mathfrak{B}(z_1 = 0) \supset \mathfrak{B}(z_1 = \gamma) \quad \text{for } |\gamma| \leq \delta,\ \mathrm{Re}(\gamma) > 0 \tag{2.39}$$

28) This type of boundary point appears for example in certain domains with maximum surface. Compare *Über eine in gewissen Bereichen mit Maximumfläche gültige Integraldarstellung der Funktionen zweier komplexer Variablen I*, Mathematische Zeitschrift 89 (1934), p. 77, in particular the domains in §5.

29) In the following, the primes will be omitted in z_1', z_2'.

30) Note that on p. 23 in K, $A(z_2, \bar{z}_2)$ must be replaced by $A(y_1, z_2, \bar{z}_2)$ and in (6.3*), $\lim_{z_1 \to 0} A(z_2, \bar{z}_2)$ must be replaced by $\lim_{y_1 \to 0} A(y_1, z_2, \bar{z}_2)$.

holds,[31)] where $m(|\gamma|) = 1 + N|\gamma|^{\frac{1}{\tau}}$ and $N < \infty$, $\tau > 0$ are suitable constants.

4. For all points in $\mathfrak{B} + \mathfrak{b}^3$, $z_1 + \bar{z}_1 \geq 0$ holds.

By an approach A^{V} towards the boundary point $Q(0, a_2)$, with a_2 in $\mathfrak{H}^2$, we mean the convergence $\{z_1, z_2\} \to \{0, a_2\}$, where the z_1-coordinates of the point $\{z_1, z_2\}$ of $\mathfrak{B}$ remains inside the angular domain

$$\mathfrak{W}_\alpha^2: \qquad x_1 > 0, \quad \frac{|z_1|}{z_1} < \frac{1}{\cos(\alpha)}, \quad |\alpha| < \frac{\pi}{2}. \tag{2.40}$$

In analogy to our previous usage, $\mathfrak{W}_{c\alpha}^2$ denotes the subdomain of $\mathfrak{W}_\alpha^2$ that satisfies (2.8) in addition to (2.40). Without further mention, once fixed, α shall be considered constant in the following.

To overlook the changes in the limit formulas that appear in the variation of the limit point in the interior of $\mathfrak{H}^2$, we drop the assumption that Q_2 is the coordinate origin and merely assume that Q_2 is an inner point of $\mathfrak{H}^2$, that is, Q_2 has coordinates $0, a_2$ (with $a_2 \in \mathfrak{H}^2$).

For those functions that converge uniformly to 0 under an approach A^{V} to $\{0, a_2\}$ ($a_2 \in \mathfrak{H}^2$ fixed) and whose exact value is not of interest for our investigations, we shall throughout use the symbol O, where the uniformly bounded functions under the same approach are usually denoted by B.

Theorem II. *At a limit point $Q_2(0, a_2)$, $a_2 \in \mathfrak{H}^2$, the following limit relations hold under the approach A^{V}:*

$$\begin{aligned} \lim (z_1 + \bar{z}_1)^3 \frac{\mathrm{d}\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \bar{z}_1, \bar{z}_2)}{\mathrm{d}z_1} &= -\frac{2}{\pi^2 \mathsf{P}(a_2)^2}, \\ \lim (z_1 + \bar{z}_1)^3 \frac{\mathrm{d}\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \bar{z}_1, \bar{z}_2)}{\mathrm{d}z_2} &= 0, \end{aligned} \tag{2.41}$$

$$\lim (z_1 + \bar{z}_1)^2 \mathrm{d}s^2 = \lim (z_1 + \bar{z}_1)^2 \sum_{m,n=1}^{2} \mathsf{T}_{m\bar{n}} \mathrm{d}z_m \mathrm{d}\bar{z}_n = 2|\mathrm{d}z_1|^2, \tag{2.42}$$

[31)] The assumptions made on p. 38 can be replaced by the weaker conditions stated here in 2. and 3. The following arguments in K do not have to be modified.

Moreover, note that the work cited in footnote 28) and its second part (to appear soon in Mathematische Zeitschrift) studies certain domains that are bounded by finitely many analytic hypersurfaces. As shown in §7 there, for a large class of such domains, every “leaf” of the analytic boundary surface can be transformed into our normal form here, where assumptions 1. to 4. are satisfied.

$$\lim \boldsymbol{I}_{\mathfrak{B}}(z_1, z_2; \bar{z}_1, \bar{z}_2) = \frac{1}{4\pi^2}, \tag{2.43}$$

where $\mathsf{P}(a_2)$ *is the image radius of* $\mathfrak{H}^2$ *with respect to* a_2*.*[32)]

PROOF: To construct the comparison domain of $\mathfrak{B}$ for Q_2, we need a function $f_\nu(z_2)$ that was already introduced in (6.14*) in K. It maps the right z-halfplane to an isosceles triangle AOB. The sides AO and BO of this triangle are symmetric with respect to the real axis, the point O lies on this axis and has abscissa 1. The angle $\sphericalangle AOB$ has the value $\frac{\pi}{\nu}$. To determine the parameter ν, we choose a positive $\kappa < \frac{a}{Z_2^{\max}} \leq 1$, where a is the distance of the point a_2 from the boundary of $\mathfrak{H}^2$, and $Z_2^{\max}$ is the maximum of the absolute values of all z_2-coordinates in $\mathfrak{B}$. By $\mathfrak{J}_\nu^2(\kappa)$ we denote the domain in the right z-halfplane on which $|f_\nu(z)| > \kappa$ holds. Now determine ν large enough for the inequalities (6.16*), (6.17*), (6.29*), (6.30*) and (6.31*) to hold (the last three inequalities make use of the constants in assumptions 1. to 3.). All these inequalities yield a lower boundary for ν.

As the circle $\mathfrak{J}^2$ introduced in assumption 2. can always be replaced by a small circle (touching the y-axis in the origin), we assume that $\mathfrak{J}^2 \subset \mathfrak{J}_\nu^2(\kappa)$ to begin with.

Let $\mathfrak{J} = \mathfrak{J}^2 \times \mathfrak{H}^2$. The domain $\mathfrak{J}_\nu$ obtained from $\mathfrak{J}$ via the transformation $z_1' = z_1$, $z_2' = z_2 f_\nu(z_1)$ is an inner comparison domain of $\mathfrak{B}$: Firstly, at the point Q_2 the inner normal will coincide with the x_1-axis, and secondly, as $\mathfrak{J}_\nu$ is contained in the domain $\mathfrak{D}^{(\nu)} \subset \mathfrak{B}$ (introduced in K), it follows that $\mathfrak{J}_\nu \subset \mathfrak{B}$.

As an outer comparison domain we use the domain $\mathfrak{A}_\nu$ given in K (denoted by $\mathfrak{A}^{(\nu)}$ there), which is obtained from $\mathfrak{A} = \mathfrak{E}^2 \times \mathfrak{H}^2$ (with $\mathfrak{E}^2$ the right z_1-halfplane) via $z_1' = z_1$, $z_2' = \frac{z_2}{f_\nu(z_1)}$.

We will now compute the coefficients of the Jacobian reductions $\boldsymbol{J}(X_{00}, X_{10}, X_{01})$ and $\boldsymbol{H}(X_{00}, X_{01})$ for $\mathfrak{J}_\nu$ and $\mathfrak{A}_\nu$.

By (6.14*) and by §2 of the article *Über die ausgezeichneten Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen*[33)], where a function

[32)]By the *image radius* of a domain $\mathfrak{H}^2$ with respect to a_2 we mean the radius of the circle to which the domain can be mapped simply and conformally by that function which has derivative 1 at a_2. Compare Bieberbach, *Lehrbuch der Funktionentheorie II* (Berlin, 1927), p. 322. In the following paragraph we refer to §6 of K, where instead of A_1, O_1, B_1, $\mathfrak{T}_\nu(\kappa)$ we use A, O, B, $\mathfrak{J}_\nu^2(\kappa)$. On p. 25, line 1, instead of "z-plane" it should read "right z-halfplane", and in (6.20*), $M_2 \leq 1$ should be replaced by $M_2 \leq 1 + \varepsilon_\nu$, $\lim_{\nu\to\infty} \varepsilon_\nu = 0$.

[33)]Mathematische Annalen 104 (1931), p. 611. This work in referred to as A in the following.

$t_\nu(z)$ related to $f_\nu(z)$ was studied, $f_\nu(z_1)$ has an expansion

$$f_\nu(z_1) = 1 - A_\nu z_1^{\frac{1}{\nu}} + \alpha_1 z_1^{1+\frac{1}{\nu}} + \dots, \quad A_\nu = \frac{\nu\Gamma(\frac{1}{2}+\frac{1}{2\nu})}{\Gamma(\frac{1}{\nu})\Gamma(\frac{1}{2}-\frac{1}{2\nu})}, \tag{2.44}$$

near $z_1 = 0$, which implies

$$f_\nu(0) = 1, \quad \lim_{z_1\to 0}(z_1+\overline{z}_1)f_\nu'(z_1) = 0. \tag{2.45}$$

If $w(z_2)$, $w(0) = 0$, $w'(0) > 0$ denote the function that maps $\mathfrak{S}^2$ to the circle $|w| < 1$, and if we write $w \equiv w^{(0)}$, $\frac{\mathrm{d}w}{\mathrm{d}z_2} =\equiv w^{(1)}$, etc. for short, then by the Mean Value Theorem,

$$w^{(k)}\left(\frac{z_2}{f_\nu(z_1)}\right) = w^{(k)}(z_2)+z_1^{\frac{1}{\nu}}K_k(z_1,z_2) \quad \text{and} \quad w^{(k)}(z_2)(z_2 f_\nu(z_1)) = w^{(k)}(z_2)+z_1^{\frac{1}{\nu}}G_k(z_1,z_2), \tag{2.46}$$

where

$$K_k = -w^{(k+1)}\left(\frac{z_2}{f_\nu(\theta z_1)}\right)\frac{z_2 f_\nu^{(1)}(\theta z_1)}{f_\nu(\theta z_1)^2}, \quad G_k = w^{(k+1)}(z_2 f_\nu(\theta z_1))z_2 f_\nu^{(1)}(\theta z_1), \tag{2.47}$$

with $0<\theta<1, k=0,1,$ $f_\nu^{(\kappa)} \equiv \frac{\mathrm{d}^\kappa f_\nu}{\mathrm{d}(z_1^{\frac{1}{\nu}})^\kappa}$. Since now by (1.15*) and (4.11*)

$$\begin{aligned} \mathsf{K}_{\mathfrak{F}_\nu} &= \frac{|w'(\frac{z_2}{f_\nu(z_1)})|^2}{\pi^2(z_1+\overline{z}_1-\sigma|z_1|^2)^2(1-|w(\frac{z_2}{f_\nu(z_1)})|^2)^2|f_\nu(z_1)|^2} \\ \mathsf{K}_{\mathfrak{A}_\nu} &= \frac{|w'(z_2 f_\nu(z_1))|^2|f_\nu(z_1)|^2}{\pi^2(z_1+\overline{z}_1)^2(1-|w(z_2 f_\nu(z_1))|^2)^2}, \end{aligned} \tag{2.47a}$$

we obtain[34)]

$$\mathsf{K} = \frac{B}{\pi^2(z_1+\overline{z}_1)^2}(1+\mathrm{O}), \quad \mathsf{K}_{1000} = \frac{\mathsf{K}}{z_1+\overline{z}_1}(-2+\mathrm{O}), \quad \mathsf{K}_{0100} = \mathsf{K}B_1, \tag{2.48}$$

[34)]The relation for $\mathsf{K}_{1000}^{\mathfrak{F}_\nu}$ is obtained from

$$\mathsf{K}_{\mathfrak{F}_\nu}^{1000} = \mathsf{K}\left(-\frac{w^{(2)}(\frac{z_2}{f_\nu(z_1)})z_2 f_\nu'(z_1)}{w^{(1)}(\frac{z_2}{f_\nu(z_1)})f_\nu(z_1)^2} - 2\frac{1-\varrho z_1}{z_1+\overline{z}_1-\varrho|z_1|^2} - 2\frac{\overline{w(\frac{z_2}{f_\nu(z_1)})}w^{(1)}(\frac{z_2}{f_\nu(z_1)})z_2 f_\nu'(z_1)}{(1-|w(\frac{z_2}{f_\nu(z_1)})|^2)f_\nu(z_1)^2} - \frac{f_\nu'(z_1)}{f_\nu(z_1)}\right)$$

Similarly, using (2.44) and (2.46), the other formulas are established. If $\mathsf{K}_{mn\overline{pq}}$ does not have an upper indes $\mathfrak{F}_\nu$ or $\mathfrak{A}_\nu$, then the formula holds for both domains.

where, as already stated,

$$B = \frac{|w^{(1)}(z_2)|^2}{(1-|w(z_2)|^2)^2} = \frac{1}{\mathsf{P}(z_2)^2}, \quad B_1 = \frac{w^{(2)}(z_2)}{w^{(1)}(z_2)} + 2\frac{\overline{w(z_2)}w'(z_2)}{1-|w(z_2)|^2} + \mathrm{O}.$$

Furthermore,

$$\mathsf{K}_{01\overline{1}\overline{0}} = \mathsf{K}\left(\frac{6+\mathrm{O}}{(z_1+\overline{z}_1)^2}\right), \quad \mathsf{K}_{010\overline{1}} = \mathsf{K}(B_2+\mathrm{O}), \quad \mathsf{K}_{100\overline{1}} = -\frac{\mathsf{K}}{z_1+\overline{z}_1}(2\overline{B}_1+\mathrm{O}), \tag{2.49}$$

where $B_2 = |B_1|^2 + 2B$. We obtain

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{1000} \\ \mathsf{K}_{00\overline{1}0} & \mathsf{K}_{10\overline{1}0} \end{vmatrix} = \frac{2\mathsf{K}^2}{(z_1+\overline{z}_1)^2}(1+\mathrm{O}), \tag{2.50}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{0100} \\ \mathsf{K}_{000\overline{1}} & \mathsf{K}_{010\overline{1}} \end{vmatrix} = 2\mathsf{K}^2(B+\mathrm{O}), \tag{2.51}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{0100} \\ \mathsf{K}_{00\overline{1}0} & \mathsf{K}_{01\overline{1}0} \end{vmatrix} = \frac{\mathsf{K}^2\mathrm{O}}{z_1+\overline{z}_1}, \tag{2.52}$$

$$\begin{vmatrix} \mathsf{K} & \mathsf{K}_{1000} & \mathsf{K}_{0100} \\ \mathsf{K}_{00\overline{1}0} & \mathsf{K}_{10\overline{1}0} & \mathsf{K}_{01\overline{1}0} \\ \mathsf{K}_{000\overline{1}} & \mathsf{K}_{100\overline{1}} & \mathsf{K}_{010\overline{1}} \end{vmatrix} = \frac{4\mathsf{K}^3(B+\mathrm{O})}{(z_1+\overline{z}_1)^2}. \tag{2.53}$$

In analogy to the limit points of third order, it follows from (2.23), (2.48) and (2.50) that

$$\lambda_{\mathfrak{B}}^{10} = \frac{1}{2\mathsf{K}}(z_1+\overline{z}_1)^2(1+\mathrm{O}), \tag{2.54}$$

which implies

$$\lim (z_1+\overline{z}_1)^2 \mathsf{T}^{\mathfrak{B}}_{1\overline{1}} = \lim \frac{(z_1+\overline{z}_1)^2}{\mathsf{K}_{\mathfrak{B}}\lambda_{\mathfrak{B}}^{10}} = 2. \tag{2.55}$$

Analogously, by (2.51), (2.50) and (2.53),

$$\lim \frac{\mu_{\mathfrak{B}}^{01}}{(z_1+\overline{z}_1)^2} = \frac{\pi^2}{2B^2}, \tag{2.56}$$

$$\lim \frac{\lambda_{\mathfrak{B}}^{01}}{(z_1+\overline{z}_1)^2} = \frac{\pi^2}{2B^2}, \tag{2.57}$$

which implies

$$\lim \mathsf{T}^{\mathfrak{B}}_{2\overline{2}} = 2B \tag{2.58}$$

and (2.43).

