On the groups of motions in Euclidean spaces

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Contents

Part I

Generalities and groups with infinite fundamental domain

Introduction

The groups of motions in two- and three-dimensional Euclidean space have been the subject of many studies. The first mainly because the complex analytic interest they offer, the latter because of their relevance to crystallography.[1)] In both cases a theorem was found stating that only finitely many groups of motions with a finite fundamental domain exist.

Hilbert was the first to point to the relevance of this fact by suspecting in it a general property of Euclidean spaces. He thus posed the investigation of this question as a problem in his talk on mathematical problems given at the Paris Congress 1900.[2)]

The present article, instigated by Hilbert, aims to develop the details of the proof of Hilbert's aforementioned theorem, whose main ideas I already sketched in a note in the Göttinger Nachrichten 1910.

The proof is similar in spirit to the aforementioned proof in the three-dimensional case due to Schoenflies, in the sense that here at first the existence of a translation subgroup consisting of n linear independent translations is proved, a fact on which the rest of the proof relies. Also, the use of finite groups of orthogonal substitutions is taken from the aforementioned proof. A new element is introduced only through the consistent use of finite groups of integral linear orthogonal substitutions and the related theory of positive quadratic forms. This theory proved to be a valuable tool which allows to essentially reduce the proof of finiteness to a theorem of Minkowski from the theory of positive quadratic forms, namely the theorem stating there are only finitely many unimodular substitutions with integral coefficients which are capable of transforming positive reduced forms again

[1)]For details and references I refer to: Schoenflies, *Krystallsysteme und Krystallstruktur* (Leipzig 1891).

[2)]Hilbert, *Mathematische Probleme*; talk given at the second international mathematical congress Paris 1900. First published in Göttinger Nachrichten 1900.

into positive reduced forms.[3)]

The first part presents general facts and treats the groups with infinite fundamental domain. It prepares the treatment of groups with finite fundamental domain in that the existence of the aforementioned translations subgroup with n linear independent translations is proven (Theorem XV, §10).

The main results on groups with infinite fundamental domain are contained in §§9, 10, 11. According to them, a group of motions with infinite fundamental domain is either homogeneous or finite, or it is decomposable (see the beginning of §8).

1 Euclidean motions and orthogonal substitutions

A *Euclidean motion* or *motion of the n-dimensional Euclidean space* is a linear substitution

$$x_i' = \sum_{k=1}^{n} a_{ik} x_k + \alpha_i \quad (i = 1, \ldots, n) \tag{1.1}$$

which transforms the line element

$$\mathrm{d}s^2 = \sum_{k=1}^{n} \mathrm{d}x_k^2 \tag{1.2}$$

into itself. For this it is necessary and sufficient that the following relations hold for the coefficients a_{ik}:

$$\begin{aligned} \sum_{i=1}^{n} a_{ik}^2 &= 1 \qquad (k = 1, \ldots, n), \\ \sum_{i=1}^{n} a_{ih} a_{ik} &= 0 \qquad (h \neq k). \end{aligned} \tag{1.3}$$

If we let A denote the substitution obtained from (1.1) by setting the α_k to 0, that is, the substitution

$$x_i' = \sum_{k=1}^{n} a_{ik} x_k \qquad (i = 1, \ldots, n), \tag{1.4}$$

[3)]Minkowski, *Diskontinuitätsbereich für arithmetische Äquivalenz*; J. f. Math. 129 (1905).

then this is an *orthogonal substitution*. We call A the orthogonal part of the motion. For short, we write

$$A = (a_{ik}) \text{ or } = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

denote the determinant of A by $\Delta_A = |a_{ik}|$ and set

$$D_A(\varrho) = \begin{vmatrix} a_{11} - \varrho & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - \varrho \end{vmatrix}.$$

If A^{-1} denotes the inverse operation to A, and A_1 the transpose, so that $A_1 = (a_{ki})$, then $A^{-1} = A_1$ by the relation (1.3). But A^{-1} also transforms the differential form (1.2) into itself. So the following relations hold, which are a consquence of (1.3) and which conversely imply (1.3):

$$\begin{aligned} &\sum_{k=1}^{n} a_{ik}^2 = 1 \qquad (i = 1, \ldots, n), \\ &\sum_{k=1}^{n} a_{ik} a_{hk} = 0 \qquad (h \neq i). \end{aligned} \tag{1.5}$$

The determinant $|a_{ik}|$ coincides with the determinant $|a_{ki}|$. Now, $A_1 A = 1$ (here and in the following, this equation expresses that the operation obtained by first applying A and then A_1 equals the identity operation). Hence $\Delta_A = +1$ or $\Delta_A = -1$. If $\Delta_A = +1$, then A is a *proper motion* or *operation of the first kind*. If $\Delta_A = -1$, then A is an *operation of the second kind*.

2 Representations of orthogonal substitutions by skew-symmetric matrices

I will now assume $D_A(1) \neq 0$, $\Delta_A = +1$ and want to derive Cayley's rational parameter representation for this class of orthogonal subsitutions. Let

$$x'_\alpha = \sum_{\beta=1}^{n} a_{\alpha\beta} x_\beta \qquad (\alpha = 1, \ldots, n)$$

be an orthogonal substitution. Then the z_α obtained by setting

$$2z_\alpha = x_\alpha + x'_\alpha$$

are linearly independent, because we assumed $D_A(1) \neq 0$. Then conversely

$$x_\alpha = \sum_{\beta=1}^{n} \lambda_{\alpha\beta} z_\beta \qquad (\alpha = 1, \dots, n),$$

where the $\lambda_{\alpha\beta}$ are suitable constants. Among these there exist certain relations which we want to derive now. It holds that

$$\sum_{\alpha=1}^{n} (x'_\alpha)^2 = \sum_{\alpha=1}^{n} x_\alpha^2.$$

But

$$x'_\alpha = 2z_\alpha - x_\alpha,$$

so that

$$4\sum_{\alpha=1}^{n} z_\alpha^2 - 4\sum_{\alpha=1}^{n} z_\alpha x_\alpha + \sum_{\alpha=1}^{n} x_\alpha^2 = \sum_{\alpha=1}^{n} x_\alpha^2,$$

or

$$\sum_{\alpha=1}^{n} z_\alpha^2 = \sum_{\alpha=1}^{n} z_\alpha \Big(\sum_{\beta=1}^{n} \lambda_{\alpha\beta} z_\beta \Big).$$

This implies

$$\lambda_{\alpha\alpha} = +1, \quad \lambda_{\alpha\beta} = -\lambda_{\beta\alpha},$$

that is, the matrix of the $\lambda_{\alpha\beta}$ in

$$x_\alpha - \sum_{\beta=1}^{n} \lambda_{\alpha\beta} z_\beta$$

is *skew-symmetric*, and so is the matrix of the $\mu_{\alpha\beta}$ in

$$x'_\alpha = \sum_{\beta=1}^{n} \mu_{\alpha\beta} z_\beta,$$

because $x'_\alpha = 2z_\alpha - x_\alpha$. Moreover, the matrix of the $\mu_{\alpha\beta}$ is the transpose of the matrix of the $\lambda_{\alpha\beta}$, that is, it is obtained from the latter by exchanging rows and columns.