From

$$\frac{B}{\pi^2}\boldsymbol{J}_{\mathfrak{F}_\nu}\left(\frac{X_{00}}{z_1+\overline{z}_1},\frac{X_{10}}{(z_1+\overline{z}_1)^2}\right)\le\frac{B}{\pi^2}\boldsymbol{J}_{\mathfrak{B}}\left(\frac{X_{00}}{z_1+\overline{z}_1},\frac{X_{10}}{(z_1+\overline{z}_1)^2}\right)$$
$$\le\frac{B}{\pi^2}\boldsymbol{J}_{\mathfrak{A}_\nu}\left(\frac{X_{00}}{z_1+\overline{z}_1},\frac{X_{10}}{(z_1+\overline{z}_1)^2}\right)\tag{2.59}$$

it follows that

$$|X_{00}|^2+\frac{1}{2}|2X_{00}+X_{10}|^2\le|X_{00}|^2+\frac{1}{2}\left|-\lim(z_1+\overline{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}X_{00}+X_{10}\right|^2$$
$$\le|X_{00}|^2+\frac{1}{2}|2X_{00}+X_{10}|^2,\tag{2.60}$$

from which we obtain

$$\lim(z_1+\overline{z}_1)\frac{\mathsf{K}_{1000}}{\mathsf{K}}=-2,\qquad\text{that is, }\lim(z_1+\overline{z}_1)^3\mathsf{K}_{1000}=-\frac{2B}{\pi^2}.\tag{2.61}$$

Multiplying (2.33) by $\frac{B}{\pi^2}\frac{1}{(z_1+\overline{z}_1)^2}$ and taking the limi $z_1\to0$, $z_2\to a_2$, we obtain

$$1+\frac{1}{2}\lim\left|\frac{\mathsf{K}_{1000}}{\mathsf{K}}\right|^2\le1+\frac{1}{2}|B_1|^2,$$

which implies

$$\lim\left|\frac{\mathsf{K}_{1000}}{\mathsf{K}}\right|=|B_1|,\qquad\text{that is, }\lim(z_1+\overline{z}_1)^2|\mathsf{K}_{1000}|=\frac{B|B_1|}{\pi^2}.\tag{2.62}$$

Multiplying (2.35) by $\frac{2B}{\pi^2(z_1+\overline{z}_1)^4}$ and taking the limit $z_1\to0$, $z_2\to a_2$, we obtain

$$1+\frac{1}{B}\left|\frac{1}{z_1+\overline{z}_1}\frac{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{0100}\\\mathsf{K}_{00\overline{1}\overline{0}}&\mathsf{K}_{01\overline{1}\overline{0}}\end{vmatrix}}{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{1000}\\\mathsf{K}_{00\overline{1}\overline{0}}&\mathsf{K}_{10\overline{1}\overline{0}}\end{vmatrix}}\right|^2\le1+\frac{1}{B}|\mathsf{O}|^2,$$

which implies

$$\lim\frac{1}{z_1+\overline{z}_1}\frac{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{0100}\\\mathsf{K}_{00\overline{1}\overline{0}}&\mathsf{K}_{01\overline{1}\overline{0}}\end{vmatrix}}{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{1000}\\\mathsf{K}_{00\overline{1}\overline{0}}&\mathsf{K}_{10\overline{1}\overline{0}}\end{vmatrix}}=0,\qquad\text{that is, }\lim(z_1+\overline{z}_1)|\mathsf{T}^{\mathfrak{B}}_{1\overline{2}}|=\lim(z_1+\overline{z}_1)\left|\frac{\begin{vmatrix}\mathsf{K}&\mathsf{K}_{0100}\\\mathsf{K}_{00\overline{1}\overline{0}}&\mathsf{K}_{01\overline{1}\overline{0}}\end{vmatrix}}{\mathsf{K}^2}\right|=0\tag{2.63}$$

Now (2.42) follows from (2.55), (2.58) and (2.63).

§ 3

In §§3 to 5, we give limit relations under an approach to the limit points Q_3 of third order for certain families of functions, and moreover we will show that the minimal function $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2) = \frac{\mathsf{K}_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)}{\mathsf{K}_{\mathfrak{B}}(t_1, t_2; \bar{t}_1, \bar{t}_2)}$ of $\mathfrak{B}$ with base point $\{t_1, t_2\}$ belongs to this family of functions.

By t_1, t_2 and z_1, z_2 we will always denote normal coordinates for the boundary point Q_3, which we will not mention explicitely. As we remarked in §2, the boundary hypersurface in a neighborhood of Q_3 is given by (2.3). Through a transformation of type (2.5) we obtain the canonical replacement domain $\mathfrak{B}^*$, where the equation of the boundary hypersurface $\mathfrak{b}^{*3}$ in a neighborhood of Q_3 is brought into the special form (2.6).

We will now show: From the assumptions made on ψ_3 in §2 follows the existence of a constant A such that

$$|\psi_3(y_1^*, z_2^* \bar{z}_2^*)| \leq A(|y_1^*|^3 + |z_2^*|^3). \tag{3.1}$$

It is enough to show that (3.1) holds in a sufficiently small neighborhood of Q_3.

If in (2.4) we replace, according to (2.5), x_1, y_1 by the real and imaginary parts $(x_1^* + \mathrm{i}y_1^*) + 2b(z_1^* + \mathrm{i}y_1^*)(x_2^* + \mathrm{i}y_2^*) + \ldots$, x_2, y_2 by x_2^*, y_2^*, then we obtain an equation of the form $\Phi^*(x_1^*, y_1^*, x_2^*, y_2^*) = 0$. As the pair of transformations (2.5) maps the point Q_3 to itself and a sufficiently small neighborhood of Q_3 in $z_1 z_2$-space bijectively and continuously into such a neighborhood of Q_3 in $z_1^* z_2^*$-space, it follows that

$$\Phi^*(x_1^*, y_1^*, x_2^*, y_2^*) \not\equiv 0, \quad \left(\frac{\partial \Phi^*(x_1^*, y_1^*, x_2^*, y_2^*)}{\partial x_1^*}\right)_{x_1^*=y_1^*=x_2^*=y_2^*=0} \neq 0, \quad \Phi^*(0,0,0,0) = 0$$

hold. By the Implicit Function Theorem[35] $\Phi^* = 0$ can be represented in a sufficiently small neighborhood $|y_1^*| \leq \delta$, $|x_2^*| \leq \delta$, $|y_2^*| \leq \delta$ by

$$x_1^* = \psi^*(y_1^*, x_2^*, y_2^*). \tag{3.2}$$

As Φ was assumed three-times continuously differentiable and the pair of transformations (2.5) is analytic, Φ^* and hence ψ^* is three-times continuously differentiable. A formal computation shows that the terms of second degree are of

[35]Compare for example Osgood, *Funktionentheorie I*, p. 69.

the form given in (2.6), whereas $\psi_3^*(y_1^*, x_2^*, \overline{z}_2^*)$ is a three-times continuously differentiable function in $y_1^* = z_2^* = 0$ for which the first two derivatives in all variables vanish.

By the first Mean Value Theorem, we can set

$$\psi_3^*(y_1^*, z_2^*, \overline{z}_2^*) = A_1 z_1^{*3} + A_2 y_1^{*2} z_2^* + \ldots + A_8 z_2^{*2}\overline{z}_2^* + A_9 z_2^* \overline{z}_2^{*2},$$

where the $A_k \equiv A_k(y_1^*, x_2^*, \overline{z}_2^*)$ are uniformly bounded functions in a neighborhood of $y_1^* = z_2^* = 0$. If $\frac{A}{18}$ denotes an upper bound for the A_k, then we obtain (3.1), which was to be shown.[36)]

THE p-COUPLED APPROACH OF A PAIR OF POINTS. By a *p-coupled* pair of points ($p > 0$)[37)]

$$\{z_1, z_2, t_1, t_2\}, \quad z_k = x_k + \mathrm{i}y_k, \quad t_k = u_k + \mathrm{i}v_k, \quad x_1 > 0,\ u_1 > 0$$

we mean a sequence in which the x_1- and u_1-coordinates of corresponding points $\{z_1, z_2\}$ and $\{t_1, t_2\}$ always satisfy the inequality

$$0 < m \leq \frac{u_1^p}{x_1} \leq M < \infty, \tag{3.3}$$

where m, M are suitable constants, independent of the positions of the points.

If moreover the points $\{z_1, z_2\}, \{t_1, t_2\}$ converge to Q_3 in the sense of the approach A^{I}, then we will speak of a *p-coupled A^{I}-approach* of the pair of points, or simply of a *p-coupled A^{I}-sequence of points*.

As remarked in K, §3, we can change from one normal coordinate system z_1, z_2 to another one z_1', z_2' via a transformation of the form (3.3*).

Now we wish to show that *the given definition of the p-coupled A^{I}-approach is independent of the choice of normal coordinate system.*

Since, as was expounded in §3 of K in more detail, a sequence of points converging in the sense A^{I} in z_1z_2-space is mapped into such a sequence in $z_1'z_2'$-space, we only need to show that (3.3) implies an analogous estimate for $\frac{u_1'^p}{x_1'}$.

---

36)As the domain $\mathfrak{B}^*$ is finite, we may assume that all of the boundary $\mathfrak{b}^{*3}$ of $\mathfrak{B}^*$ is given by (2.6), where ψ_3^* statisfies the inequality (3.1) (possibly after enlarging A) and ψ_3 has the stated property in a neighborhood of Q_3. (Of course, ψ_3^* can be ambiuous outside of the neighborhood of Q_3.

37)In the following, we omit the $*$ in z_1^*, z_2^*, with the exception of the arguments on p. 86.

Now[38)] (compare (3.6*) and (3.4*)),

$$\frac{(2u_1')^p}{x_1'} = \frac{(t_1' + \bar{t}_1')^p}{z_1' + \bar{z}_1'} = \frac{\left(\tau + \frac{g_{12}(t_1,t_2)+\overline{g_{12}(t_1,t_2)}}{\sqrt{|t_1|^2+|t_2|^2}} \cdot \frac{\sqrt{|t_1|^2+|t_2|^2}}{t_1+\bar{t}_1}\right)^p}{\tau + \frac{g_{12}(z_1,z_2)+\overline{g_{12}(z_1,z_2)}}{\sqrt{|z_1|^2+|z_2|^2}} \cdot \frac{\sqrt{|z_1|^2+|z_2|^2}}{z_1+\bar{z}_1}} \frac{(t_1 + \bar{t}_1)^p}{z_1 + \bar{z}_1}$$

$$= \frac{\left(\tau + \Omega \cdot \frac{\sqrt{|t_1|^2+|t_2|^2}}{t_1+\bar{t}_1}\right)^p}{\tau + \Omega \cdot \frac{\sqrt{|z_1|^2+|z_2|^2}}{z_1+\bar{z}_1}} \frac{(t_1 + \bar{t}_1)^p}{z_1 + \bar{z}_1}.$$

As (2.7) holds under the approach A^{I}, we have for a sufficiently small neighborhood of Q_3

$$km \leq k\frac{u_1^p}{x_1} \leq \frac{u_1'^p}{x_1'} \leq K\frac{u_1^p}{x_1} \leq KM, \tag{3.4}$$

where $0 < k < K < \infty$. As relation (3.3) only needs to be shown for sufficiently small values of the variables, it follows from (3.4) that the sequence $\{z_1', z_2', t_1', t_2'\}$ is also p-coupled, as was to be shown.

For the following, it is helpful to give different characterization for the p-coupled A^{I}-sequences of pairs of points.

Let $\mathfrak{M}$ denote a subdomain of $\mathfrak{W}_\alpha$ for which in addition to (2.7) also

$$0 < c \leq X_1 \leq C < \infty \tag{3.5}$$

holds, where c and C are suitable positive constants, small enough such that $\mathfrak{W}_{C\alpha}$ lies in $\mathfrak{B}^*$.

By capital letters Z_1, Z_2 ($Z_k = X_k + \mathrm{i}Y_k$) and T_1, T_2 ($T_k = U_k + \mathrm{i}V_k$) we will always denote points in $\mathfrak{M}$.

We will now show that for every p-coupled sequence of pairs of points $\{z_1^{(\nu)}, z_2^{(\nu)}, t_1^{(\nu)}, t_2^{(\nu)}\}$, $\nu = 1, 2, \ldots$, we can construct a suitable domain $\mathfrak{M}$ such that

$$z_k^{(\nu)} = \frac{Z_k^{(\nu)}}{n_\nu^p}, \quad t_k^{(\nu)} = \frac{T_k^{(\nu)}}{n_\nu}, \quad \lim_{\nu\to\infty} n_\nu = \infty,$$

where $\{Z_1^{(\nu)}, Z_2^{(\nu)}\}$ and $\{T_1^{(\nu)}, T_2^{(\nu)}\}$, $\nu = 1, 2, \ldots$, are two point sequences in $\mathfrak{M}$. We choose $n_\nu = \frac{d}{u_1^{(\nu)}}$, that is, we set $U_1^{(\nu)} = d$. From $\lim u_1^{(\nu)} = 0$ follows $\lim_{\nu\to\infty} n_\nu = \infty$. From (3.3) it moreover follows that

$$\frac{d^p}{M} \leq X_1^{(\nu)} = x_1^{(\nu)} n_\nu^p \leq \frac{u_1^{(\nu)p} n_\nu^p}{m} = \frac{d^p}{m}. \tag{3.6}$$

38) See §3 in K for details of this argument.

Since on the other hand $\frac{\sqrt{|Z_1^{(\nu)}|^2+|Z_2^{(\nu)}|^2}}{X_1^{(\nu)}} = \frac{\sqrt{|z_1^{(\nu)}|^2+|z_2^{(\nu)}|^2}}{x_1^{(\nu)}} < \frac{1}{\cos(\alpha)}$, we now conclude that the sequences of points $\{Z_1^{(\nu)}, Z_2^{(\nu)}\}$ and $\{T_1^{(\nu)}, T_2^{(\nu)}\}$ lie in a domain $\mathfrak{M}$ with

$$c = \min\left(d, \frac{d^p}{M}\right), \quad C = \max\left(d, \frac{d^p}{M}\right).$$

Conversely, if a sequence of pairs of points $\left\{\frac{Z_1^{(\nu)}}{n_\nu^p}, \frac{Z_2^{(\nu)}}{n_\nu^p}, \frac{T_1^{(\nu)}}{n_\nu^p}, \frac{T_2^{(\nu)}}{n_\nu^p}\right\}$ is given, in which (3.5) holds for $X_1^{(\nu)}$ and $U_1^{(\nu)}$, then $\frac{c^p}{C} \leq \frac{u_1^{(\nu)p}}{x_1^{(\nu)}} \leq \frac{C^p}{c}$, that is, the sequence is p-coupled in the sense of our first definition.

§ 4

Theorem III. *Suppose to every point $\{t_1, t_2\}$ of a sequence of points $\mathfrak{P}^0$ lying in $\mathfrak{W}_\alpha$ with $\lim\{t_1, t_2\} = Q_3$ we assign a function $f(z_1, z_2; t_1, t_2)$ in the complex variables z_1, z_2 that is regular on $\mathfrak{B}$ and square-integrable. Assume further that every function in this family has the following two properties:*

1. *It holds that*
$$f(t_1, t_2; t_1, t_2) = 1. \tag{4.1}$$

2. *The integrals $\int_{\mathfrak{B}} |f(z_1, z_2; t_1, t_2)|^2 \mathrm{d}\omega_z$ satisfy the inequality*
$$\int_{\mathfrak{B}} |f(z_1, z_2; t_1, t_2)|^2 \mathrm{d}\omega_z \leq \frac{\pi^2}{2\sigma}(t_1 + \bar{t}_1)^3(1 + C(t_1 + \bar{t}_1)^r), \tag{4.2}$$
where $C < \infty$, $r > 0$ are fixed constants independent of $\{t_1, t_2\}$, and σ is the characteristic quantity of the boundary point Q_3 appearing in the expansion (2.4). Then for every p with $\max(1 - \frac{r}{3}, \frac{14}{15}) < p \leq 1$ it holds that under a p-coupled A^{I}-approach of the pair of points $\{z_1, z_2, t_1, t_2\}$ uniformly[39] *(with respect to any path of approach within $\mathfrak{W}_\alpha$)*
$$\lim \frac{z_1^3}{(t_1 + \bar{t}_1)^3} f(z_1, z_2; t_1, t_2) = 1 \quad \textit{for } p < 1, \tag{4.3}$$
$$\lim \frac{(z_1 + \bar{z}_1)^3}{(t_1 + \bar{t}_1)^3} f(z_1, z_2; t_1, t_2) = 1 \quad \textit{for } p = 1. \tag{4.4}$$

[39] By assumption, α is fixed for our angular domain $\mathfrak{W}_\alpha$ (that is, α in (2.7)).

PROOF: For the proof, we use claims I to IV, which will be proved later in §5.

Via a transformation (2.5) we replace $\mathfrak{B}$ by its canonical replacement domain $\mathfrak{B}^*$ (for which the boundary hypersurface in a neighborhood of $Q_3^* = Q_3$ is given by (2.6)). If on $\mathfrak{B}^*$ we consider the function

$$f^*(z_1^*, z_2^*; t_1^*, t_2^*) = f\left(z_1(z_1^*, z_2^*), z_2(z_1^*, z_2^*); t_1(t_1^*, t_2^*), t_2(t_1^*, t_2^*)\right) \frac{E(z_1^*, z_2^*)}{E(t_1^*, t_2^*)}, \tag{4.5}$$

with

$$E(z_1^*, z_2^*) = \frac{\partial(z_1, z_2)}{\partial(z_1^*, z_2^*)},$$

for which $f^*(t_1^*, t_2^*; t_1^*, t_2^*) = 1$ holds, then (as will be shown in Ia) the integrals $\int_{\mathfrak{B}} |f^*(z_1^*, z_2^*; t_1^*, t_2^*)|^2 d\omega_z$ satisfy an inequality of the form (4.2). On the other hand, from the fact that f^* satisfies the limit relations (4.3) and (4.4) we can easily deduce that the same limit relations hold for f (see Ib).

To prove relations (4.3) and (4.4) for f^*,[40] we construct a sequence of inner comparison domains $\mathfrak{I}_n$ (described in more detail later in II on p. 88). The investigations in III and II (compare p. 96) lead to the following result: Let $\{Z_1, Z_2\}$ and $\{T_1, T_2\}$ be arbitrary points in $\mathfrak{M}$. Then $\left\{\frac{T_1}{n}, \frac{T_2}{b}\right\} \in \mathfrak{I}_n$ and

$$\int_{\mathfrak{I}_n} \left| \mathsf{M}_{\mathfrak{I}_n}\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right)\right|^2 d\omega_z = \frac{\pi^2}{2n^3\sigma}(T_1+\overline{T}_1)^3 \left(1 + \frac{B(n, T_1, T_2; \overline{T}_1, \overline{T}_2}{n^{\frac{1}{5}}}\right), \tag{4.6}$$

where B is a uniformly bounded function in $n > n_0$ and $\{T_1, T_2\} \in \mathfrak{M}$. Moreover for every p with $\frac{4}{5} \leq p \leq 1$,

$$\lim_{n\to\infty} \left(\frac{Z_1 + \frac{\overline{T}_1}{n^{1-p}}}{n^p}\right)^3 \left(\frac{n}{T_1 + \overline{T}_1}\right)^3 \mathsf{M}_{\mathfrak{I}_n}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}, \frac{T_1}{n}, \frac{T_2}{n}\right) = 1. \tag{4.7}$$

We now introduce the sequence of auxiliary functions

$$h\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right) = f\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right) - \mathsf{M}_{\mathfrak{I}_n}\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right), \tag{4.8}$$

of which each one is regular in its corresponding $\mathfrak{I}_n$. Using Lemma II (to be proved later), we find that for $n > n_0$ (where n_0 can be chosen independently of $\{T_1, T_2\}$ in $\mathfrak{M}$) it follows by using (4.2) and (4.6) (compare p. **??**)[41]

$$\int_{\mathfrak{I}_n} \left| h\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right)\right|^2 d\omega_z \leq \frac{c_1}{n^{3+\varrho}}, \quad \varrho = \min\left(r, \frac{1}{5}\right). \tag{4.9}$$

[40] With the exception of the aruguments in I, we omit the $*$ in $f^*, z_1^*, z_2^*, t_1^*, t_2^*$ from now on.

[41] The c_k, $k = 1, 2, 3, 4$, are fixed constants, independent of $n > n_0$ and $\{T_1, T_2\} \in \mathfrak{M}$.

On the other hand, as will be shown in III, there exists for every point $\{Z_1, Z_2\}$ of $\mathfrak{M}$ and for every $n > n_0$ a bicylinder $\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right)$, centered at $\left\{\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right\}$ and of volume $\frac{c_2}{n^{3p}}$, that is completely contained in $\mathfrak{J}_n$. Here, $c_2 = \pi^2\tau^3$, where τ is the constant given in III.

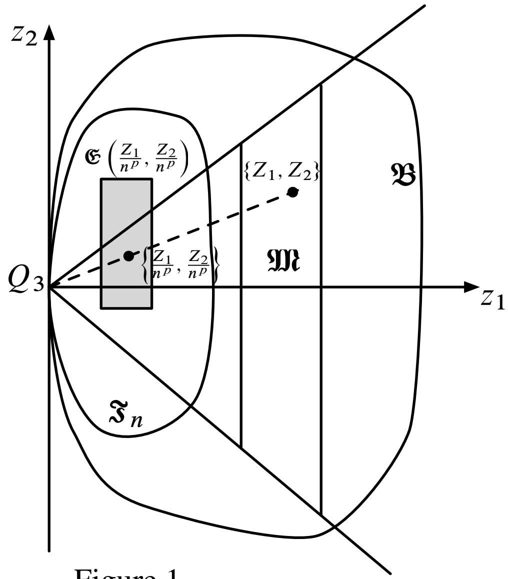

Figure 1

For every regular function $g(Z_1, Z_2)$ on a Reinhardt circle domain $\mathfrak{K}$ a theorem (proved earlier) holds, stating that the value of g at the center (ζ_1, ζ_2) of $\mathfrak{K}$ satisfies the inequality[42)]

$$|g(\zeta_1, \zeta_2)|^2 \le \frac{\int_{\mathfrak{B}} |g(Z_1, Z_2)|^2 \mathrm{d}\omega}{\mathrm{vol}(\mathfrak{K})}. \tag{4.10}$$

Since $\{Z_1, Z_2\} \in \mathfrak{M}$ (as shown in III) implies $\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right) \subset \mathfrak{J}_n$, for $h\left(z_1, z_2, \frac{T_1}{n}, \frac{T_2}{n}\right)$, which is regular in $\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right) \subset \mathfrak{J}_n$, it holds by (4.10) that

$$\begin{aligned} \left| h\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}; \frac{T_1}{n}, \frac{T_2}{n}\right) \right| &\le \sqrt{\frac{\int_{\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right)} |h|^2 \mathrm{d}\omega}{\mathrm{vol}\left(\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right)\right)}} \\ &\le \sqrt{\frac{\int_{\mathfrak{J}_n} |h|^2 \mathrm{d}\omega}{\mathrm{vol}\left(\mathfrak{E}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\right)\right)}} \le \sqrt{\frac{c_3}{n^{3+\varrho-3p}}} = \frac{c_4}{n^{\frac{3}{2}+\frac{1}{2}\varrho-\frac{3}{2}p}}, \end{aligned} \tag{4.11}$$

42) See H, Hilfssatz I, p. 649.t

that is,

$$\left| n^{3-3p} h\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}; \frac{T_1}{n}, \frac{T_2}{n}\right) \right| \leq \frac{c_4}{n^{\frac{3}{2}p+\frac{1}{2}\varrho-\frac{3}{2}}}. \tag{4.12}$$

Both functions $\mathsf{M}_{\mathfrak{Z}_n}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}; \frac{T_1}{n}, \frac{T_2}{n}\right)$ and $h\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}; \frac{T_1}{n}, \frac{T_2}{n}\right)$ are defined for any n and any $\{Z_1, Z_2\} \in \mathfrak{M}$. If we now choose $1 > p > \max\left(1 - \frac{r}{3}, \frac{14}{15}\right)$, then from (4.7) and (4.12) the limit relations (4.3) and (4.4) follows, which, assuming claims I to IV, proves Theorem III.

The minimal function $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ satisfies (4.1) by definition, and by (1.12*) and (1.11*) and the relations (5.40) below also the inequality (4.2). Hence:

Corollary. *Under a p-coupled A^{I}-approach of the pair of points $\{z_1, z_2, t_1, t_2\} \to \{0, 0, 0, 0\}$, the limit relations stated in* (4.3) *and* (4.4) *hold for the minimal function $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ of $\mathfrak{B}$.*

§ 5

Now we give the proofs for the statements I to IV.

I. We show that

(a) relation (4.2) holds for the function $f^*(z_1^*, z_2^*; t_1^*, t_2^*)$[43] introduced in (4.5)

(b) from the limit relations (4.3) and (4.4) for the function $f^*(z_1^*, z_2^*; t_1^*, t_2^*)$, the same limit relations follow for $f(z_1, z_2; t_1, t_2)$.

PROOF: (a) As a consequence of the regularity of $t_1(t_1^*, t_2^*)$ in a neighborhood of $t_1^* = 0, t_2^* = 0$, it follows from the Mean Value Theorem that

$$t_1 = t_1^* + A_1 t_1^{*2} + A_2 t_1^* t_2^* + A_3 t_2^{*2}, \quad E(t_1^*, t_2^*) = 1 + A_4 t_1^* + A_5 t_1^*,$$

where $A_k = A_k(t_1^*, t_2^*)$, $k = 1, \ldots, 5$, is a function in t_1^*, t_2^* that is uniformly bounded on a small neighborhood of $\{0, 0\}$. By (2.7), in $\mathfrak{W}_{\alpha^*}$ it holds that

$$\frac{|t_k^*|}{t_1^* + \bar{t}_1^*} \leq \frac{1}{\cos(\alpha^*)}, \quad |\alpha^*| < \frac{\pi}{2}.$$

Hence there exists a finite constant A such that for the given neighborhood

$$t_1 + \bar{t}_1 \leq (t_1^* + \bar{t}_1^*)(1 + A(t_1^* + \bar{t}_1^*)), \quad \frac{1}{|E(t_1^*, t_2^*)|} \leq 1 + A(t_1^* + \bar{t}_1^*) \tag{5.1}$$

[43] Recall that the coordinates z_1^*, z_2^* refer to the replacement domain, z_1, z_2 refer to the original domain.

holds. Hence, for $r < 1$,[44)]

$$
\begin{aligned}
\int_{\mathfrak{B}^*} |f^*(z_1^*, z_2^*; t_1^*, t_2^*)|^2 \mathrm{d}\omega_{z^*} &= \frac{1}{|E(t_1^*, t_2^*)|^2} \int_{\mathfrak{B}} |f(z_1, z_2; t_1, t_2)|^2 \mathrm{d}\omega_z \\
&\leq \frac{\pi^2}{2\sigma} \frac{1}{|E(t_1^*, t_2^*)|^2} (t_1 + \bar{t}_1)^3 (1 + C(t_1 + \bar{t}_1)^r) \\
&\leq \frac{\pi^2}{2\sigma} (t_1^* + \bar{t}_1^*)^3) 1 + A(t_1^* + \bar{t}_1^*))^2 \big(1 + C(t_1^* + \bar{t}_1^*)^r (1 + A(t_1^* + \bar{t}_1^*))^r\big) \\
&\leq \frac{\pi^2}{2\sigma} (t_1^* + \bar{t}_1^*)^3 (1 + C^*(t_1^* + \bar{t}_1^*)^r),
\end{aligned}
\tag{5.2}
$$

where C^* is a suitable constant. This proves (a).

(b) Analogously, we can write for z_1^* and $\frac{1}{D(z_1, z_2)} = E(z_1^*, z_2^*)$ as follows:

$$
z_1^* = z_1 + B_1 z_1^2 + B_2 z_1 z_2 + B_3 z_2^2, \quad \frac{1}{D(z_1, z_2)} = 1 + B_4 z_1 + B_5 z_2,
$$

where $B_k \equiv B_k(z_1, z_2)$ a uniformly bounded functions in z_1, z_2 on a sufficiently small neighborhood of the coordinate origin. It thus follows firstly that under an A^{I}-approach

$$
\lim \frac{z_1^*}{z_1} = 1 \qquad \left(\text{and thus } \lim \frac{t_1^* + \bar{t}_1^*}{t_1 + \bar{t}_1} = 1\right) \tag{5.3}
$$

holds. Moreover, under a 1-coupled A^{I}-approach of the pair of points $\{z_1, z_2, t_1, t_2\}$,

$$
\frac{z_1 + \bar{t}_1^*}{z_1 + \bar{t}_1} = 1 + (z_1 + \bar{t}_1) \frac{B_1 z_1^2 + \overline{B}_1 \bar{t}_1^2 + B_2 z_1 z_2 + \overline{B}_2 \bar{t}_1 \bar{t}_2 + B_3 z_2^2 + \overline{B}_3 \bar{t}_2^2}{(z_1 + \bar{t}_1)^2}.
$$

On the other hand, $|z_1 + \bar{t}_1| > x_1$ (as $x_1 > 0, u_1 > 0$), where

$$
|z_k| \leq \frac{x_1}{\cos(\alpha)}, \quad |t_k| \leq \frac{u_1}{\cos(\alpha)} \leq \frac{M x_1}{\cos(\alpha)},
$$

from which the uniform boundedness of the coefficients in $z_1 + \bar{t}_1$ and thus in the case $p = 1$ in (3.3)

$$
\lim \frac{z_1^* + \bar{t}_1^*}{z_1 + \bar{t}_1} = 1 \tag{5.4}
$$

follows.

[44)] Without loss of generality, we may assume $r < 1$.

II. We turn to setting up the comparison domains $\mathfrak{I}_n$ and $\mathfrak{A}_n$, $n = 1, 2, 3 \ldots$.[45)] These arise from hyperspheres (2.12) (denoted by $\mathfrak{K}_{\sigma_{kn}}$) of the radii

$$\frac{1}{\sigma_{1n}} = \frac{1}{\sigma + \frac{D_1}{n^{\frac{1}{5}}}} \quad \text{and} \quad \frac{1}{\sigma_{2n}} = \frac{1}{\sigma - \frac{D_2}{n^{\frac{1}{5}}}} \tag{5.5}$$

via the transformations (2.13) and (2.14), respectively, where we have to put

$$\beta_{kn} = C_k n^{\frac{4}{5}}. \tag{5.6}$$

The α_k appearing in (2.13) and (2.14), as well as D_k, C_k, $k = 1, 2$, are constants, independent of n, with whose determination we will be concerned later. We turn to their determination and the proof that the thus obtained domains are the inner and outer comparison domains $\mathfrak{B}^*$ for Q_3. As all of these domains have the coordinate origin Q_3 as a boundary point and $x = 0$ as a common tangent plane, all we need to show is that

$$\mathfrak{I}_n \subset \mathfrak{B}^*, \tag{5.7}$$

and

$$\mathfrak{B}^* \subset \mathfrak{A}_n \tag{5.8}$$

holds. In K, §5, the same relations (compare (5.16*) and (5.33*)) were proved by firstly constructing a domain $\mathfrak{K}_{\alpha_1}$ from $\mathfrak{K}_{\sigma_{1n}}$ via the transformation (5.5*): $z_1' = \frac{1}{1+\alpha_1 z_1}$, $z_2' = z_2$, and showing that the part of $\mathfrak{K}_{\alpha_1}$ that belongs to the bicylinder

$$|z_1| < l_1 \tag{5.9}$$

lies in $\mathfrak{B}^*$. By another transformation (5.13*): $z_1'' = z_1'$, $z_2'' = \frac{z_2'}{1+\beta_1 z_1'}$ we achieved that the remaining part of $\mathfrak{K}_{\alpha_1}$ was contracted such that for the thus arising domain $\mathfrak{K}_{\alpha_1\beta_1}$, now denoted by $\mathfrak{I}_n$, (5.7) holds. By a similar method, (5.8) was proved. By $\mathfrak{B}_k^2$ we denoted the totality of the points of $\mathfrak{B}^*(z_2 = 0)$ for which $|z_1| \leq l_k$, $k = 1, 2$, holds (see footnote 23)).

The situation is different from §5 in K because we now have an infinite sequence of domains $\mathfrak{I}_n$ and $\mathfrak{A}_n$, respectively, that (since $\lim_{n\to\infty} \beta_{kn} = \infty$) converge for $n \to \infty$ to a disc and a domain extending to infinity, respectively, and because now the l_{kn}, $k = 1, 2$, converge to zero.

Therefore, we have obtain some sharper estimates as in K, §5, that allow the application of the described method.

[45)]The outer comparison domain will be used in the corollary for the proof that the minimal functions satisfies relation (4.2). All quantities that vary with n will be given an index n.

We begin by establishing some inequalities.

1. *There exists a constant* $P_1 \equiv P_1(\alpha_1)$ *independent of* n *such that for all points* $\{z_1, z_2\}$ *in every* $\mathfrak{K}_{\alpha_1 n}$ *the inequality*

$$|z_2| \leq P_1(\alpha_1)|z_1|^{\frac{1}{2}} \tag{5.10}$$

holds.