3 Canonical forms of orthogonal substitutions

We consider all real orthogonal substitutions arising from a given one, A, by means of transformation by real orthogonal subsitutions, that is, those of the form BAB^{-1}. Among these, we are looking for a particular one with a very simple system of coefficients. I will prove the following theorem:

I *For a given real orthogonal substitution $O = (o_{ik})$ one can always find another real orthogonal substitution P such that $Q = POP^{-1} = (q_{ik})$ has the following matrix:*

(a) $\Delta_0 = +1, n$ *even:* $Q = Q_1|Q_2|\dots|Q_{\frac{n}{2}}$,

(b) $\Delta_0 = +1, n$ *odd:* $Q = Q_1|Q_2|\dots|Q_{\frac{n-1}{2}}|1$,

(c) $\Delta_0 = -1, n$ *even:* $Q = Q_1|Q_2|\dots|Q_{\frac{n-2}{2}}|\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$,

(d) $\Delta_0 = 1, n$ *odd:* $Q = Q_1|Q_2|\dots|Q_{\frac{n-2}{2}}|-1$.

Here, $Q_1|Q_2|\dots|Q_k$ denotes a matrix of the following form:

$$\begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Q_k \end{pmatrix};$$

here,

$$Q_i = \begin{pmatrix} \cos(\vartheta_i 2\pi) & -\sin(\vartheta_i 2\pi) \\ \sin(\vartheta_i 2\pi) & \cos(\vartheta_i 2\pi) \end{pmatrix}$$

and 0 denotes the matrix with all entries 0. This form of an orthogonal substitution is called the *normal form*, and the ϑ_i are called the orthogonal substitution's *angles of rotation*.

This theorem is proved in the theory of elementary divisors.[4)] Different proofs were given by Schläfli[5)] and Goursat[6)] ($n = 4$). I want to give a proof based on Cayley's representation of orthogonal substitutions by skew-symmetric matrices and on the reality of the roots of the characteristic equation, because the train of thoughts on which this proof is based will be useful later on.

[4)]Muth, *Theorie und Anwendung der Elementarteiler* (Leipzig 1899), p. 176.
[5)]Schläfli, J. f. Math. 65 (1866), p. 185.
[6)]Goursat, Ann. éc. norm. sup. (3) 6 (1889).

First, I want to reduce the case (b) to the case (a). So let n be odd and $\Delta_0 = +1$. Then there always exist points other than $x_1 = \ldots = x_n = 0$ which are fixed by the application of the orthogonal transformation. For this to hold, it is necessary and sufficient that $D_0(1) = 0$. By multiplying this determinant with that of A, we get $\Delta_0 D_0(1) = -D_0(1)$; but as $\Delta_0 = +1$, it follows that $D_0(1) = 0$.

So in this case one can always apply a real linear transformation to ensure that

$$o_{1n} = \ldots = o_{n-1,n} = 0$$

and

$$o_{n1} = \ldots = o_{n,n-1} = 0, \quad o_{nn} = +1$$

hold. For if we pick a point fixed by O other than $x_1 = \ldots = x_n = 0$, then we now have to pick P as an orthogonal substitution transforming this point to a point on the x_n-axis, that is, into a point $x_1 = \ldots = x_{n-1} = 0$.

Then the elements on the last column of the matrix $POP^{-1} = Q$ except for q_{nn} are all 0. But according to §1, $q_{nn} = +1$, because a point on the x_n-axis is fixed. This implies again by §1 that the remaining elements in the last row vanish.

We have thus reduced the case (b) of our theorem to case (a). By an analoguous argument which I omit here we can reduce the cases (c) and (d) of Theorem I to case (a).

The following idea will lead us to the proof of case (a) of the theorem: In general, one cannot find a point other than 0 fixed by the orthogonal substitution. So we look for the planes stabilised by the orthogonal substitution. Once we have found one of these, it is always possible to move this one into the plane $x_1 = x_2 = 0$ by means of an orthogonal transformation. Then the proof of the theorem is reduced to the analogous theorem for orthogonal substitutions in $n-2$ variables. Now the proof of the theorem follows by induction.

So what we need to investigate is whether we can determine the A_i, B_i in

$$\begin{aligned} \sum_{i=1}^{n} A_i x_i &\equiv A, \\ \sum_{i=1}^{n} B_i x_i &\equiv B \end{aligned} \tag{3.1}$$

in such a way that these two linear forms are transformed among each other by the orthogonal transformation. Let

$$O \equiv x'_\alpha = \sum_{i=1}^{n} o_{\alpha\beta} x_\beta \qquad (\alpha = 1, \dots, n)$$

the orthogonal transformation. The system (3.1) is transformed into

$$\sum_{i=1}^{n} A_i \Big(\sum_{\beta=1}^{n} o_{i\beta} x_\beta \Big) \equiv A',$$
$$\sum_{i=1}^{n} B_i \Big(\sum_{\beta=1}^{n} o_{i\beta} x_\beta \Big) \equiv B'$$

If A and B are linear forms of the kind we are looking for, then there exist real numbers $\lambda_1, \mu_1, \lambda_2, \mu_2$ such that

$$\begin{aligned} A' &= \lambda_1 A + \mu_1 B, \\ B' &= \lambda_2 A + \mu_2 B \end{aligned} \tag{3.2}$$

holds identically in the x_i. The condition (3.2) implies certain linear equations for the A_i, B_i. I will label the coefficients of A' by A'_i and those of B' by B'_i. Then

$$A'_\beta = \sum_{i=1}^{n} A_i o_{i\beta}, \quad B'_\beta = \sum_{i=1}^{n} B_i o_{i\beta} \qquad (\beta = 1, \dots, n). \tag{3.3}$$