The hypersphere (2.12) for which $\sigma_{kn} = \sigma$ shall be denoted by $\mathfrak{K}_\sigma$. The domain obtained from $\mathfrak{K}_\sigma$ by (2.13) shall be called $\mathfrak{K}_{\alpha_1}$. Since $\mathfrak{K}_{\sigma_1 n} \subset \mathfrak{K}_\sigma$, and hence $\mathfrak{K}_{\alpha_1 n} \subset \mathfrak{K}_{\alpha_1}$, $\mathfrak{K}_{\alpha_1 n}(z_1 = \gamma) \subset \mathfrak{K}_{\alpha_1}(z_1 = \gamma)$, it is sufficient to prove (5.10) for $\mathfrak{K}_{\alpha_1}$. For a sufficiently small neighborhood of Q_3, the interior of $\mathfrak{K}_{\alpha_1}$ is given by

$$2x_1 - (2\alpha_1 + \sigma)y_1^2 - \sigma|z_2|^2 + \varphi_3(y_1, z_2, \overline{z}_2) > 0, \tag{5.11}$$

where φ_3 satisfies[46)]

$$|\varphi_3(y_1, z_2, \overline{z}_2)| < B(|y_1|^3 + |z_2|^3), \quad B < \infty. \tag{5.12}$$

At every point of $\mathfrak{K}_{\alpha_1}$ belonging to this domain, it holds that

$$2x_1 - (2\alpha_1 + \sigma)y_1^2 - \sigma|z_2|^2 + B|y_1|^3 + B|z_2|^3 > 0, \tag{5.13}$$

which implies (as we may assume $l_1 < 1$ and $|y_1| < 1$)

$$|z_2|^2(\sigma - B|z_2|) < 2x_1 - (2\alpha_1 + \sigma)y_1^2 + By_1^3 \leq 2x_1 + By + 1 \leq (2 + B)|z_1|.$$

Since now $\mathfrak{K}_{\alpha_1}(z_1 = \gamma)$ falls into an arbitrarily small neighborhood of $z_2 = 0$, it follows that for sufficiently small z_1, say $|z_1| \leq l = l(\alpha_1)$, $B|z_2| \leq \frac{1}{2}\sigma$ holds,

46) The existence of a finite B in (5.12) can be deduced as follows: As $\mathfrak{K}_{\alpha_1}$ is a finite domain, we only need to prove (5.12) for sufficiently small values of the arguments, for example, if simultaneously $|y_1| < l$, $|z_2| < l$, where l is a sufficiently small positive number. Now, the boundary of $\mathfrak{K}_{\alpha_1}$ is given by

$$\Phi \equiv z_1 + \overline{z}_1 - (2\alpha_1 + \sigma)|z_1|^2 - \sigma|1 - \alpha_1 z_1|^2|z_2|^2 = 2x_1 - (2\alpha_1 + \sigma)(z_1^2 + x_1^2) - \sigma|1 - \alpha_1(x_1 + \mathrm{i}y_1)^2||z_2|^2 = 0.$$

As $\left(\frac{\partial \Phi}{\partial x_1}\right)_{\substack{y_1=0\\ z_2=0}} \neq 0$, we can invert the equation $\Phi = 0$ for sufficiently small y_1, z_2, say for $|y_1| \leq l_1$, $|z_2| \leq l_1$, and we obtain $2x_1 = \psi(y_1, z_2, \overline{z}_2)$, where ψ is an infinitely differentiable function on the given domain. Application of the Mean Value Theorem yields

$$\varphi_3(y_1, z_2, \overline{z}_2) = B_1 y_1^3 + B_2 y_1^2 z_2 + B_3 y_1 z_1^2 + \ldots + B_8 z_2 \overline{z}_2^2 + B_9 \overline{z}_2^3,$$

where the B_k are uniformly bounded functions in $y_1, z_2, \overline{z}_2$ on the given domain. If B is large enough such that $|B_k(y_1, z_2, \overline{z}_2)| \leq \frac{B}{18}$ on the given domain, then we obtain (5.12).

and thus in this inequality,

$$|z_2| \leq \sqrt{\frac{2(2+B)}{\sigma}}|z_1|^{\frac{1}{2}} \tag{5.14}$$

holds. As $\mathfrak{K}_{\alpha_1}$ is bounded, (5.10) already follows from (5.14) by setting $P_1(\alpha_1) = \max\left(\sqrt{\frac{2(2+B)}{\sigma}}, \frac{M}{\sqrt{l(\alpha_1)}}\right)$ where M is the maximum of the absolute values of the z_2-coordinates in $\mathfrak{K}_{\alpha_1}$.

Analogous reasoning leads to the following result:

2. *There exists a constant $P_2 < \infty$ such that for all points $\{z_1, z_2\}$ of $\mathfrak{B}^*$,*

$$|z_2| \leq P_2|z_1|^{\frac{1}{2}} \tag{5.15}$$

holds.

3. *For a sufficiently large E we can determine a positive constant P_3 such that for every point $\{z_1, z_2\}$ on the boundary $\mathfrak{b}^{*3}$ of $\mathfrak{B}^*$ whose z_1-coordinates satisfy*

$$\left|z_1 - \frac{1}{E}\right| \leq \frac{1}{E}, \tag{5.16}$$

it holds that

$$|z_2| \geq P_3(E)|z_1|. \tag{5.17}$$

We choose E large enough such that the circle (5.16) lies in the interior of $\mathfrak{B}^*(z_2 = 0)$, with the exception of the boundary point $z_1 = 0$. Then $E > a$ (see (2.6) for a), and we can determine a sufficiently small positive p such that $E > a + Ap$, and moreover the part of $\mathfrak{b}^{*3}$ whose z_1-coordinate satisfies $|z_1| < p$ can be represented in the form (2.6), where ψ_3 satisfies the inequality (2.6). If we decompose the circle (5.16) into two parts $\mathfrak{E}_1^2 + \mathfrak{E}_2^2$ such that $\mathfrak{E}_1^2$ contains all points $\{z_1, z_2\}$ with $|z_1| < p$, then $\mathfrak{E}_2^2$ is closed and contained in the interior of $\mathfrak{B}^*(z_2 = 0)$. For all points $\{z_1, z_2\}$ of $\mathfrak{b}^{*3}$ whose z_1-coordinates belong to $\mathfrak{E}_2^2$, the corresponding $|z_1|$ have a lower bound ϱ, so that $|z_2| \geq \frac{E\varrho}{2}|z_1|$ holds for this part of $\mathfrak{b}^{*3}$. So (5.17) only needs to be proved for those $\{z_1, z_2\}$ for which $|z_1| < p$ holds. Now this part of $\mathfrak{b}^{*3}$ is given by (2.6). By (2.6) and (3.1),

$$2x_1 - ay_1^2 = \sigma|z_1|^2 + \psi_3(y_1, z_2, \bar{z}_2) \leq \sigma|z_2|^2 + A|y_1|^3 + A|z_2|^3$$

and thus (as we may assume $|z_1| < 1$)

$$|z_2|^2(\sigma + A) \geq 2x_1 - y_1^2(a + A|y_1|). \tag{5.18}$$

On the other hand, $|z_2|^2 \leq \frac{2x_1}{E}$ by (5.16), and from (5.18) follows

$$|z_2|^2(\sigma + A) \geq E|z_1|^2 - y_1^2(a + A|y_1|). \tag{5.19}$$

If we now set $P_3 = \min\left(\frac{E\varrho}{2}, \sqrt{\frac{E-a-Ap}{\sigma+A}}\right)$, then we obtain (5.17).

If $\mathfrak{K}_{\alpha_2}$ denotes the domain obtained from $\mathfrak{K}_\sigma$ via $z_1' = \frac{z_1}{1-\alpha_2 z_1}$, $z_2' = z_2$, then $\mathfrak{K}_{\alpha_2} \subset \mathfrak{K}_{\alpha_2 n}$ and $\mathfrak{K}_{\alpha_2}(z_1 = \gamma) \subset \mathfrak{K}_{\alpha_2 n}(z_1 = \gamma)$. With the analogous argument above, we conclude:

4. *For every $F > \sigma - 2\alpha_2$ there exists a positive constant $P_4 \equiv P_4(F)$ independent of n_0 such that for all points $\{z_1, z_2\}$ the boundary $\mathfrak{k}^3_{\alpha_2 n}$ of $\mathfrak{K}_{\alpha_2 n}$, for which*

$$\left|z_1 - \frac{1}{F}\right| \leq \frac{1}{F} \tag{5.20}$$

holds, satisfies the inequality

$$|z_2| \geq P_4(F)|z_1|. \tag{5.21}$$

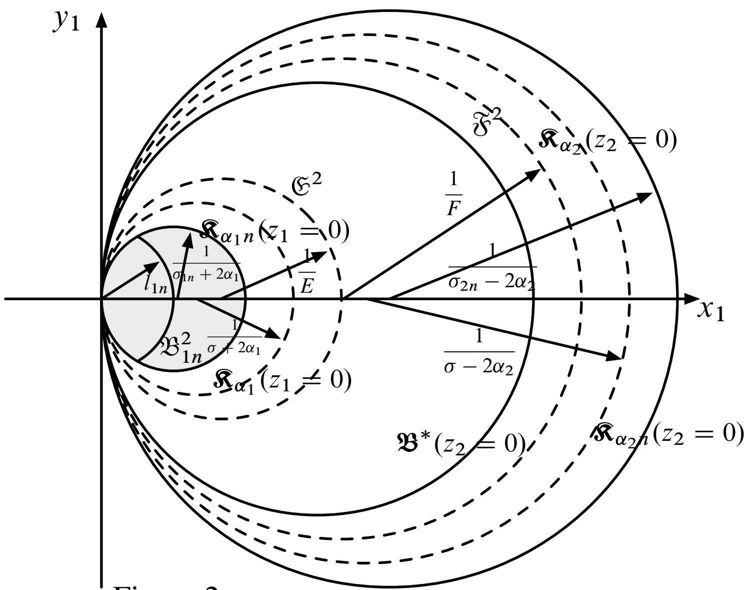

Figure 2

Now that claims 1 to 4 have been derived, we turn to the proof of (5.7) and (5.8). Here we determine α_1 large enough such that $\mathfrak{K}_{\alpha_1}(z_1 = 0)$ lies in the circle (5.16). Then

$$\sigma + 2\alpha_1 > E > a. \tag{5.22}$$

Moreover, the constants l_1, D_1 and C_1 appearing in (5.9), (5.5), (5.6) are set to

$$l_{1n} = \frac{1}{n^{\frac{2}{5}}}, \quad D_1 = (A+B)P_1, \text{ that is, } \sigma_{1n} = \sigma + \frac{(A+B)P_1}{n^{\frac{1}{5}}}, \quad C_1 = \frac{1}{P_3\sigma}, \tag{5.23}$$

where P_1, P_3 mean the $P_1(\alpha_1)$, $P_3(E)$ belonging to the just chose α_1 and E, respectively, and A, B are the quantities given in (3.1) and (5.12).

For sufficiently large n, every point $\{z_1, z_2\}$ of $\mathfrak{K}_{\alpha_1 n}$ with $|z_1| \le l_{1n}$ lies in $\mathfrak{B}^*$, for it follows from (5.23) that for $|z_1| \le l_{1n} \equiv \frac{1}{n^{\frac{2}{5}}}$

$$\sigma_{1n} - BP_1|z_1|^{\frac{1}{2}} > \sigma + AP_1|z_1|^{\frac{1}{2}},$$

and together with (5.10),

$$\sigma_{1n} - B|z_2| > \sigma + A|z_2|. \tag{5.24}$$

Since moreover $y_1 \to 0$ for sufficiently large n and thus by (5.22)

$$\sigma_{1n} + 2\alpha_1 - B|y_1| > \alpha + A|y_1|,$$

it follows from (5.24) and (5.22):

$$\begin{aligned} &2x_1 - (\sigma_{1n} + 2\alpha_1)y_1^2 - \sigma_{1n}|z_2|^2 + \varphi_3(y_1, z_2\overline{z}_2) \\ &\le 2x_1 - (\sigma_{1n} + 2\alpha_1)y_1^2 - \sigma_{1n}|z_2|^2 + B|y_1|^3 + B|z_2|^3 \\ &\le 2x_1 - ay_1^2 - \sigma|z_2|^2 - A|y_1|^3 - A|z_2|^3 \\ &\le 2x_1 - ay_1^2 - \sigma|z_2|^2 + \psi_3(y_1, z_2, \overline{z}_2), \end{aligned} \tag{5.25}$$

that is, every point $\{z_1, z_2\}$ of $\mathfrak{K}_{\alpha_1 n}$ with $|z_1| \le l_1$ lies in $\mathfrak{B}^*$ (for large enough n). By (5.2) and (2.12), $\mathfrak{K}_{\alpha_1 n}(z_2 = 0)$: $\left|z_1 - \frac{1}{\sigma_n + 2\alpha_1}\right| \le \frac{1}{\sigma_{1n} + 2\alpha_1}$ lies in the circle (5.16). For the points $\{z_1, z_2\}$ on the boundary $\mathfrak{b}^{*3}$ of $\mathfrak{B}^*$ whose z_1-coordinates belong to (5.16), (5.17) holds. If ϱ_n^* denotes the minimum of absolute values of z_2 on $\mathfrak{b}^{*3}$, whose z_1-coordinates belong to $\mathfrak{B}_{1n}^2$, then by (5.17) and (5.23),

$$\varrho_n^* = \min_{z_1 \in \mathfrak{B}_{1n}^2} |z_2| \ge \frac{P_3}{n^{\frac{2}{5}}}. \tag{5.26}$$

The inequality (5.14*) of K, which is a sufficient condition that the domain $\mathfrak{I}_n (= \mathfrak{K}_{\alpha_1\beta_1}^{(n)})$, obtained from $\mathfrak{K}_{\alpha_1 n}$ via (5.13), lies in the interior of $\mathfrak{B}^*$,[47] is clearly

[47]For the proof that (5.14*) implies $\mathfrak{K}_{\alpha_1\beta_1}^{(n)} \subset \mathfrak{B}^*$, see p. 28.

satisfied; for by (5.23), (2.13), (5.6), (5.26),

$$\frac{1}{\sigma_{1n}|1+\beta_{1n}z_1|} \le \frac{1}{\sigma|\beta_{1n}z_1|} \le \frac{1}{\sigma\left|\frac{n^{\frac{4}{5}}}{P_3\sigma}\cdot\frac{1}{n^{\frac{2}{5}}}\right|} = \frac{P_3}{n^{\frac{2}{5}}} \le \varrho_n^*. \tag{5.27}$$

Thus, (5.7) is proved and we turn to the outer comparison domain. We choose F such that (5.20) contains the domain $\mathfrak{B}^*(z_2 = 0)$ in the interior (which implies $a > F$), and determine α_2 such that

$$a > F > \sigma - 2\alpha_2,$$

and for the constants appearing in (5.5) and (5.6) we substitute the values

$$l_{2n} = \frac{1}{n^{\frac{2}{5}}}, \quad D_2 = (A+B)P_2, \text{ that is, } \sigma_{2n} = \sigma - \frac{(A+B)P_2}{n^{\frac{1}{5}}}, \quad C_2 = \frac{\mathsf{P}}{P_4}, \tag{5.28}$$

where P is an upper bound for the absolute values of z_2 in $\mathfrak{B}^*$ and P_4 is the $P_4(F)$ belonging to F in 4. ($\mathfrak{B}^*$ is bounded, hence P is finite). From (5.28) it follows that for the points $\{z_1, z_2\}$ of $\mathfrak{B}^*$ that satisfy $|z_1| \le l_{2n} \equiv \frac{1}{n^{\frac{2}{5}}}$,

$$\sigma - AP_2|z_1|^{\frac{1}{2}} \ge \sigma_{2n} + BP_2|z_1|^{\frac{1}{2}}$$

holds, and in connection with (5.15)

$$\sigma - A|z_2| \ge \sigma_{2n} + B|z_2|, \tag{5.29}$$

from which, using (5.5) and the fact that $a - A|y_1| > \sigma - 2\alpha_2 + B|y_1|$ for sufficiently large n, it follows that

$$\begin{aligned} 2x_1 - ay_1^2 - \sigma|z_2|^2 + \psi_3(y_1, z_2, \overline{z}_2) &\le 2x_1 - ay_1^2 - \sigma|z_2|^2 + A|y_1|^3 + A|z_2|^3 \\ &\le 2x_1 - (\sigma_{2n} - 2\alpha_2)y_1^2 - \sigma_{2n}|z_2|^2 - B|y_1|^3 - B|z_2|^3 \\ &\le 2x_1 - (\sigma_{2n} - 2\alpha_2)y_1^2 - \sigma_{2n}|z_2|^2 + \varphi_3(y_1, z_2, \overline{z}_2) \end{aligned} \tag{5.30}$$

holds, that is, every point $\{z_1, z_2\}$ of $\mathfrak{B}^*$ with $|z_1| \le l_{2n}$ lies in $\mathfrak{K}_{\alpha_2 n}$.