Now we turn to the skew-symmetric form:

$$A'_\alpha = \sum_{\beta=1}^{n} \lambda_{\alpha\beta} z_\beta, \quad B'_\alpha = \sum_{\beta=1}^{n} \lambda_{\alpha\beta} z'_\beta,$$
$$A'_\beta = \sum_{\alpha=1}^{n} \lambda_{\alpha\beta} z_\alpha, \quad B'_\beta = \sum_{\alpha=1}^{n} \lambda_{\alpha\beta} z'_\alpha$$

where $\lambda_{\alpha\alpha} = 1$ and $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$ for $\alpha \neq \beta$. We rewrite this by multiplying these equations with inderterminates and adding them:

$$A' = S_1(z), \quad B' = S_1(z'),$$
$$A = S(z), \quad B = S(z').$$

Now it follows from (3.2) that

$$S_1(z) = \lambda_1 S(z) + \mu_1 S(z'),$$
$$S_1(z') = \lambda_2 S(z) + \mu_2 S(z').$$

We need to prove that there exist real $\lambda_1, \mu_1, \lambda_2, \mu_2$ such that these $2n$ linear equations can be solved for the $2n$ unknowns z, z'. These in turn will yield the A_i, B_i. At first, I somewhat manipulate the system of equations. I compute $S(z)$ from the first equation and plug it into the second equation:

$$S_1(z) = \lambda'_1 S(z) - \mu'_1 S(z'),$$
$$S_1(z) = -\lambda'_2 S_1(z) + \mu'_2 S(z').$$

This system of equations is equivalent to the original one, and the $\lambda'_1, \mu'_1, \lambda'_2, \mu'_2$ arise from the $\lambda_1, \mu_1, \lambda_2, \mu_2$ by a real substitution. For existence of $\lambda'_1, \mu'_1, \lambda'_2, \mu'_2$ allowing to solve the equations for z, z', the $\lambda'_1, \mu'_1, \lambda'_2, \mu'_2$ must satisfy the equation obtained by setting the determinant to 0. This is:

$$\begin{vmatrix} 1-\lambda'_1 & -\lambda_{12}(1+\lambda'_1) & \cdots & -\lambda_{1n}(1+\lambda'_1) & \mu'_1 & \mu'_1\lambda_{12} & \cdots & \mu'_1\lambda_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \lambda_{1n}(1+\lambda'_1) & \lambda_{2n}(1+\lambda'_1) & \cdots & 1-\lambda'_1 & -\mu'_1\lambda_{1n} & & \cdots & \mu'_1 \\ \lambda'_2 & -\lambda'_2\lambda_{12} & \cdots & -\lambda'_2\lambda_{1n} & 1-\mu'_2 & -\lambda_{12}(1+\mu'_2) & \cdots & -\lambda_{1n}(1+\mu'_2) \\ \vdots & \vdots & & \vdots & & & & \vdots \\ \lambda'_2\lambda_{1n} & & \cdots & \lambda'_2 & \lambda_{1n}(1+\mu'_2) & & \cdots & 1-\mu'_2 \end{vmatrix} = 0.$$

Now, we make the ansatz $\lambda'_1 = -1$, $\mu'_2 = -1$. Then the equation becomes

$$\begin{vmatrix} 2 & 0 & \cdots & 0 & \mu'_1 & \mu'_1\lambda_{12} & \cdots & \mu'_1\lambda_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 2 & -\mu'_1\lambda_{1n} & & \cdots & \mu'_1 \\ \lambda'_2 & -\lambda'_2\lambda_{12} & \cdots & -\lambda'_2\lambda_{1n} & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \lambda'_2\lambda_{1n} & & \cdots & \lambda'_2 & 0 & 0 & \cdots & 2 \end{vmatrix} = 0.$$

Moreover, we require $\mu'_1 = \lambda'_2$. Then, if we write $x = \frac{1}{\mu'_1}$,

$$\begin{vmatrix} 2x & 0 & \cdots & 0 & 1 & \lambda_{12} & \cdots & \lambda_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 2x & -\lambda_{1n} & & \cdots & 1 \\ 1 & -\lambda_{12} & \cdots & -\lambda_{1n} & 2x & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \lambda_{1n} & \lambda_{2n} & \cdots & 1 & 0 & 0 & \cdots & 2x \end{vmatrix} = 0.$$

But this is a characteristic equation, and as we know it has only real roots. So we learn from our ansatz that there exist real numbers λ_1', μ_1', λ_2', μ_2' allowing to solve the $2n$ equations for z, z'. The λ_1, μ_1, λ_2, μ_2 depend on the λ_i', μ_i' as follows:

$$\lambda_1 = \lambda_1', \quad \mu_1 = -\mu_1', \quad \lambda_2 = -\lambda_2'\lambda_1', \quad \mu_2 = \mu_1'\lambda_1' + \mu_2'.$$

We assumed:

$$\lambda_1' = -1, \quad \mu_2' = -1, \quad \mu_1' = \lambda_2'.$$

This implies

$$\lambda_1 = -1, \quad \mu_1 = -\mu_1', \quad \lambda_2 = -\mu_1', \quad \mu_2 = (\mu_2')^2 - 1.$$

For these λ_i, μ_i we have solutions A_i, B_i of our linear equations. There are two possible cases. Either the associated linear forms A, B are linearly independent or not. If they are linearly independent, then we can always assume that the following relations between the A_i, B_i hold:

$$\sum A_i^2 = 1, \quad \sum B_i^2 = 1, \quad \sum A_i B_i = 0.$$

Then the following relations between the A_i', B_i' hold:

$$\sum (A_i')^2 = 1, \quad \sum (B_i')^2 = 1, \quad \sum A_i' B_i' = 0,$$

and the λ_i, μ_i are coefficients of a linear orthogonal transformation (if these relations do not hold to begin with, we can achieve it via a linear combination of the A, B). One can always state an orthogonal substitution whose last rows comprise the A_i, B_i. If we transform our original matrix with this one, it takes the following form:

$$A_{n-2} \Bigg| \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}.$$

But this is just what we wanted to prove.

If A and B are not linearly independent, then $A = bB$ identically in the x. We may assume: $\sum A_i^2 = 1, \sum B_i^2 = 1$. So $b = \pm 1$. Because of the equations (3.3),

$$A_\beta' = \sum_{i=1}^{n} A_i o_{i\beta}, \quad B_\beta' = \sum_{i=1}^{n} B_i o_{i\beta} \qquad (\beta = 1, \ldots, n),$$

it also holds that

$$A' = bB'. \tag{3.4}$$

But now

$$A' = \lambda_1 A + \mu_1 B = -A - \mu_1' B,$$
$$B' = \lambda_2 A + \mu_2 B = \mu_1' A + ((\mu_1')^2 - 1)B, \tag{3.5}$$

so that either

$$-bB - \mu_1' B = b(\mu_1' b + ((\mu_1')^2 - 1)B$$

or

$$-\mu_1' = b^2 \mu_1' + b(\mu_1')^2. \tag{3.6}$$

Here, two cases have to be distinguished. Either $\mu_1' = 0$ or $\mu_1' \neq 0$:

(a) $\mu_1' = 0$: By (3.5), $A' = -A$, $B' = -B$. Choose the A_i as the last row of an otherwise arbitrary orthogonal transformation and transform O with it. The the transform of Q becomes

$$Q = Q'| - 1.$$

Q' now has an uneven number of rows and a negative determinant. Then $Q' = Q''| - 1$ by §3, and O can thus be transformed into the form

$$Q''\Big|\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have thus reduced the proof of our theorem for matrices with n rows to the case of matrices with $n - 2$ rows.

(b) $\mu_1' \neq 0$: By (3.6): $-1 = b^2 + b\mu_1'$. For this to be satisfied by a $b = \pm 1$, $\mu_1' = \pm 2$ has to hold. Then $b = \mp 1$ and by (3.5) in both cases $A' = A$, $B' = B$. This means, because of equation (3.3), that there exist points fixed by our orthogonal transformation O. So O can be transformed into the form $O'|1$, and as O' has an uneven number of rows and positive determinant, O can be transformed into

$$O''\Big|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we have reached our goal and proved Theorem I in general, as it clearly holds for binary orthogonal substitutions.

4 Commuting orthogonal substitutions

We now wish to continue the investigations of §3 for orthogonal substitutions with positive determinant. First, I will prove a theorem of the theory of elementary divisors by following the preceding procedure.

II *The absolute values of the angles of rotation* ϑ_i *are invariants of the orthogonal transform, that is, in whatever way the transformation is brought into its normal form, there will always be the same absolute values of the angles of rotation, and their respective multiplicities are the same in every normal form.*

To prove this theorem, I will apply the procedure of §3 to an orthogonal transformation in normal form. So the task is to find pairs of linear form

$$\begin{aligned} &\sum_{i=1}^{n} A_i x_i \equiv A, \quad \sum A_i^2 = 1, \\ &\sum_{i=1}^{n} B_i x_i \equiv B, \quad \sum B_i^2 = 1, \end{aligned} \qquad \sum A_i B_i = 0,$$

which transform into each other under the subsitution which was brought into normal form. Let the orthogonal transformation of positive determinant be

$$C = C_2^{(1)} | C_2^{(2)} | \dots | C_2^{(k)} | 1_h \quad (2k + h = n),$$

where

$$C_2^{(i)} = \begin{pmatrix} \cos(2\pi\vartheta_i) & -\sin(2\pi\vartheta_i) \\ \sin(2\pi\vartheta_i) & \cos(2\pi\vartheta_i) \end{pmatrix}.$$

(Here and in the following, the lower index indicates the number of rows in a matrix. 1_h is the identical transform of h elements.) C transforms our linear forms (A, B) into

$$A' \equiv \sum_{i=1}^{k}\big((A_{2i-1}\cos(2\pi\vartheta_i) + A_{2i}\sin(2\pi\vartheta_i))x_{2i-1}$$

$$+ (-A_{2i-1}\sin(2\pi\vartheta_i) + A_{2i}\cos(2\pi\vartheta_i)x_{2i}\big) + \sum_{\lambda=1}^{h} A_{2k+\lambda}x_{2k+\lambda},$$

$$B' \equiv \sum_{i=1}^{k}\big((B_{2i-1}\cos(2\pi\vartheta_i) + B_{2i}\sin(2\pi\vartheta_i))x_{2i-1}$$

$$+ (-B_{2i-1}\sin(2\pi\vartheta_i) + B_{2i}\cos(2\pi\vartheta_i)x_{2i}\big) + \sum_{\lambda=1}^{h} B_{2k+\lambda}x_{2k+\lambda}.$$

As in §3, there exists a real φ such that

$$\begin{aligned} A' &= A\cos(2\pi\varphi) - B\sin(2\pi\varphi) \\ B' &= A\sin(2\pi\varphi) + B\cos(2\pi\varphi) \end{aligned}$$

are identical in $x_1, x_2, \ldots, x_n$. From this the following system of linear equation follows:

$$\begin{aligned} A_{2i-1}\cos(2\pi\vartheta_i) + A_{2i}\sin(2\pi\vartheta_i) &= A_{2i-1}\cos(2\pi\varphi) - B_{2i-1}\sin(2\pi\varphi), \\ -A_{2i-1}\sin(2\pi\vartheta_i) + A_{2i}\cos(2\pi\vartheta_i) &= A_{2i}\cos(2\pi\varphi) - B_{2i}\sin(2\pi\varphi), \\ B_{2i-1}\cos(2\pi\vartheta_i) + B_{2i}\sin(2\pi\vartheta_i) &= A_{2i-1}\sin(2\pi\varphi) + B_{2i-1}\cos(2\pi\varphi), \\ -B_{2i-1}\sin(2\pi\vartheta_i) + B_{2i}\cos(2\pi\vartheta_i) &= A_{2i}\sin(2\pi\varphi) + B_{2i}\cos(2\pi\varphi), \\ &\text{(for } i = 1, 2, \ldots, n) \\ A_{2k+\lambda} &= A_{2k+\lambda}\cos(2\pi\varphi) - B_{2k+\lambda}\sin(2\pi\varphi), \\ B_{2k+\lambda} &= A_{2k+\lambda}\sin(2\pi\varphi) + B_{2k+\lambda}\cos(2\pi\varphi), \\ &\text{(for } \lambda = 1, 2, \ldots, h). \end{aligned}$$

The questions is now, for which values of φ there exist A_i, B_i, not all vanishing, such that this system of equation is satisfied? I will show that only the values above, $\varphi = \pm\vartheta_i$, 0, satisfy this requirement. The unknowns A_i, B_i fall into distinct systems of four and two unknowns, respectively, which satisfy certain linear equations among themselves, as given above. It is therefore sufficient to consider a single one of these systems, for if not all A_i, B_i are to vanish, then at least one of these system has solutions other than 0.