Now it remains to show that for the choice of

$$\beta_{2n} = \frac{\mathsf{P}}{P_4} n^{\frac{4}{5}} \tag{5.31}$$

in (5.6) and (5.28) the inequality (5.30*) of K is satisfied. This inequality is a sufficient condition to ensure that the part of $\mathfrak{B}^*$ outside of the bicylinder[48)] $|z_1| \leq l_{2n}$ is also contained in $\mathfrak{A}_n$.[49)] But (5.30*) is satisfied, since the lower bound r of the absolute values of the z_2-coordinates of the boundary points $\{z_1, z_2\}$ of $\mathfrak{k}^3_{\alpha_2 n}$ whose z_1-coordinates belong to $\mathfrak{B}^2_{2n}$ satisfies by (5.21)

$$r \geq P_4 l_{2n} \geq \frac{P_4}{n^{\frac{2}{5}}}. \tag{5.32}$$

From (5.31), (5.32) follows

$$r|1 + \beta_{2n} z_1| \geq r|\beta_{2n} z_1| \geq \frac{P_4}{n^{\frac{2}{5}}} \cdot \frac{\mathsf{P}}{P_4} n^{\frac{4}{5}} \cdot \frac{1}{n^{\frac{2}{5}}} = \mathsf{P}, \tag{5.33}$$

which finally proves (5.8).

Now we turn to establishing the kernel functions of $\mathfrak{J}_n$ and $\mathfrak{A}_n$, and the minimal function of $\mathfrak{J}_n$, and moreover the investigation of its behavior under an approach to Q_3. By applying (1.15*), (4.20*), (2.13*), we obtain

$$\begin{aligned}&\frac{1}{\mathsf{K}_{\mathfrak{J}_n}(z_1, z_2; \bar{t}_1, \bar{t}_2)}\\ &= \frac{\pi^2}{2\sigma_{1n}} \frac{\left(z_1\bar{t}_1 - (2\alpha_1 + \sigma_{1n}) z_1\bar{t}_1 - \sigma_{1n} z_2\bar{t}_2 (1 + \beta_{1n} z_1)(1 + \beta_{1n}\bar{t}_1)(1 - \alpha_1 z_1)(1 - \alpha_1\bar{t}_1)\right)^3}{(1 - \alpha_1 z_1)(1 + \beta_{1n} z_1)(1 - \alpha_1\bar{t}_1)(1 + \beta_{1n}\bar{t}_1)}\end{aligned} \tag{5.34}$$

If we now set[50)]

$$z_k = \frac{Z_k}{n^p}, \quad t_k = \frac{T_k}{n}, \tag{5.35}$$

then by (5.5) and (5.6),

$$\frac{1}{\mathsf{K}_{\mathfrak{J}_n}\left(\frac{Z_1}{n^p}, \frac{Z_2}{n^p}, \frac{\overline{T}_1}{n}, \frac{\overline{T}_2}{n}\right)} = \frac{\pi^2}{2\left(\sigma + \frac{D_1}{n^{\frac{1}{5}}}\right) n^{3p}} \left(Z_1 + \frac{\overline{T}_1}{n^{1-p}}\right)^3 \frac{\left(1 - \frac{1}{n} H_1(n, Z_1, Z_2; \overline{T}_1, \overline{T}_2)\right)^3}{1 - \frac{1}{n^{p-\frac{4}{5}}} H_2(n, Z_1; \overline{T}_1)}, \tag{5.36}$$

where

$$\begin{aligned}&H_1(n, Z_1, Z_2; \overline{T}_1, \overline{T}_2)\\ &= \frac{\left(2\alpha_1 + \sigma + \frac{D_1}{n^{\frac{1}{5}}}\right) Z_1\overline{T}_1 + \left(\sigma + \frac{D_1}{n^{\frac{1}{5}}}\right) Z_2\overline{T}_2 \left(1 + \frac{C Z_1}{n^{p-\frac{4}{5}}}\right)\left(1 + \frac{C_1\overline{T}_1}{n^{\frac{1}{5}}}\right)\left(1 - \frac{\alpha_1\overline{T}_1}{n}\right)}{Z_1 + \frac{\overline{T}_1}{n^{1-p}}},\end{aligned}$$

[48)]See p. 93 for the meaning of l_{2n}.
[49)]Compare part I, p. 31.
[50)]Recall that Z_1, Z_2 and T_1, T_2 denote coordinates in $\mathfrak{M}$.

$$
\begin{aligned}
&H_2(n, Z_1; \overline{T}_1)\\
&= \frac{\alpha_1 Z_1}{n^{\frac{4}{5}}} + C_1 Z_1 - \frac{\alpha_1 \overline{T}_1}{n^{\frac{4}{5}-p}} + \frac{C_1 \overline{T}_1}{n^{1-p}} - \frac{\alpha_1 C_1 Z_1^2}{n^p} + \frac{\alpha_1^2 Z_1 \overline{T}_1}{n^{\frac{4}{5}}} - \frac{\alpha_1 C_1 \overline{T}_1 Z_1}{n} - \frac{\alpha_1 C_1 Z_1 \overline{T}_1}{n}.
\end{aligned}
$$

Since $\mathfrak{M}$ is a bounded domain, and by (3.3), $Z_1 + \overline{Z}_1 > 2c$, both H_1 and H_2 are uniformly bounded for every $n > 1$ and all $\{Z_1, Z_2\} \in \mathfrak{M}$, $\{T_1, T_2\} \in \mathfrak{M}$. Since by (1.10*) and (1.11*)

$$
\begin{aligned}
\mathsf{M}_{\mathfrak{Z}_n}(z_1, z_2; t_1, t_2) &= \frac{\mathsf{K}_{\mathfrak{Z}_n}(z_1, z_2; \bar{t}_1, \bar{t}_2)}{\mathsf{K}_{\mathfrak{Z}_n}(t_1, t_2; \bar{t}_1, \bar{t}_2)},\\
\int_{\mathfrak{Z}_n} |M_{\mathfrak{Z}_n}(z_1, z_2; \bar{t}_1, \bar{t}_2)|^2 \mathrm{d}\omega_z &= \frac{1}{\mathsf{K}_{\mathfrak{Z}_n}(t_1, t_2; \bar{t}_1, \bar{t}_2)},
\end{aligned}
\tag{5.37}
$$

we obtain from (5.36) the result stated on p. 83.

In the outer comparison domain, we are only interested in the limit expression for the kernel function. By (1.15*) and (2.14*), we have

$$
\begin{aligned}
&\frac{1}{\mathsf{K}_{\mathfrak{A}_n}(t_1, t_2; \bar{t}_1, \bar{t}_2)}\\
&\quad = \frac{\pi^2}{2\sigma_{2n}} \frac{\left((t_1 + \bar{t}_1 = (\sigma_{2n} - 2\alpha_2)|t_1|^2)(|1 + \beta_{2n} t_1|^2) - \sigma_{2n}|t_2|^2|1 + \alpha_2 t_1|^2\right)^3}{|1 + \alpha_2 t_1|^2 |1 + \beta_{2n} t_1|^4}.
\end{aligned}
\tag{5.38}
$$

If we set $t_k = \frac{T_k}{n}$, then, considering (5.5), (5.6) and (5.28), we obtain

$$
\frac{1}{\mathsf{K}_{\mathfrak{A}_n}\left(\frac{T_1}{n}, \frac{T_2}{n}; \frac{\overline{T}_1}{n}, \frac{\overline{T}_2}{n}\right)} = \frac{\pi^2}{2\left(\sigma - \frac{D_2}{n^{\frac{1}{5}}}\right) n^3} (T_1 + \overline{T}_1)^3 \frac{\left(1 + \frac{1}{n^{\frac{1}{5}}} L_1(n, T_1, T_2; \overline{T}_1, \overline{T}_2)\right)}{1 + \frac{1}{n^{\frac{1}{5}}} L_2(n, T_1; \overline{T}_1)},
\tag{5.39}
$$

where

$$
\begin{aligned}
&L_1(n, T_1, T_2; \overline{T}_1, \overline{T}_2)\\
&= \frac{(C_2 T_1 + \overline{C_2 T_1})(T_1 + \overline{T}_1) \frac{1}{n^{\frac{1}{5}}} C_2^2 |T_1|^2 (T_1 + \overline{T}_1) + \frac{1}{n^{\frac{4}{5}}} \left(\sigma - \frac{D_2}{n^{\frac{1}{5}}} - 2\alpha_2\right) |T_1|^2 \left|1 + \frac{C_2 T_1}{n^{\frac{1}{5}}}\right|^2 - \left(\sigma - \frac{D_2}{n^{\frac{1}{5}}}\right) \frac{|T_2|^2}{n^{\frac{4}{5}}} \left|1 + \alpha_2 \frac{T_1}{n}\right|^2}{T_1 + \overline{T}_1},
\end{aligned}
$$

and

$$
L_2(n, T_1; \overline{T}_1) = 2C_2(T_1 + \overline{T}_1) + \frac{1}{n^{\frac{1}{5}}} C_2^2((T_1 + \overline{T}_1)^2 + 2|T_1|^2) + \ldots + \frac{1}{n^{\frac{13}{5}}} C_2^4 \alpha_2^2 |T_1|^6.
$$

This implies

$$\frac{1}{\mathsf{K}_{\mathfrak{A}_n}\left(\frac{T_1}{n},\frac{T_2}{n};\frac{\overline{T}_1}{n},\frac{\overline{T}_2}{n}\right)}=\frac{\pi^2}{2\sigma n^3}(T_1+\overline{T}_1)^3\left(1+\frac{1}{n^{\frac{1}{5}}}L_1(n,T_1,T_2;\overline{T}_1,\overline{T}_2)\right), \tag{5.40}$$

where L is a uniformly bounded function in n, T_1, T_2 for $n > 1$ and $\{T_1, T_2\} \in \mathfrak{M}$.

III. By (2.12) and (2.13), the inner comparison domain $\mathfrak{I}_n$ is given by

$$\frac{M(z_1,z_2;\overline{z}_1,\overline{z}_2)}{|1-\alpha_1 z_1|^2}$$
$$\equiv\frac{1}{|1-\alpha_1 z_1|^2}\big(z_1+\overline{z}_1-(\sigma_{1n}+2\alpha_1)|z_1|^2-\sigma_{1n}|z_2|^2|1+\beta_{1n}z_1|^2|1-\alpha_1 z_1|^2\big)>0. \tag{5.41}$$

Let $\{Z_1, Z_2\}$ be any point in $\mathfrak{M}$ and let $p > \frac{4}{5}$. There exists a positive τ, independent of $\{Z_1, Z_2\}$, such that for all $n > n_0(p)$ the bicylinder

$$\mathfrak{C}\left(\frac{Z_1}{n^p},\frac{Z_2}{n^p}\right):\quad \text{(a)}\left|\zeta_1-\frac{Z_1}{n^p}\right|\le\frac{\tau}{n^p},\quad \text{(b)}\left|\zeta_2-\frac{Z_2}{n^p}\right|\le\frac{\tau}{n^{\frac{p}{2}}} \tag{5.42}$$

with center $\{\frac{Z_1}{n^p}, \frac{Z_2}{n^p}\}$ lies in $\mathfrak{I}_n$. (Compare Figure 1.)

PROOF: $(1-\alpha_1\frac{Z_1}{n^p})$ clearly will not vanish at all $\{Z_1, Z_2\} \in \mathfrak{M}$ for sufficiently large n. So it is enough to show that for $\{\zeta_1, \zeta_2\}$ defined by (5.42), $M > 0$ holds. We choose positive τ and ε' small enough such that

$$2\tau+\sigma_{1n}(\tau+\varepsilon')^2<\frac{c}{2}, \tag{5.43}$$

where c is the constant in (3.5). From (5.42) (a) it follows that

$$|\zeta_1|\le\frac{\tau+|Z_1|}{n^p}\le\frac{K}{n^p},\quad K=\tau+\frac{C}{\cos(\alpha)} \tag{5.44}$$

and

$$\left|\operatorname{Re}\left(\zeta_1-\frac{Z_1}{n^p}\right)\right|\le\frac{\tau}{n^p},\qquad \text{that is, } \zeta_1+\overline{\zeta}_1\ge\frac{Z_1+\overline{Z}_1-2\tau}{n^p}. \tag{5.45}$$

For the meaning of C and α, see (3.5) and (2.7). From (5.42) (b) it follows for sufficiently large n that

$$|\zeta_2|\le\frac{\tau+|Z_2|n^{-\frac{p}{2}}}{n^{\frac{p}{2}}}\le\frac{\tau+\varepsilon'}{n^{\frac{p}{2}}}. \tag{5.46}$$

Hence

$$\begin{aligned} M(\zeta_1, \zeta_2; \overline{\zeta}_1, \overline{\zeta}_2) \geq & \frac{Z_1 + \overline{Z}_1 - 2\tau}{n^p} - \left(\sigma + \frac{D}{n^{\frac{1}{5}}} + 2\alpha_1\right) \frac{K^2}{n^{2p}} \\ & - \sigma_{1n} \frac{(\tau + \varepsilon')^2}{n^p} \left(1 + \frac{C_1 K}{n^{p-\frac{4}{5}}}\right)^2 \left(1 + \alpha_1 \frac{K}{n^p}\right)^2 \\ = & \frac{1}{n^p} \left(Z_1 + \overline{Z}_1 - 2\tau - \sigma_{1n}(\tau + \varepsilon')^2 - \frac{1}{n^{p-\frac{4}{5}}} G(n)\right), \end{aligned} \tag{5.47}$$

where

$$G(n) = \left(-\left(\sigma + \frac{D_1}{n^{\frac{1}{5}}} + 2\alpha_1\right) \frac{K^2}{n^{\frac{4}{5}}} + \sigma_{1n}(\tau + \varepsilon')^2 (2S + S^2)\right) \leq \Gamma < \infty,$$

and

$$S = C_1 K + \frac{\alpha_1 K}{n^{\frac{4}{5}}} + \frac{C_1 K^2}{n^p}.$$

If we choose

$$n > n_0 \equiv \max\left(\left(\frac{K}{\varepsilon'}\right)^{\frac{2}{p}}, \left(\frac{2\Gamma}{c}\right)^{\frac{5}{5p-4}}, (\alpha_1 K)^{\frac{1}{p}}\right),$$

then it follows from (5.47) and (3.3) that

$$M(\zeta_1, \zeta_2; \overline{\zeta}_1, \overline{\zeta}_2) \geq \frac{c}{n^p} > 0, \tag{5.48}$$

which proves the claim.

IV. In the proof of Theorem III we used the following

Lemma II. *Let $n(z_1, z_2)$ be a regular and square-integrable function on $\mathfrak{G}$ that vanishes at the point $\{t_1, t_2\} \in \mathfrak{G}$, that is,*

$$n(t_1, t_2) = 0 \tag{5.49}$$

holds. Then $n(z_1, z_2)$ is orthogonal to the minimal function $\mathsf{M}_{\mathfrak{G}}(z_1, z_2; t_1, t_2)$ of $\mathfrak{G}$ with base point in $\{t_1, t_2\}$, that is,

$$\int_{\mathfrak{G}} n(z_1, z_2) \overline{\mathsf{M}_{\mathfrak{G}}(z_1, z_2; t_1, t_2)} \mathrm{d}\omega_z = 0 \tag{5.50}$$

holds.