For example, if we consider the system $i = l$, then $(A_{2l-1}, A_{2l}, B_{2l-1}, B_{2l}) \neq (0,0,0,0)$ if and only if the determinant of this system vanishes. This determinant is

$$\nabla = \begin{vmatrix} \cos(2\pi\vartheta_l)-\cos(2\pi\varphi) & \sin(2\pi\vartheta_l) & \sin(2\pi\varphi) & 0 \\ \sin(2\pi\vartheta_l) & \cos(2\pi\vartheta_l)-\cos(2\pi\varphi) & 0 & \sin(2\pi\varphi) \\ -\sin(2\pi\varphi) & 0 & \cos(2\pi\vartheta_l)-\cos(2\pi\varphi) & \sin(2\pi\vartheta_l) \\ 0 & -\sin(2\pi\vartheta_l) & -\sin(2\pi\vartheta_l) & \cos(2\pi\vartheta_l)-\cos(2\pi\varphi) \end{vmatrix}.$$

If I set $\alpha = \cos(2\pi\vartheta_l) - \cos(2\pi\varphi)$, this becomes

$$\nabla = \alpha^4 + 2\alpha^2(\sin(2\pi\vartheta_l)^2 + \sin(2\pi\varphi)^2) + (\sin(2\pi\varphi)^2 - \sin(2\pi\vartheta_l)^2)^2,$$

a sum of squares. Every term must vanish on its own, and in particular $\alpha = \cos(2\pi\vartheta_l) - \cos(2\pi\varphi) = 0$. Hence

$$\varphi = \pm\vartheta_l.$$

If this is the case, then also

$$\nabla = 0.$$

Next consider $\lambda = l$. For

$$(A_{2k+l}, B_{2k+l}) \neq (0,0)$$

to hold, it is required that

$$\begin{vmatrix} \cos(2\pi\varphi) - 1 & -\sin(2\pi\varphi) \\ \sin(2\pi\varphi) & \cos(2\pi\varphi) - 1 \end{vmatrix} = 0.$$

But this is the case if and only if $\varphi = 0$. This proves Theorem II. In a normal form, only those angles can appear for which the equations above are solvable.

Our investigation immediately yield the correctness of the following theorem:

III *Except for those linear subspaces distinguished by the normal form, the orthogonal subsitution preserves other linear subspaces if and only if one or more of the ϑ_i have the same absolute value.*

For if the conditions of the theorem are not satisfied, then only one quadruple of equations is solvable for a given φ. We obtain the following equations for a fixed plane:

$$\begin{aligned} A_1x_1 + A_2x_2 &= 0, \\ -A_2x_1 + A_1x_2 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} A_1x_1 + A_2x_2 &= 0, \\ A_2x_1 - A_1x_2 &= 0. \end{aligned}$$

This implies $x_1 = x_2 = 0$, a linear subspace already distinguished by the normal form. But if some of the ϑ_i have the same absolute value, then some systems of equations are simultaneously solvable, and then there exist additional planes that are preserved by the orthogonal transformation. For example, if $n = 4$ and $\vartheta_1 = \vartheta_2$, then the first two quadruples of equations are solvable. We obtain equations for a preserved linear subspace such as the following:

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0,$$
$$-A_2x_1 + A_1x_2 - A_4x_3 + A_3x_4 = 0,$$

where $\sum A_i^2 = 1$. It is easy to see that the A_i can be determined such that this plane coincides either with $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$. If we choose these coefficients to be the last two rows of an orthogonal transformation and we choose other coefficients of the same form for the first two rows, then this transformation commutes with the normal form and we obtain a new transformation of into the normal form. I will not continue these investigations here, but rather directly state the theorem they are leading up to. Suppose the orthogonal transformation A is brought into normal form, such that the angles of rotation are ordered by their absolute values. I write

$$A = A_{2\alpha_1}|A_{2\alpha_2}|\dots|A_{2\alpha_k}|1_\lambda,$$

where the angles of rotation of $A_{2\alpha_i}$ all have the same absolute value and are distinct in absolute value from those of $A_{2\alpha_j}$, $j \neq i$, and from 0. The the following theorem holds:

IV *If A and B commute, the B is necessarily of the following form:*

$$B = B_{2\alpha_1}|B_{2\alpha_2}|\dots|B_{2\alpha_k}|B_\lambda,$$

Our method allows us to determine the sufficient form of the coefficients. As this is not used in the following, it shall be omitted here.

I will now prove the following theorem:

V *If and A and B have positive determinant and commute, then A and B can be simultaneously brought into the following form:*

$$A = A_{2\beta_1}|A_{2\beta_2}|\dots|A_{2\beta_\nu}|1_\lambda,$$
$$B = B_{2\beta_1}|B_{2\beta_2}|\dots|B_{2\beta_\nu}|B_\lambda,$$

where the absolute values of the angles of rotation of an $A_{2\beta_i}$ and a $B_{2\beta_i}$ are identical, and moreover B is in normal form.

Note that it is sufficient to prove the special case of this theorem in which all angles of A have the same absolute value. Then apply a transformation to A and B simultaneously that brings B into a normal form, such that its angles of rotation are ordered by absolute value. It now follow from Theorem IV that A is of the form claimed in Theorem V.

5 A theorem on motions

Every motion the first or second kind can be brought into the following form:

$$A = A_{2k} \left| \begin{pmatrix} 1 & \cdots & 0 & T_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & T_h \end{pmatrix} \right.,$$

where A_{2k} does not fix any point other than $x_1 = x_2 = \ldots = x_{2k} = 0$. We then have the following theorem:

VI *It is a necessary condition for BAB^{-1} to take the given form for A simultaneously with A is that B is of the following form:*

$$B = B_{2k} \left| \begin{pmatrix} b_{11} & \cdots & b_{1h} & T'_1 \\ \vdots & \ddots & \vdots & \vdots \\ b_{h1} & \cdots & b_{hh} & T'_h \end{pmatrix} \right..$$

That the orthogonal part of B must be of the given form follows from the fact that B maps all the points which are fixed by orthogonal part of A to themselves, because the orthogonal part of BAB^{-1} fixes the same points as A does.

That the first $2k$ translational components of B vanish can be proved as follows: I assume the first $2k$ translational components to be

$$b_1, b_2, \ldots, b_{2k}.$$

Moreover, let

$$A_{2k} = (\alpha_{ik}), \quad B_{2k} = (\beta_{ik}).$$

Then those $2k$ translational components of BAB^{-1} that are assumed to vanish become

$$\sum_{g=1}^{2k} \beta_{g\nu} \sum_{i=1}^{2k} \alpha_{gi} b_i - \sum_{g=1}^{2k} \beta_{g\nu} b_g \qquad (\nu = 1, 2, \dots, 2k).$$

As the determinant of the $\beta_{g\nu}$ is different from 0, these $2k$ expressions can vanish only if

$$\sum_{i=1}^{2k} \alpha_{gi} b_i - b_g = 0 \qquad (g = 1, 2, \dots, 2k).$$

As A_{2k} fixes no point other than 0, these equations can hold only if

$$b_g = 0 \qquad (g = 1, 2, \dots, 2k).$$

But this is just what we wanted to prove.

6 Fundamental domains and infinitesimal operations

In this work we investigate groups of motions which have a fundamental domain. This means the following: We call two points *equivalent* if they can be mapped to one another by a motion of the group. A *fundamental domain* of a group is a connected domain, that is, a part of space of non-vanishing size, which contains precisely one equivalent point to every point of a domain into which it is transformed by the group. If a group has a fundamental domain, then for each of its points it is possible to find a domain around it which contains no two equivalent points; conversely, if there is any point contained in a domain that does not contain any two equivalent points, then the group has a fundamental domain. Firstly, it is clear that every point equivalent to the first one is contained in a domain with the same property, and this domain is obtained from the original domain by a motion in the group. We can the let these domains grow simultaneously such that they congruent to each other and never overlap. As soon as two domains touch each other, we stop. As all domains are congruent, there can never be an accumulation of them in the infinite. Through this procedure we obtain a division of the whole space into congruent parts which arise from each other by motions in the group. Every such domain contains precisely one equivalent point for every point in space. Such a domain is called a fundamental domain of the group.

To prove the existence of a fundamental domain, it is thus sufficient to know of only one point at which not equivalent points accumulate; for as soon as such a point is known, we can distinguish a domain around it which contains no two equivalent points. As the given point is not a point of accumulation of equivalent points, there exists a closest equivalent point to it. If I now consider a sphere around this point whose radius is less than half the distance to the closest equivalent point, then this is such a domain containing no two equivalent points. If I apply the motions of the group to this sphere, then I obtain congruent sphere which have no point in common with the original sphere, which would have to be the case of there were two equivalent points in the first sphere.

When I say a group contains *infinitesimal operations*, then it means that it is possible to find a sequence of operations $A_1, \dots, A_m, \dots$ in the group such that for every given ε there is some index m such that the coefficients of A_i (for $i > m$) deviate from the identity matrix by less than ε.

It is clear that a group with fundamental domain can contain no infinitesimal operations, for otherwise there would be an equivalent point arbitrarily close to any given point. But the converse is also true, for the following theorem holds:

VII *A group of motions without infinitesimal operations has a fundamental domain.*

Note first that a motion in n-dimensional Euclidean space is uniquely determined if $n+1$ points $Q_0, \dots, Q_n$ are known, into which $n+1$ given points $P_0, \dots, P_n$ are transformed by the motion, given that not all points belong to a space of dimension less than n. If the group does not have a fundamental domain, then every point in space is an accumulation point of its equivalent points. For if there was even one point for which this was not true, then we could conclude as above that the group has a fundamental domain and thus does not contain any infinitesimal transformations, contradicting our assumption. Hence P_0 has to be such an accumulation point. Choose a sequence of equivalent points accumulating at this point, $Q_0^{(1)}, \dots, Q_0^{(n)}, \dots$. Moreover, choose any motion $B^{(i)}$ out of all motions mapping P_0 to $Q_0^{(i)}$. This way we obtain a sequence $B^{(1)}, \dots, B^{(n)}, \dots$ of motions. I will show that this leads to infinitesimal transformations. First, pick a transformation $T^{(i)}$ mapping $Q_0^{(i)}$ to P_0. Then $B^{(i)}T^{(i)} = \Gamma^{(i)}$ is an orthogonal transformation, and the sequence of the $T^{(i)}$ leads to infinitesimal transformations. Now assume $P_1, \dots, P_n$ are mapped to $Q_1^{(i)}, \dots, Q_n^{(i)}$ by $\Gamma^{(i)}$. Then there are two possibilities: Either the $Q_k^{(i)}$ coincide with the P_i from a certain index i on. In this case, $\Gamma^{(i)}$ is the identity from this index on, so $B^{(i)} = (T^{(i)})^{-1}$ from then on. So then our sequence leads to an infinitesimal operation. Or there are arbitrarily large indices i for which $(Q_1^{(i)}, \dots, Q_n^{(i)})$ is different from $(P_1, \dots, P_m)$. Then we can pick a sequence, again denoted by $(Q_1^{(i)}, \dots, Q_n^{(i)})$, such that $Q_1^{(i)}$ converges to P_1, $Q_2^{(i)}$ converges to $P_2, \dots$, and $Q_n^{(i)}$ converges to P_n. Let $\vartheta_k^{(i)}$ denote the angle $(Q_k^{(i)} P_0 P_k)$. Then $\lim_{k\to\infty} \vartheta_k^{(i)} = 0$. We may now assume that the points $P_1, \dots, P_n$ are situated on the axes of a right-angled coordinate system with origin P_0 and distance 1 from P_0. Mapping $(P_1, \dots, P_n)$ to $(Q_1^{(i)}, \dots, Q_n^{(i)})$ corresponds to a change of coordinates. Let $(x'_1, \dots, x'_n)$ be the coordinate system associated to $(Q_1^{(i)}, \dots, Q_n^{(i)})$, and $(x_1, \dots, x_n)$ the one associated to $(P_1, \dots, P_n)$. Then

$$x'_h = \sum_{k=1}^{n} a_{hk}^{(i)} x_k \qquad (h = 1, 2, \dots, n).$$

But here is $a_{hh} = \cos(\vartheta_h^{(i)})$. Hence $\lim_{k\to\infty} a_{hh}^{(i)} = 1$ and therefore $\lim_{k\to\infty} a_{kh}^{(i)} = 0$ for $k \neq h$ by the relations given in §1. Therefore the sequence $\Gamma^{(i)}$ and hence $B^{(i)}$ leads to infinitesimal transformations. So if the group does not have a fundamental domain, then it contains infinitesimal transformations. From this Theorem VII follows.

7 Proof of two lemmas

In this paragraph I want to prove two lemmas to be used in the next paragraph. The first one is:

VIII *Let A and B be two commuting orthogonal operations of even row number. Suppose the angles of rotation of A are all irrational, those of B either irrational or 0, and both cases occur. Then there exist two exponents $\omega_1 > 0$ and $\omega_2 > 0$ such that the angles of rotation of $A^{\omega_1} B^{\omega_2}$ are all irrational.*