PROOF: By definition, the integral $\int_{\mathfrak{G}} |f(z_1, z_2)|^2 \mathrm{d}\omega$ assumes its minimal value at the minimal function, namely the value $\frac{1}{\mathsf{K}_{\mathfrak{G}}(t_1,t_2;\overline{t}_1,\overline{t}_2)}$. Here, all regular and square-integrable functions $h(z_1, z_2)$ in $\mathfrak{G}$ are considered that satisfy the condition

$$h(t_1, t_2) = 1. \tag{5.51}$$

The function

$$m(z_1, z_2) = \mathsf{M}_{\mathfrak{G}}(z_1, z_2; t_1, t_2) + n(z_1, z_2)\frac{\int_{\mathfrak{G}} n\overline{\mathsf{M}}_{\mathfrak{G}} \mathrm{d}\omega}{\int_{\mathfrak{G}} |n|^2 \mathrm{d}\omega} \tag{5.52}$$

satisfies the condition (5.51) by (5.49), and is square-integrable by the Schwartz inequality. On the other hand

$$\int_{\mathfrak{G}} |m|^2 \mathrm{d}\omega = \int_{\mathfrak{G}} |\mathsf{M}_{\mathfrak{G}}|^2 \mathrm{d}\omega - \frac{\left|\int_{\mathfrak{G}} n\overline{\mathsf{M}}_{\mathfrak{G}} \mathrm{d}\omega\right|^2}{\int_{\mathfrak{G}} |n|^2 \mathrm{d}\omega} = \frac{1}{\mathsf{K}_{\mathfrak{G}}(t_1, t_2; \overline{t}_1, \overline{t}_2)} - \frac{\left|\int_{\mathfrak{G}} n\overline{\mathsf{M}}_{\mathfrak{G}} \mathrm{d}\omega\right|^2}{\int_{\mathfrak{G}} |n|^2 \mathrm{d}\omega}. \tag{5.53}$$

If (5.50) was not satisfied, then $\int_{\mathfrak{G}} |m|^2 \mathrm{d}\omega < \frac{1}{\mathsf{K}_{\mathfrak{G}}(t_1,t_2;\overline{t}_1,\overline{t}_2)}$ would hold, which contradicts the minimality property of the minimal function.

By (4.2), (4.6), (4.8), Lemma II and (4.1),

$$\begin{aligned} &\int_{\mathfrak{F}_n} \left| h\left(z_1, z_2; \frac{T_1}{n}, \frac{T_2}{n}\right)\right|^2 \mathrm{d}\omega_z \\ &= \frac{\pi^2}{2\sigma} \frac{(T_1 + \overline{T}_1)^3}{n^3} \left(\frac{C(T_1 + \overline{T}_1)^r}{n^r} - \frac{B(n, T_1, T_2; \overline{T}_1, \overline{T}_2)^{\frac{1}{5}}}{n}\right) \leq \frac{c_1}{n^{3+\varrho}}, \end{aligned} \tag{4.9}$$

where $\varrho = \min(r, \frac{1}{5})$, $c_1 \leq \frac{\pi^2}{\sigma} 4C^3 \left(\frac{C(2C)^r}{n^{r-\varrho}} + \frac{B}{n^{\frac{1}{5}-\varrho}}\right)$ is a constant independent of n, B is an upper bound for the function $B(n, T_1, T_2; \overline{T}_1, \overline{T}_2)$, $n > 1$, $\{T_1, T_2\} \in \mathfrak{M}$ and C is the constant given in (3.5).

§ 6

In the present paragraph, we study the behavior of a certain family of functions in a neighborhood of a limit point Q_2 of second order. We begin with some preliminary remarks.

By a *p-coupled* A^{V}*-approach* of a pair of points $\{z_1, z_2, t_1, t_2\}$, $z_k = x_k + \mathrm{i}y_k$, $t_k = u_k + \mathrm{i}v_k$, to $\{0, a_2, 0, b_2\}$ we again mean an approach under which the points

$\{z_1, z_2\}$ and $\{t_1, t_2\}$ converge to $\{0, a_2\}$ and $\{0, b_2\}$ in the sense of A^{V}, respectively, and in addition inequality (3.3) holds for the coordinates x_1 and u_1. As before, it is helpful to present another interpretation of a p-coupled sequence of pairs of points $\{z_1^{(\mu)}, z_2^{(\mu)}, t_1^{(\mu)}, t_2^{(\mu)}\}$.

Let $\mathfrak{U}_2^2$ denote a simply connected domain contained completely in the interior of $\mathfrak{H}^2$ that contains the point $z_2 = 0$, and let $\mathfrak{U}_1^2$ denote a subdomain of $\mathfrak{W}_\alpha^2$ for which (3.5) holds as well as $\frac{|z_1|}{x_1} < \frac{1}{\cos(\alpha)}$, where c and C, with $C > c$, are arbitrary constants subject to the condition that $\mathfrak{W}_{C\alpha}^2$ lies in $\mathfrak{J}^2$ and $\mathfrak{U}_2^2 \times \mathfrak{W}_{C\alpha}^2$ lies in $\mathfrak{B}$.[51)] See pp. 74 for the meaning of $\mathfrak{W}_\alpha^2$ and $\mathfrak{J}_n^2$. The product domain $\mathfrak{U}_1^2 \times \mathfrak{U}_2^2$ is denoted by $\mathfrak{U}$, the coordinates of its points by capital letters Z_1, Z_2 and T_1, T_2, respectively. Now, given a sequence of points $\{Z_1^{(\mu)}, Z_2^{(\mu)}, T_1^{(\mu)}, T_2^{(\mu)}\}$ with $\lim_{\mu\to\infty} Z_2^{(\mu)} = a_2$, $\lim_{\mu\to\infty} T_2^{(\mu)} = b_2$, where $\lim_{\mu\to\infty} n_\mu = \infty$, the sequence $\left\{\frac{Z_1^{(\mu)}}{n_\mu^p}, Z_2^{(\mu)}, \frac{T_1^{(\mu)}}{n_\mu^p}, T_2^{(\mu)}\right\}$ is a p-coupled A^{V}-sequence of points, since

$$\frac{c^p}{C} < \frac{\left(\frac{U_1^{(\mu)}}{n_\mu}\right)^p}{\frac{X_1^{(\mu)}}{n_\mu^p}} < \frac{C^p}{c}$$

holds.

Conversely, if a p-coupled sequence $\{z_1^{(\mu)}, z_2^{(\mu)}, t_1^{(\mu)}, t_2^{(\mu)}\}$ is given, we again set $n_\mu = \frac{d}{u_1^{(\mu)}}$, where $d > 0$ and is sufficiently small, then similar to p. 83, $c = \min\left(d, \frac{d^p}{M}\right)$, $C = \max\left(d, \frac{d^p}{m}\right)$. If we choose $\mathfrak{U}_2^2$ as a domain containing the points a_2 and b_2 in its interior, then by (2.38) we can choose d small enough such that $\mathfrak{W}_{C\alpha}^2 \times \mathfrak{U}_2^2$ lies in $\mathfrak{B}$.

In the investigation of the limit points of second order, the parameter ν appeared (the auxiliary function $f_\nu(z)$ introduced in §2 depends on ν). This parameter satisfies the inequalities (6.28*), (6.17*), (6.23*), (6.29*), (6.30*), (6.31*), by which it is bounded from below. The quantities appearing in these inequalities depend in a complicated manner on the structure of the boundary at Q_2. The lower of the quantities satisfying the above inequalities, that is the lower bound of the admissible values of the parameter ν, is denoted by ν_0. With regard to the continuing investigation, the parameter ν must moreover satisfy the inequality

$$a\cos\left(\frac{\alpha}{\nu}\right) - 4B\sin\left(\frac{\alpha}{\nu}\right) > 0, \tag{6.1}$$

51) From (2.38) one easily deduces that for every closed $\mathfrak{U}_2^2$ we can find a sufficiently small C such that the domain $\mathfrak{U}_2^2 \times \mathfrak{W}_{C\alpha}^2$ lies in $\mathfrak{B}$.

where a, B are the constants appearing in condition 1. (p. 74) and α is the angular aperture of $\mathfrak{W}_\alpha^2$ appearing in (2.40). By $\tau(\alpha)$ we denote the lower bound of the quantities given by (6.1).

Theorem IV. *Assign to each point $\{t_1, t_2\}$ of an A^{V}-convergent sequence $\mathfrak{P}^0$ of points with limit $Q_2(0, b_2)$, $b_2 \in \mathfrak{S}^2$, a regular and square-integrable function $f(z_1, z_2; t_1, t_2)$ on $\mathfrak{B}$ in the complex variables z_1, z_2. Every function of this family shall have the following two properties:*

1. *We have*
$$f(t_1, t_2; t_1, t_2) = 1, \tag{6.2}$$

2. *and the integrals $\int_{\mathfrak{B}} |f(z_1, z_2; t_1, t_2)|^2 \mathrm{d}\omega_z$ satisfy for $\{t_1, t_2\} \in \mathfrak{P}^0$ the inequality*
$$\int_{\mathfrak{B}} |f(z_1, z_2; t_1, t_2)|^2 \mathrm{d}\omega_z \leq \pi^2 (t_1 + \bar{t}_1)^2 \left(\mathsf{P}(t_2)^2 + C(t_1 + \bar{t}_1)^{\frac{1}{\nu^*}}\right) \tag{6.3}$$
where $C < \infty$, $\nu^* > 0$ are fixed constants independent of $\{t_1, t_2\}$.

Then for every p with $1 - \frac{1}{2\nu} < p \leq 1$ under a p-coupled A^{V}-approach of the pair of points $\{z_1, z_2, t_1, t_2\}$ we have uniformly

$$\lim \frac{z_1^2}{(t_1 + \bar{t}_1)^2} f(z_1, z_2; t_1, t_2) = \mathsf{M}_{\mathfrak{S}^2}(a_2; b_2) \quad \text{for } p < 1, \tag{6.4}$$

$$\lim \frac{(z_1 + \bar{t}_1)^2}{(t_1 + \bar{t}_1)^2} f(z_1, z_2; t_1, t_2) = \mathsf{M}_{\mathfrak{S}^2}(a_2; b_2) \quad \text{for } p = 1, \tag{6.5}$$

where $\mathsf{M}_{\mathfrak{S}^2}(z; t)$ denotes the minimal function of $\mathfrak{S}^2$ with base point in $\{t\}$, and $\nu = \max(\nu_0, \nu^*, \tau(\alpha))$.[52)]

The proof of our theorem is based on the same idea as in the case of a limit point of third order. As the situation is slightly different here, the arguments shall be worked out in detail.

I. In §2 we introduced the inner comparison domain $\mathfrak{I}_\nu$. By (2.47a), the reciprocal kernel function of $\mathfrak{I}_\nu$ assumes the value

$$\frac{1}{\mathsf{K}_{\mathfrak{I}_\nu}(z_1, z_2; t_1, t_2)} = \frac{\pi^2 (z_1 + \bar{t}_1 + \varrho z_1 \bar{t}_1)^2 \left(1 - w\left(\frac{z_2}{f_\nu(z_1)}\right) \overline{w\left(\frac{t_2}{f_\nu(t_1)}\right)}\right)^2 f_\nu(z_1) \overline{f_\nu(t_1)}}{w'\left(\frac{z_2}{f_\nu(z_1)}\right) \overline{w'\left(\frac{t_2}{f_\nu(t_1)}\right)}} \tag{6.6}$$

[52)]Note that (6.3) remains true if ν is increased (we have $(t_1 + \bar{t}_1)^{\frac{1}{\nu}} < (t_1 + \bar{t}_1)^{\frac{1}{\nu'}}$ if $\nu < \nu'$).

By (2.46), (6.6) equals

$$
\begin{aligned}
&\frac{\pi^2(z_1+\bar t_1)^2\left(1+\frac{\varrho z_1\bar t_1}{z_1+\bar t_1}\right)^2\left(1-\left(w(z_2)+z_1^{\frac{1}{\nu}}K_1(z_1,z_2)\right)\left(\overline{w(t_2)}+\overline{t_1^{\frac{1}{\nu}}K_1(t_1,t_2)}\right)\right)^2\left(1+z_1^{\frac{1}{\nu}}P_1(z_1)\right)\left(1+\overline{t_1^{\frac{1}{\nu}}P_1(t_1)}\right)}{\left(w'(z_2)+z_1^{\frac{1}{\nu}}K_2(z_1,z_2)\right)\left(\overline{w'(t_2)}+\overline{t_1^{\frac{1}{\nu}}K_2(t_1,t_2)}\right)}\\
&=\frac{\pi^2(z_1+\bar t_1)^2\left(1+\frac{\varrho z_1\bar t_1}{z_1+\bar t_1}\right)^2(1-w(z_2)\overline{w(t_2)}+z_1^{\frac{1}{\nu}}B_5(z_1,z_2;\bar t_1,\bar t_2)+\bar t_1^{\frac{1}{\nu}}B_6(z_1,z_2;\bar t_1,\bar t_2))^2(1-z_1^{\frac{1}{\nu}}B_7(z_1,\bar t_1)-\bar t_1^{\frac{1}{\nu}}B_8(z_1,\bar t_1))}{w'(z_2)\overline{2'(t_2)}\left(1+z_1^{\frac{1}{\nu}}\frac{K_2(z_1,z_2)}{w'(z_2)}\right)\left(1+\bar t_1^{\frac{1}{\nu}}\frac{K_2(\bar t_1,\bar t_2)}{\overline{w'(t_2)}}\right)}\\
&=\frac{\pi^2(z_1+\bar t_1)^2(1-w(z_2)\overline{2(t_2)})^2}{w'(z_2)\overline{w'(t_2)}}\left(1+z_1^{\frac{1}{\nu}}B_9(z_1,z_2;\bar t_1,\bar t_2)+\bar t_1^{\frac{1}{\nu}}B_{10}(z_1,z_2;\bar t_1,\bar t_2)\right).
\end{aligned}
\tag{6.7}
$$

Since $|w_2(z_2)| \le q_1 < 1$, $w'(z_2) \ge q_2 > 0$ and $|\frac{z_1}{z_1+\bar t_1}|$ are bounded for $z_1, t_1 \in \mathfrak{W}^2_{C\alpha}$, the functions $B_k(z_1,z_2;\bar t_1,\bar t_2)$ on $\{z_1,z_2\}$, $\{t_1,t_2\}$ is uniformly bounded on $\mathfrak{W}^2_{C\alpha}\times\mathfrak{U}^2_2$ by (2.47) and (2.44).

For $\frac{1}{\mathsf{K}_{\mathfrak{F}_\nu}(z_1,z_2;t_1,t_2)}$ we need a sharper estimate. Using the abbreviations introduced on p. 77, $w^{(k)}(X)\equiv\frac{\mathrm{d}^k w(X)}{\mathrm{d}X}$ and applying the Mean Value Theorem, we write

$$
w^{(k)}\left(\frac{t_2}{f_\nu(t_1)}\right)=w^{(k)}(t_2)+t_1^{\frac{1}{\nu}}(t_2)A_\nu t_2+t_1^{\frac{2}{\nu}}K_{2+k}(t_1,t_2),\tag{6.8}
$$

where

$$
\begin{aligned}
&K_{k+2}(t_1,t_2)\\
&=\frac{1}{2}w^{(k+2)}\left(\frac{t_2}{f_\nu(t_1)}\right)\frac{t_2^2 f_\nu^{(1)}(\vartheta t_1)^2}{f_\nu(\vartheta t_1)^4}+2w^{(k+1)}\left(\frac{t_2}{f_\nu(\vartheta t_1)}\right)\frac{t_2 t_\nu^{(1)}(\vartheta t_1)^2}{f_\nu(\vartheta t_1)^3}-w^{(k+1)}\left(\frac{t_2}{f_\nu(\vartheta t_1)}\right)\frac{t_2 f_\nu^{(2)}(\vartheta t_1)}{f_\nu(\vartheta t_1)^2},
\end{aligned}
\tag{6.9}
$$

with

$$
0<\vartheta<1,\quad f_\nu^{(\kappa)}=\frac{\mathrm{d}^\kappa f_\nu}{\mathrm{d}(t_1^{\frac{1}{\nu}})^\kappa},\quad \kappa=1,2.
$$

Hence:

$$
\begin{aligned}
&\frac{1}{\mathsf{K}_{\mathfrak{F}_\nu}(t_1,t_2;\bar t_1,\bar t_2)}\\
&=\frac{\pi^2(t_1+\bar t_1)^2\left(1+\frac{\varrho|t_1|^2}{t_1+\bar t_1}\right)^2\left(1-|w(t_2)+t_1^{\frac{1}{\nu}}w^{(1)}(t_2)A_\nu t_2+t_1^{\frac{2}{\nu}}K_3(t_1,t_2)|^2\right)^2\left|1-A_\nu t_1^{\frac{1}{\nu}}+t_1^{\frac{2}{\nu}}G_4(t_1)\right|^2}{\left|w^{(1)}(t_2)+t_1^{\frac{1}{\nu}}A_2t_2w^{(2)}(t_2)+t_1^{\frac{2}{\nu}}K_4(t_1,t_2)\right|^2}\\
&=\frac{\pi^2(t_1+\bar t_1)^2\left(1-|w(t_2)|^2-2\operatorname{Re}\left(t_1^{\frac{1}{\nu}}A_\nu t_2w^{(1)}(t_2)\overline{w(t_2)}\right)+t_1^{\frac{2}{\nu}}B_{11}(t_1,t_2)+\overline{t_1^{\frac{2}{\nu}}B_{12}(t_1,t_2)}\right)^2\left(1+\frac{\varrho|t_1|^2}{t_1+\bar t_1}\right)^2\left(1-2\operatorname{Re}\left(A_\nu t_1^{\frac{1}{\nu}}\right)+t_1^{\frac{2}{\nu}}G_5(t_1)+t_1^{\frac{2}{\nu}}G_6(t_1)+|t_1|^{\frac{2}{\nu}}|A_\nu|^2\right)}{|w^{(1)}(t_2)|^2\left(1+2\operatorname{Re}\left(\frac{t_1^{\frac{1}{\nu}}A_\nu t_2w^{(2)}(t_2)}{w^{(1)}(t_2)}\right)+t_1^{\frac{2}{\nu}}B_{13}(t_1,t_2)+\overline{t_1^{\frac{2}{\nu}}B_{14}(t_1,t_2)}+|t_1|^{\frac{2}{\nu}}|t_2w^{(2)}(t_2)|^2\right)}\\
&=\frac{\pi^2(t_1+\bar t_1)^2(1-|w(t_2)|^2)^2}{w^{(1)}(t_2)|^2}\left(1+\frac{\varrho|t_1|^2}{t_1+\bar t_1}\right)^2\left(1-2A_\nu\operatorname{Re}\left(t_1^{\frac{1}{\nu}}\left(\frac{2t_2w^{(1)}(t_2)\overline{w(t_2)}}{1-|w(t_2)|^2}+1+\frac{t_2w^{(2)}(t_2)}{w^{(1)}(t_2)}\right)\right)+t_1^{\frac{1}{\nu}}B_{15}(t_1,t_2)+\bar t_1^{\frac{2}{\nu}}B_{16}(t_1,t_2)+|t_1|^{\frac{2}{\nu}}B_{17}(t_1,t_2)\right)\\
&=\pi^2(t_1+\bar t_1)^2\left(\mathsf{P}(t_2)^2-2\operatorname{Re}\left(A_\nu t_1^{\frac{1}{\nu}}\left(\mathsf{P}(t_2)^2-t_2\frac{\mathrm{d}\mathsf{P}(t_2)^2}{\mathrm{d}t_2}\right)\right)+t_1^{\frac{2}{\nu}}B_{15}(t_1,t_2)+\bar t_1^{\frac{2}{\nu}}B_{16}(t_1,t_2)+|t_1|^{\frac{2}{\nu}}B_{17}(t_1,t_2)\right)\left(1+\frac{\varrho|t_1|^2}{t_1+\bar t_1}\right)^2,
\end{aligned}
\tag{6.10}
$$

where $B_k(z_1, z_2; t_1, t_2)$ are uniformly bounded functions in $\{t_1, t_2\} \in \mathfrak{U}$.

By setting $t_1 = \frac{T_1}{n}$, $t_2 = T_2$, $z_1 = \frac{Z_1}{n}$, $z_2 = Z_2$, we obtain the following result:

For the minimal function and the reciprocal kernel function of the innter comparison domain $\mathfrak{I}_\nu$, we have:

$$\mathsf{M}_{\mathfrak{I}_\nu}\left(\frac{Z_1}{n^p}, Z_2; \frac{T_1}{n}, T_2\right)$$
$$= n^{2p-2} \frac{w^{(1)}(Z_1)(T_1+\overline{T}_1)^2(1-|w(T_2)|^2)^2\left(1+\frac{1}{n^{\frac{1}{\nu}}}B_{17}(n, T_1, T_2; \overline{T}_1, \overline{T}_2)\right)}{\left(Z_1+\frac{\overline{T}_1}{n^{1-p}}\right)^2(1-w(Z_2)\overline{w(T_2)})^2 w^{(1)}(T_2)\left(1+\frac{1}{n^{\frac{p}{\nu}}}B_{18}(n, T_1, T_2; \overline{T}_1, \overline{T}_2)\right)} \tag{6.11}$$

$$\frac{1}{\mathsf{K}_{\mathfrak{I}_\nu}\left(\frac{Z_1}{n^p}, Z_2; \frac{T_1}{n}, T_2\right)}$$
$$= \frac{\pi^2(T_1+\overline{T}_1)^2}{n^2}\left(\mathsf{P}(T_2)^2 - \frac{2A_\nu}{n^{\frac{1}{\nu}}}\operatorname{Re}\left(T_1^{\frac{1}{\nu}}\left(\mathsf{P}(T_2)^2 - T_2\frac{\mathrm{d}\mathsf{P}(T_2)^2}{\mathrm{d}T_2}\right)\right) + \frac{1}{n^{\frac{p}{\nu}}}B_{19}(n, T_1, T_2; \overline{T}_1, \overline{T}_2)\right), \tag{6.12}$$

where $B_{18}(n, Z_1, Z_2; T_1, T_2)$ and $B_k(n, T_1, T_2; \overline{T}_1, \overline{T}_2)$, $k = 17, 19$, for $\{Z_1, Z_2\}$ and $\{T_1, T_2\}$ in $\mathfrak{U}$ and $n > 1$, are uniformly bounded functions in all variables, and $\mathsf{P}(T_2)$ means the mapping radius of $\mathfrak{H}^2$ in the point T_2.

In the corollary in which we obtain the limit formulas for $\mathsf{M}_{\mathfrak{B}}$, we need an estimate for the kernel function of the outer comparison domain introduced in §2. By (2.47a),

$$\frac{1}{\mathsf{K}_{\mathfrak{A}_\nu}(t_1, t_2; \overline{t}_1, \overline{t}_2)} = \frac{\pi^2(t_1+\overline{t}_1)^2(1-|w(t_2 f_\nu(t_1))|^2)^2}{|w^{(1)}(t_2 f_\nu(t_1))|^2 |f_\nu(t_1)|^2}. \tag{6.13}$$

By the first Mean Value Theorem, again

$$w^{(k)}(t_2 f_\nu(t_1)) = w^{(k)}(t_2) - t_1^{\frac{1}{\nu}} A_\nu t_2 w^{(k+1)}(t_2) + t_1^{\frac{2}{\nu}} K_5(t_1, t_2), \tag{6.14}$$

where

$$K_5(t_1, t_2) = w^{(k+2)}(t_2 f_\nu(\vartheta t_1)) t_2^2 (f_\nu^{(1)}(\vartheta t_1))^2 + w^{(k+1)}(t_2 f_\nu(\vartheta t_1)) t_2 f_\nu^{(2)}(\vartheta t_1), \tag{6.15}$$

with $0 < \vartheta < 1$, $f_\nu^{(\kappa)} = \frac{\mathrm{d} f_\nu(t_1)}{\mathrm{d}\left(t_1^{\frac{1}{\nu}}\right)^\kappa}$, is a uniformly bounded function in $\{t_1, t_2\} \in \mathfrak{U}$.

We thus obtain for (6.13)

$$\frac{\pi^2(t_1+\bar t_1)^2\left(1-|w(t_2)|^2+2\,\mathrm{Re}\left(t_1^{\frac{1}{\nu}}A_\nu t_2 w^{(1)}(t_2)\overline{w(t_2)}\right)+t_1^{\frac{2}{\nu}}B_{20}+t_1^{\frac{2}{\nu}}B_{21}\right)^2}{|w^{(1)}(t_1)|^2\left|1-t_1^{\frac{1}{\nu}}\frac{A_\nu t_2 w^{(2)}(t_2)}{w^{(1)}(t_2)}+t_1^{\frac{2}{\nu}}B_{22}\right|^2\left|1-A_\nu t_1^{\frac{1}{\nu}}+t_1^{\frac{2}{\nu}}B_{23}\right|^2}$$
$$=\frac{\pi^2(t_1+\bar t_1)^2(1-|w(t_2)|^2)^2}{|w^{(1)}(t_2)|^2}\left(1+2\,\mathrm{Re}\left(t_1^{\frac{1}{\nu}}A_\nu\left(\frac{2t_2w^{(1)}(t_2)\overline{w(t_2)}}{1-|w(t_2)|^2}+\frac{t_2w^{(2)}(t_2)}{w^{(1)}(t_2)}+1\right)\right)+t_1^{\frac{2}{\nu}}B_{24}+t_1^{\frac{2}{\nu}}B_{25}+|t_1|^{\frac{2}{\nu}}B_{26}\right)$$
$$=\pi^2(t_1+\bar t_1)^2\left(\mathsf{P}(t_2)^2 2\,\mathrm{Re}\left(A_\nu t_1^{\frac{1}{\nu}}\left(\mathsf{P}(t_2)^2-t_2\frac{\mathrm{d}\mathsf{P}(t_2)^2}{\mathrm{d}t_2}\right)\right)+\mathrm{Re}\left(t_1^{\frac{2}{\nu}}B_{26}\right)\right), \tag{6.16}$$

where the B_k are uniformly bounded functions for $\{t_1,t_2\}\in\mathfrak{W}^2_{C\alpha}\times\mathfrak{U}^2_2$.

If once more we set $t_1=\frac{T_1}{n}$, $t_2=\frac{T_2}{n}$, we obtain the result:

For the reciprocal kernel function of the outer comparison domain $\mathfrak{A}_\nu$, we have

$$\frac{1}{\mathsf{K}_{\mathfrak{A}_\nu}\left(\frac{T_1}{n},T_2;\frac{\overline{T}_1}{n},T_2\right)}$$
$$=\frac{\pi^2(T_1+\overline{T}_1)^2}{n^2}\left(\mathsf{P}(T_2)^2+\frac{1}{n^{\frac{1}{\nu}}}\mathrm{Re}\left(A_\nu T_1^{\frac{1}{\nu}}\left(\mathsf{P}(T_2)^2+T_2\frac{\mathrm{d}\mathsf{P}(T_2)^2}{\mathrm{d}T_2}\right)\right)+\frac{1}{n^{2\nu}}B_{27}(n,T_1,T_2;\overline{T}_1,\overline{T}_2)\right), \tag{6.17}$$

where $B_{27}(n,T_1,T_2;\overline{T}_1,\overline{T}_2)$ is a uniformly bounded function for $n>1$ and $\{T_1,T_2\}\in\mathfrak{U}$. Then there exists a τ independent of $\{Z_1,Z_2\}$ and n, such that for all $n>n_0$, the bicylinder

$$\mathfrak{E}\left(\frac{Z_1}{n},Z_2\right):\quad\left|\zeta_1-\frac{Z_1}{n^p}\right|<\frac{\tau}{n^p},\quad|\zeta_2-Z_2|<\tau \tag{6.18}$$

with center in $\{\frac{Z_1}{n^p},Z_2\}$ lies in $\mathfrak{I}_\nu$.

PROOF: $\mathfrak{I}_\nu(z_1=\gamma)$ is a star-shaped domain whose boundary is given by the equation

$$R=h^*(\gamma,\theta)\equiv h(\theta-\vartheta_\nu(\gamma))|f_\nu(\gamma)|, \tag{6.19}$$

$\gamma=r\mathrm{e}^{\mathrm{i}\varphi}$, $\vartheta_\nu(\gamma)=\mathrm{arc}(f_\nu(\gamma))$ (6.46*). We now wish to show that there exists an $r_0=r_0(\alpha,\nu)$ independent of θ and φ, such that for $r<\min(r_0,\frac{2}{\varrho}\cos(\alpha))$,

$$\frac{\mathrm{d}h^*(\gamma,\theta)}{\mathrm{d}r}<0,\quad\gamma=r\mathrm{e}^{\mathrm{i}\varphi},0\le\theta\le2\pi \tag{6.20}$$

holds.

We have

$$\frac{\mathrm{d}h^*(\gamma,\theta)}{\mathrm{d}r}=h(\theta-\vartheta_\nu(\gamma))\frac{\mathrm{d}|f_\nu(\gamma)|}{\mathrm{d}r}-h'(\theta-\vartheta_\nu(\gamma))\frac{\mathrm{d}\vartheta_\nu(\gamma)}{\mathrm{d}r}|f_\nu(\gamma)|. \tag{6.21}$$

We will now obtain some estimates for $\frac{\mathrm{d}|f_\nu(\gamma)|}{\mathrm{d}r}$ and $\frac{\mathrm{d}\vartheta_\nu(\gamma)}{\mathrm{d}r}$, where we at first assume that $r \le (\frac{1}{2})^{2\nu}$.

By the formulas derived on p. 619 and p. 620 in article A and by (2.44),

$$f_\nu(z) = 1 - A_\nu z^{\frac{1}{\nu}} - G_4(z), \tag{6.22}$$

where

$$G_4(z) = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)\Gamma\left(\frac{1}{2} - \frac{1}{2\nu}\right)} z^{\frac{1}{\nu}} \left(\sum_{\kappa=1}^{\infty} \binom{-\frac{1}{2} - \frac{1}{2\nu}}{\kappa} \frac{z\mathrm{i}^\kappa}{\kappa + \frac{1}{\nu}}\right). \tag{6.23}$$

Since now

$$\frac{A_\nu}{\nu} \equiv \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)\Gamma\left(\frac{1}{2} - \frac{1}{2\nu}\right)} = \frac{1}{\nu} + \mathrm{O}\left(\frac{1}{\nu^2}\right), \quad \left|\binom{-\frac{1}{2} - \frac{1}{2\nu}}{\kappa} \frac{z\mathrm{i}^\kappa}{\kappa + \frac{1}{\nu}}\right| \le \frac{|z|^\kappa}{\kappa},$$

there exist constants L, M independent of ν ($L < 1 + \varepsilon_\nu$ with $\lim_{\nu\to\infty} \varepsilon_\nu = 0$), such that

$$|G_4(r\mathrm{e}^{\mathrm{i}\varphi})| \le \frac{L}{\nu} r^{1+\frac{1}{\nu}}, \quad \left|\frac{\mathrm{d}G_4(r\mathrm{e}^{\mathrm{i}\varphi})}{\mathrm{d}r}\right| \le \frac{M}{\nu} r^{\frac{1}{\nu}}, \tag{6.24}$$

from which also

$$\left|\frac{\mathrm{d}|G_4(r\mathrm{e}^{\mathrm{i}\varphi})|}{\mathrm{d}r}\right| \le \frac{M}{\nu} r^{\frac{1}{\nu}}$$

follows. On the other hand,

$$1 - A_\nu r^{\frac{1}{\nu}} \cos\left(\frac{\varphi}{\nu}\right) - |G_4| \le |f_\nu(r\mathrm{e}^{\mathrm{i}\varphi})| \le 1 - A_\nu r^{\frac{1}{\nu}} \left(\cos\left(\frac{\varphi}{\nu}\right) - \left|\sin\left(\frac{\varphi}{\nu}\right)\right|\right) + \mathrm{Im}\, G_4 + \mathrm{Re}\, G_4. \tag{6.25}$$

From (6.24) and (6.25) we deduce that $|f_\nu(r\mathrm{e}^{\mathrm{i}\varphi})| > \frac{1}{2}$, and moreover,

$$\mathrm{Re}\, f_\nu(r\mathrm{e}^{\mathrm{i}\varphi}) = 1 - A_\nu r^{\frac{1}{\nu}} \cos\left(\frac{\varphi}{\nu}\right) + \mathrm{Re}\, G_4, \quad \mathrm{Im}\, f_\nu(r\mathrm{e}^{\mathrm{i}\varphi}) = -A_\nu r^{\frac{1}{\nu}} \sin\left(\frac{\varphi}{\nu}\right) + \mathrm{Im}\, G_4, \tag{6.26}$$

$$|f_\nu|^2 = 1 - 2r^{\frac{1}{\nu}} A_\nu \cos\left(\frac{\varphi}{\nu}\right) + A_\nu^2 r^{\frac{2}{\nu}} + 2\,\mathrm{Re}\, G_4 + |G_4|^2 + 2r^{\frac{1}{\nu}} |G_4| \cos\left(\frac{\varphi}{\nu} - \mathrm{arc}(G_4)\right) \tag{6.27}$$

which implies

$$2|f_\nu| \frac{\mathrm{d}|f_\nu|}{\mathrm{d}r} = -\frac{2A_\nu \cos\left(\frac{\varphi}{\nu}\right)}{\nu r^{1-\frac{1}{\nu}}} + K_1(\nu, r), \tag{6.28}$$

where[53)]

$$K_1(\nu,r)=\frac{2}{\nu}\frac{A_\nu^2}{r^{1-\frac{2}{\nu}}}+2\frac{\mathrm{d}\,\mathrm{Re}\,G_4}{\mathrm{d}r}+2|G_4|\frac{\mathrm{d}|G_4|}{\mathrm{d}r}+\frac{2A_\nu|G_4|}{\nu r^{1-\frac{1}{\nu}}}\cos\left(\frac{\varphi}{\nu}-\mathrm{arc}(G_4)\right)$$
$$+2A_\nu r^{\frac{1}{\nu}}\frac{\mathrm{d}|G_4|}{\mathrm{d}r}\cos\left(\frac{\varphi}{\nu}-\mathrm{arc}(G_4)\right)-2A_\nu r^{\frac{1}{\nu}}\sin\left(\frac{\varphi}{\nu}-\mathrm{arc}(G_4)\right)|G_4|\frac{\mathrm{d}\,\mathrm{arc}(G_4)}{\mathrm{d}r}.$$