Proof: By Theorem V in §4 we may assume that

$$A = A_1|A_2|\dots|A_n,$$
$$B = B_1|B_2|\dots|B_n,$$

where each B_i or A_h only has angles of rotation of the same absolute value. Let k_i denote half of the line number of A_i or B_i, respectively. Our assumption is that

$$B_{n-\mu_1} = B_{n-\mu_1+1} = \dots = B_n = 1 \qquad (\mu_1 > 0).$$

Then

$$AB = C_1|\dots|C_{n-\mu_1-1}|A_{n-\mu_1}|\dots|A_n,$$

so AB has at least $k_{n-\mu_1}+k_{n-\mu_1+1}+\dots+k_n$ irrational angles of rotation. If there are only irrational angles of rotation in the C_i, then we are done. Otherwise, there is a positive constant a_1 such that $(AB)^{a_1} = A^{a_1}B^{a_1}$ has 0 as the only rational angle of rotation. Then we can transform A and B simultaneously by a transformation of $2(k_1 + k_2 + \dots + k_{n-\mu_1+1})$ rows such that $\alpha A\alpha^{-1}$ and $\alpha(AB)^{a_1}\alpha^{-1}$ take the following form:

$$\begin{array}{l} A'_1|A'_2|\dots|A'_n \\ C'_1|C'_2|\dots|C'_{n-\mu_1-1}|A^{a_1}_{n-\mu_1}|\dots|A^{a_1}_n, \end{array}$$

where again A'_i and C'_h have only angles of rotation of the same absolute value, and $C'_{n-\mu_1} = C_{n-\mu_1+1} = \dots = C_{n-\mu_1-1} = 1$. Let $2\lambda_i$ be the number of rows in C'_i and A'_i. Then

$$\alpha A^{a_1+1}B^{a_1}\alpha^{-1} = D_1|D_2|\dots|D_{n-\mu_2-1}|A'_{n-\mu_2}|\dots|A'_{n-\mu_1}|A^{a_1+1}_{n-\mu_2}|\dots|A^{a_1+1}_n$$

and hence $A^{a_1+1}B^{a_1}$ has at least

$$\lambda_{n-\mu_2} + \dots \lambda_{n-\mu_1-1} + k_{n-\mu_1} + \dots + k_n$$

irrational angles of rotation, a number which is certainly greater than the minimal number of irrational angles of rotation in AB. If $A^{a_1+1}B^{a_1}$ still has some rational angles of rotation, then we can again determine a positive exponent a_2 such that $A^{a_2(a_1+1)}B^{a_1a_2}$ has only irrational angles of rotation except for 0. As before we can conclude that $A^{a_2(a_1+1)+1}B^{a_1a_2}$ certainly has more irrational angles than $A^{a_1+1}B^{a_1}$. We continue in this way until after finitely many steps we find an operation $A^{\omega_1}B^{\omega_2}$, $\omega_1, \omega_2 > 0$, which has only irrational angles of rotation, because in every step in which there are still rational angles left, the number of irrational angles grows and eventually has to coincide with half the line number of A. This proves the first lemma.

Now I turn to the second lemma I want to prove in this paragraph:

IX *Let $\beta_1, \beta_2, \dots, \beta_n, \dots$ be a sequence of orthogonal operations leading to infinitesimal substitutions (§6). Then there exists a number k such that no β_i (for $i > k$) can map a point of the space*

$$x_{h+1} = x_{h+2} = \dots = x_n = 0$$

to a point of the orthogonal space

$$x_1 = x_2 = \ldots = x_h = 0.$$

Proof: Let β be an orthogonal substitution

$$\begin{pmatrix} 1-\beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & 1-\beta_{nn} \end{pmatrix}.$$

If this is to transform a point

$$x_{h+1} = x_{h+2} = \ldots = x_n = 0$$

to a point

$$x_1 = x_2 = \ldots = x_h = 0,$$

then the equations

$$\begin{aligned} (1-\beta_{11})x_1 + \beta_2 x_2 + \ldots + \beta_{1h} x_h &= 0 \\ &\vdots \\ \beta_{h1} x_1 + \beta_{h2} x_2 + \ldots + (1-\beta_{hh}) x_h &= 0 \end{aligned}$$

hold. It should be possible to satsify it with $x_1, x_2, \ldots, x_h$ different from 0. Then the determinant should be

$$\begin{vmatrix} 1-\beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & 1-\beta_{nn} \end{vmatrix} = 0.$$

If we think about this determinant as an polynomial function of the β_{ik}, then it is of the form $1 + \sum_{i=1}^{m} P_i(\beta)$, where $P_i(\beta)$ is a product of at most $h < n$ of the β_{ik} with coefficient 1, and m the number of these products. As all $|\beta_{ik}| < \frac{1}{m^2}$, also each $|P_i(\beta)| < \frac{1}{m^2}$, hence $1 + \sum_{i=1}^{m} P_i(\beta) < \frac{1}{m} < 1$. But then $1 + \sum_{i=1}^{m} P_i(\beta)$ cannot vanish, since otherwise $|\sum_{i=1}^{m} P_i(\beta)| = 1$. Now I determine an index k such that in the sequence $\beta_1, \ldots, \beta_n, \ldots$ the coefficients β_i (for $i > k$) deviate from those of the identity transformation by less than $\frac{1}{m^2}$. This index k satisfies the conditions of our theorem, which is thereby proven.

8 Groups with irrational angles of rotation

A group of motions all of which operations can be simultaneously brought into the form

$$\begin{pmatrix} A_{2k} & 0 & 0 \\ 0 & A_h & T_h \end{pmatrix},$$

where k and h are independent of the individual motion and T_h denotes the translational component, is called *decomposable*.

Then the following theorem holds:

X *A group of motions that contains operations with an irrational angle of rotation and does not contain infinitesimal operations is decomposable.*

We prove this theorem by showing: Every group of motions containing operations with irrational angles of rotation and which is not decomposable contains infinitesimal operations.

So assume the group is not decomposable. Further, let A an operation with irrational angle of rotation. We can assume that all non-zero angles of rotation are irrational. In a suitable coordinate system, the motion is represented as

$$A = A_1|A_2|\dots|A_k|\begin{pmatrix} 1_{h-1} & 0_{h-1} & 0_{h-1} \\ 0 & 1 & T \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} \cos(2\pi\vartheta_i) & -\sin(2\pi\vartheta_i) \\ \sin(2\pi\vartheta_i) & \cos(2\pi\vartheta_i) \end{pmatrix}.$$

Now we can find a sequence of operations starting at A whose orthogonal parts lead to infinitesimal operations: Let λ be number of angles of rotation with different absolute values. By a theorem of Minkowski[7)] λ irrational numbers ϑ_i can be approximated simultaneously to arbitrary precision by rational numbers $\frac{x_i}{n}$ with common denominator n, such that

$$\left|\frac{x_i}{n} - \vartheta_i\right| < \frac{1}{n\sqrt[\lambda]{n}} \qquad (i = 1, 2, \dots, \lambda).$$

Let $n_1 < n_2 < \dots < n_m < \dots$ be a infinite sequence of integer numbers satisfying this condition. The angles of rotation of $A_i^{n_m}$ is $n_m\vartheta_i$ and we have $|n_m\vartheta_i - x_i| < \frac{1}{\sqrt[\lambda]{n_m}}$.