From (6.24) it follows that

$$\begin{aligned}|K_1(\nu,r)|&\le\frac{2A_\nu^2}{\nu r^{1-\frac{2}{\nu}}}+\frac{2Mr^{\frac{1}{\nu}}}{\nu}+\frac{2LMr^{1+\frac{2}{\nu}}}{\nu^2}+\frac{2A_\nu Lr^{\frac{2}{\nu}}}{\nu^2}+\frac{4A_\nu Mr^{\frac{2}{\nu}}}{\nu}\\&=\frac{1}{\nu r^{1-\frac{2}{\nu}}}\left(2A_\nu^2+2Mr^{1-\frac{1}{\nu}}+\frac{2LMr^2}{\nu}+\frac{2A_\nu Lr}{\nu}+4A_\nu Mr\right)\\&=\frac{1}{\nu r^{1-\frac{2}{\nu}}}B^*(r),\end{aligned}\tag{6.29}$$

where $B^*(r)$ is a uniformly bounded function for $r\le(\frac{1}{2})^{2\nu}$. Hence

$$2|f_\nu|\frac{\mathrm{d}|f_\nu|}{\mathrm{d}r}=-\frac{2A_\nu\cos(\frac{\varphi}{\nu})}{\nu r^{1-\frac{1}{\nu}}}\left(1+r^{\frac{1}{\nu}}K(r,\nu)\right),\tag{6.30}$$

where $K(r,\nu)=\nu r^{1-\frac{2}{\nu}}K_1(r,\nu)$, and $|K(r,\nu)|\le B^*(r)$ for $\nu>1$, $r<(\frac{1}{2})^{2\nu}$.
For $\left|\frac{\mathrm{d}\vartheta_\nu(re^{\mathrm{i}\varphi})}{\mathrm{d}r}\right|$ we obtain, using (6.24) and (6.25),

$$\begin{aligned}\left|\frac{\mathrm{d}\vartheta_\nu(re^{\mathrm{i}\varphi})}{\mathrm{d}r}\right|&=\left|\frac{\mathrm{d}\arctan\left(\frac{\mathrm{Im}\,f_\nu}{\mathrm{Re}\,f_\nu}\right)}{\mathrm{d}r}\right|\le\frac{\left|\mathrm{Re}\,f_\nu\frac{\mathrm{d}\,\mathrm{Im}\,f_\nu}{\mathrm{d}r}\right|+\left|\mathrm{Im}\,f_\nu\frac{\mathrm{d}\,\mathrm{Re}\,f_\nu}{\mathrm{d}r}\right|}{(\mathrm{Re}\,f_\nu)^2}\le4\left(\left|\mathrm{Re}\,f_\nu\frac{\mathrm{d}\,\mathrm{Im}\,f_\nu}{\mathrm{d}r}\right|+\left|\mathrm{Im}\,f_\nu\frac{\mathrm{d}\,\mathrm{Re}\,f_\nu}{\mathrm{d}r}\right|\right)\\&\le4\left(\left(1+A_\nu r^{\frac{1}{\nu}}+\frac{L}{\nu}r^{1+\frac{1}{\nu}}\right)\left(\frac{A_\nu}{\nu}\frac{\sin(\frac{\alpha}{\nu})}{r^{1-\frac{1}{\nu}}}+\frac{M}{\nu}r^{\frac{1}{\nu}}\right)+\left(A_\nu r^{\frac{1}{\nu}}\sin\left(\frac{\alpha}{\nu}\right)+\frac{L}{\nu}r^{1+\frac{1}{\nu}}\right)\left(\frac{A_\nu}{\nu}\frac{1}{r^{1-\frac{1}{\nu}}}+\frac{M}{\nu}r^{\frac{1}{\nu}}\right)\right)\\&\le\frac{4}{\nu}\Bigg(\frac{A_\nu\sin(\frac{\alpha}{\nu})}{r^{1-\frac{1}{\nu}}}+\frac{2A_\nu^2r^{\frac{1}{\nu}}\sin(\frac{\alpha}{\nu})}{r^{1-\frac{1}{\nu}}}+\frac{LA_\nu r^{1+\frac{1}{\nu}}\sin(\frac{\alpha}{\nu})}{r^{1-\frac{1}{\nu}}}+\frac{LA_\nu r^{1+\frac{1}{\nu}}}{\nu r^{1-\frac{1}{\nu}}}\\&\qquad+Mr^{\frac{1}{\nu}}\left(1+A_\nu r^{\frac{1}{\nu}}+2\frac{L}{\nu}r^{1+\frac{1}{\nu}}+A_\nu r^{\frac{1}{\nu}}\sin\left(\frac{\alpha}{\nu}\right)\right)\Bigg)\\&=\frac{4A_\nu\sin(\frac{\alpha}{\nu})}{\nu r^{1-\frac{1}{\nu}}}\left(\left(1+2A_\nu r^{\frac{1}{\nu}}+\frac{L}{\nu}r^{1+\frac{1}{\nu}}+\frac{Lr^{1+\frac{1}{\nu}}}{\nu\sin(\frac{\alpha}{\nu})}\right)+\frac{Mr}{A_\nu\sin(\frac{\alpha}{\nu})}\left(1+A_\nu r^{\frac{1}{\nu}}+2\frac{L}{\nu}r^{1+\frac{1}{\nu}}+A_\nu r^{\frac{1}{\nu}}\sin\left(\frac{\alpha}{\nu}\right)\right)\right).\end{aligned}\tag{6.31}$$

53) $|G_4|\frac{\mathrm{d}\,\mathrm{arc}(G_4)}{\mathrm{d}r}\le|G_4|\left|\frac{\mathrm{d}\log(G_4)}{\mathrm{d}r}\right|\le\left|\frac{\mathrm{d}G_4}{\mathrm{d}r}\right|$.

As we may assume $\alpha > \frac{\pi}{4}$, we have $\nu \sin(\frac{\alpha}{\nu}) > \frac{1}{2}$, and as moreover $\frac{\sqrt{r}}{\sin(\frac{\alpha}{\nu})} \leq \frac{(\frac{1}{2})^\nu}{\sin(\frac{\alpha}{\nu})}$ is uniformly bounded for $\nu > 1$, we reach the result that

$$\left|\frac{\mathrm{d}\vartheta_\nu(r\mathrm{e}^{\mathrm{i}\varphi})}{\mathrm{d}r}\right| \leq \frac{4A_\nu \sin(\frac{\alpha}{\nu})}{\nu r^{1-\frac{1}{\nu}}} \left(1 + r^{\frac{1}{\nu}} B^{**}(r)\right) \tag{6.32}$$

where $B^{**}(r)$ is a uniformly bounded function for $r \leq (\frac{1}{2})^{2\nu}$.

From (6.30) it follows that $\frac{\mathrm{d}|f_\nu|}{\mathrm{d}r} < 0$ for sufficiently small r, and as $\frac{1}{2} \leq |f_\nu| \leq 1$, it follows from (6.21), (6.30) and (6.32) that

$$\begin{aligned} \frac{\mathrm{d}h^*(\gamma,\theta)}{\mathrm{d}r} &\leq -\frac{A_\nu \cos(\frac{\alpha}{\nu})}{\nu r^{1-\frac{1}{\nu}}} \left(1 + r^{\frac{1}{\nu}} B^*(r)\right) + \frac{4BA_\nu \sin(\frac{\alpha}{\nu})}{\nu r^{1-\frac{1}{\nu}}} \left(1 + r^{\frac{1}{\nu}} B^{**}(r)\right) \\ &\leq -\frac{A_\nu}{\nu r^{1-\frac{1}{\nu}}} \left(\left(a \cos\left(\frac{\alpha}{\nu}\right) - 4B \sin\left(\frac{\alpha}{\nu}\right)\right) + r^{\frac{1}{\nu}} B^\dagger(r)\right), \end{aligned} \tag{6.33}$$

where $B^\dagger(r)$ is a uniformly bounded function for sufficiently small r (see p. 74 for a and B). From (6.1) and (6.33) we further conclude that there exists a positive $r_0^* = r_0^*(\alpha,\nu) = \min\left(r_0(\alpha,\nu), (\frac{1}{2})^{2\nu}, \frac{2}{\varrho}\cos(\alpha)\right)$ such that for every $r < r_0^*$ and every φ, $|\varphi| < \alpha$, the inequality (6.20) holds.

We now turn to proving the claim made in II. The boundary curve of $\mathfrak{J}_\nu(z_1 = \gamma)$ is given by (6.19), where for every φ with $|\varphi| < \alpha$,

$$\lim_{\nu\to\infty} |f_\nu(r\mathrm{e}^{\mathrm{i}\varphi})| = 1, \quad \lim_{\nu\to\infty} \vartheta_\nu(r\mathrm{e}^{\mathrm{i}\varphi}) = 0 \tag{6.34}$$

holds (compare also formulas (6.14*) and (14) in article A). As the set of points $Z_2^{(m)}$, $m \to \infty$, converges to an interior point of $\mathfrak{H}^2$, there exists a $\tau_2 > 0$ such that for sufficiently large m the circle $|\zeta_2 - Z_2^{(m)}| \leq 2\tau_2$ lies in $\mathfrak{H}^2$. On the other hand, by (6.35) we can determine a positive $2l \leq r_0^*\left((\frac{\alpha}{2} + \frac{\pi}{4}, \nu\right)$ such that for every $Z_2^{(m)}$ and φ, $|\varphi| < \frac{\alpha}{2} + \frac{\pi}{4}$, the circle $|\zeta_1 - Z_2^{(m)}| \leq \tau_2$ lies in $\mathfrak{J}_\nu(z_1 = 2l\mathrm{e}^{\mathrm{i}\varphi})$. Moreover, there exists $\tau_1 > 0$ such that in the angular domain $\mathfrak{W}^2_{l\alpha}$: $r < l$, $|\varphi| < \alpha$, the circle $|\zeta_1 - \frac{l}{n^p}| \leq \frac{\tau_1}{n^p}$ about the point $\frac{l}{n^p}$ lies completely in the angular domain $\mathfrak{W}^2_{2l,\frac{\alpha}{2}+\frac{\pi}{4}}$: $r < 2l$, $|\varphi| < \frac{\alpha}{2} + \frac{\pi}{4}$. Since

$$\frac{\mathrm{d}h^*(r\mathrm{e}^{\mathrm{i}\varphi})}{\mathrm{d}r} < 0 \quad \text{for } r < 2l, |\varphi| \leq \frac{\alpha}{2} + \frac{\pi}{4},$$

for every $r_1 < r_2 < 2l$ we have: $\mathfrak{J}_\nu(z_1 = r_1\mathrm{e}^{\mathrm{i}\varphi}) \subset \mathfrak{J}_\nu(z_1 = r_2\mathrm{e}^{\mathrm{i}\varphi})$. Hence, for $\gamma \in \mathfrak{W}^2_{2l,\frac{\alpha}{2}+\frac{\pi}{4}}$ and every m, the circle $|\zeta_2 - Z_2^{(m)}| \leq \tau_2$ lies in $\mathfrak{J}_\nu(z_1 = \gamma)$.

Every bicylinder (6.18) *with* $\tau = \min(\tau_1, \tau_2)$ *thus lies in* $\mathfrak{J}_\nu$.

III. We now turn to the actual proof of Theorem IV and set

$$h(z_1, z_2; t_1, t_2) = f(z_1, z_2; t_1, t_2) - \mathsf{M}_{\mathfrak{J}_\nu}(z_1, z_2; t_1, t_2). \tag{6.35}$$

By Lemma II, it follows from (6.3), (1.11*) and (6.12) that

$$\in_{\mathfrak{J}_\nu} \left| h\left(z_1, z_2; \frac{T_1}{n}, T_2\right)\right|^2 \mathrm{d}\omega \leq \frac{c_1}{n^{2+\frac{1}{\nu}}} \tag{6.36}$$

holds, with

$$c_1 = \pi^2 (T_1 + \overline{T}_1)^{2+\frac{1}{\nu}} \left(C + 2\,\mathrm{Re}\left(\left(\frac{T_1}{T_1 + \overline{T}_1} \right)^{\frac{1}{\nu}} \left(\mathsf{P}(T_2)^2 + T_2 \frac{\mathrm{d}P(T_2)^2}{\mathrm{d}T_2} \right)\right)\right).$$

If now $\{Z_1, Z_2\} \in \mathfrak{U}$, then for $\{\frac{Z_1}{n^p}, Z_2\} \in \mathfrak{J}_\nu$ for sufficiently large n, and since f is regular on the bicylinder $\mathfrak{E}(\frac{Z_1}{n^p}, Z_2)$, it follows from (4.10) that[54]

$$\begin{aligned} \left| h\left(\frac{Z_1}{n^p}, Z_2; \frac{\overline{T}_1}{n}, \overline{T}_2\right)\right| &\leq \sqrt{\frac{\int_{\mathfrak{E}(\frac{Z_1}{n^p}, Z_2)} |h|^2 \mathrm{d}\omega}{\mathrm{vol}(\mathfrak{E}(\frac{Z_1}{n^p}, Z_2))}} \\ &\leq \sqrt{\frac{\int_{\mathfrak{J}_\nu} |h|^2 \mathrm{d}\omega}{\mathrm{vol}(\mathfrak{E}(\frac{Z_1}{n^p}, Z_2))}} \leq \sqrt{\frac{1}{n^{2-2p+\frac{1}{\nu}}} \frac{c_1}{\pi^2 \tau^3}} = \frac{c_2}{n^{1-p-\frac{1}{2\nu}}}, \end{aligned} \tag{6.37}$$

that is, for every $\{Z_1, Z_2\} \in \mathfrak{U}$,

$$n^{2-2p} \left| h\left(\frac{Z_1}{n^p}, Z_2; \frac{T_1}{n^p}, T_2\right)\right| \leq \frac{c_2}{n^{p-1+\frac{1}{2\nu}}} \tag{6.38}$$

holds.

From (6.35), (6.11) and (6.38) it now follows that

$$\begin{aligned} &\mathsf{M}_{\mathfrak{J}_\nu}\left(\frac{Z_1}{n^p}, Z_2; \frac{T_1}{n}, T_2\right) \\ &= \frac{w^{(1)}(Z_2)\left(\frac{T_1}{n} + \frac{\overline{T}_1}{n}\right)^2 (1 - |w(T_2)|^2)^2}{\left(\frac{Z_1}{n^p} + \frac{\overline{T}_1}{n}\right)^2 (1 - w(Z_2)\overline{w(T_2)})} \left(-1 + \frac{B}{n^{\frac{1}{\nu}}}\right) \\ &= \frac{(T_1 + \overline{T}_1)^2 n^{2-2p}}{Z_1 + \frac{T_1}{n^{1-p}}} \mathsf{M}_{\mathfrak{H}^2}(Z_2, T_2)\left(1 + \frac{B}{n^{\frac{1}{\nu}}}\right), \end{aligned} \tag{6.39}$$

[54]By c_k we again denote constants independent of n and $\{Z_1, Z_2\}$.

which implies (6.4) and (6.5). This concludes the proof of Theorem IV.

The minimal function $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ of $\mathfrak{B}$ satisfies (6.2) by definition, and by (1.11*), (1.12*) and (6.17) the inequality (6.3). Hence the following holds:

Corollary. *The limit relations given for* $f(z_1, z_2; t_1, t_2)$ *in* (6.4) *and* (6.5) *hold for the minimal function* $\mathsf{M}_{\mathfrak{B}}(z_1, z_2; t_1, t_2)$ *under a* p*-coupled* A^{V}*-approach of the pair of points* $\{z_1, z_2, t_1, t_2\} \to \{0, a_2, 0, b_2\}$.

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