7)Minkowski, *Geometrie der Zahlen* (Leipzig 1897-1910), p. 108. *Diophantische Approximationen* (Leipzig 1907), p. 8.

n_m can be chosen such that the angles of rotation deviate from a rational number by an arbitrarily small amount, so that the coefficients of A^{n_m}, for m sufficiently large, deviate from those of the identity matrix by no more than an arbitrary given amount η. They deviate by less than $\varepsilon_m = \frac{1}{\sqrt[\lambda]{n_m}}$. The translational components of A^{n_m} are $n_m T_i$. Hence they grow unboundedly with m in such a way that $\varepsilon_m^\lambda T_i = T_i$. We agree on the following convenient convention: We say the coefficients approach those of the identity transformation like ε, and the translational components approach infinity like $\varepsilon^{-\lambda}$.

The case in which $T_i = 0$ ($i = 1, \ldots, h$) is included in the preceding arguments, and we may further assume:

$$T_1 = \ldots = T_{h-1} = 0, \quad T_h \neq 0.$$

Now everything depends on constructing from the sequence

$$\Gamma_1 = (A^{n_1}, \ldots, A^{n_m}, \ldots)$$

in an indecomposable group a new one which becomes infinitesimal also in its translational part. To this end, we choose among all oprations in the group the one with the greatest number of irrational angles of rotation. This is brought into the above normal form, and construct the sequence Γ_1 by taking powers. In the following I will operate with the product of two sequences. This means the following: Let

$$C_1 = (A_0, A_1, \ldots, A_n, \ldots),$$
$$C_2 = (B_0, B_1, \ldots, B_n, \ldots)$$

be two sequences, then their product is

$$C_1 C_2 = (A_0 B_0, A_1 B_1, \ldots, A_n B_n, \ldots).$$

Now we can always assume that there exists an operation γ_0 in the group which is not of the form

$$\begin{pmatrix} G_{2k} & 0 & 0 \\ 0 & G_h & T_h' \end{pmatrix},$$

where k is the number of irrational angles of rotation occurring in A. Otherwise, the group would be indecomposable. With such a sequence we construct the sequence

$$\Gamma_0 = (\gamma_0, \gamma_0, \ldots, \gamma_0, \ldots).$$

Then construct the sequences

$$
\begin{aligned}
\Gamma_2 &= \Gamma_0 \Gamma_1 \Gamma_0^{-1} \Gamma_1^{-1}, \\
\Gamma_3 &= \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_1^{-1}, \\
\Gamma_4 &= \Gamma_3 \Gamma_1 \Gamma_3^{-1} \Gamma_1^{-1}, \\
&\vdots
\end{aligned}
$$

Note first the dependence of the coefficients of these sequences on ε. First, consider Γ_2. Write the coefficients of an element of Γ_1 as:

$$
\begin{pmatrix}
1+\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & 0 \\
\vdots & \ddots & & \vdots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & 1+\alpha_{nn} & \varepsilon^{-\lambda} T
\end{pmatrix},
$$

where the α_{ik} converge to 0 for decreasing ε like ε itself (or are 0 themselves) and T is finite, that is, it does not grow beyond a certain boundary independent of the particular element of the sequence. How do the coefficients of the term Γ_2 depend on the α_{ik}? They are polynomial functions of the α_{ik}. If we let them tend to 0 in any way, then by definition of Γ_2, the coefficients of the orthogonal part of each term in this sequence converge to the corresponding coefficients of the identity transformation. So if we write the a term of Γ_2 as

$$
\begin{pmatrix}
1+\beta_{11} & \cdots & \beta_{1n} & B_1 \\
\vdots & \cdots & \vdots & \vdots \\
\beta_{n1} & \cdots & 1+\beta_{nn} & B_n
\end{pmatrix},
$$

then the β_{ik} are polynomial functions of the α_{ik} in which no term independent of the α_{ik} appears. The β_{ik} thus conerge to 0 like ε, but the B evidently do not tend to infinity faster than ε^{-2}, as no to terms tending to infinity are multiplied in forming the B_i and the number of terms used in forming them is independent of the particular element of the sequence.

Consider an element of Γ_3,

$$
\begin{pmatrix}
1+\gamma_{11} & \cdots & \gamma_{1n} & C_1 \\
\vdots & \cdots & \vdots & \vdots \\
\gamma_{n1} & \cdots & 1+\gamma_{nn} & C_n
\end{pmatrix}.
$$

The γ_{ik} are polynomial functions of the α_{ik} and β_{ik}. If we let one of of these two systems converge to 0, then the γ_{ik} converge to 0 as well. The γ_{ik}, as rational functions in the α, β, do not contain a term independent of the α and β. Hence the γ_{ik} converge to 0 like ε^2. Consider the C_i. They are polynomial functions of the α, β, $\varepsilon^{-2}T$, B_i. We let α and β converge to 0 simultaneously. Then the C_i also converge to 0. This implies that C_i contains no term independent of both α and β. Hence the C_i tend to infinity like $\varepsilon^{-\lambda+1}$. Now consider an element of Γ_4,

$$\begin{pmatrix} 1+\delta_{11} & \cdots & \delta_{1n} & D_1 \\ \vdots & \cdots & \vdots & \vdots \\ \delta_{n1} & \cdots & 1+\delta_{nn} & D_n \end{pmatrix}.$$

As before it follows that the δ_{ik} converge to 0 like ε^2. What about the D_i? Reasoning as for the C_i above would not yield anything new, so we need to apply a different argument. Let only the γ_{ik} converge to 0. Then

$$\begin{aligned}\Gamma_4 = &\begin{pmatrix} 1+\gamma_{11} & \cdots & \gamma_{1n} & C_1 \\ \vdots & \cdots & \vdots & \vdots \\ \gamma_{n1} & \cdots & 1+\gamma_{nn} & C_n \end{pmatrix} \cdot \begin{pmatrix} 1+\alpha_{11} & \cdots & \alpha 1n & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & 1+\alpha_{nn} & \varepsilon^{-1}T \end{pmatrix} \\ &\cdot \begin{pmatrix} 1+\gamma_{11} & \cdots & \gamma_{1n} & -C_1-\sum\gamma_{k1}C_k \\ \vdots & \cdots & \vdots & \vdots \\ \gamma_{n1} & \cdots & 1+\gamma_{nn} & -C_n-\sum\gamma_{kn}C_k \end{pmatrix} \cdot \begin{pmatrix} 1+\alpha_{11} & \cdots & \alpha 1n & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & 1+\alpha_{nn} & -\varepsilon^{-1}T \end{pmatrix}.\end{aligned}$$

Let (ε^{ϱ}) denote a term that behaves like ε^{ϱ}. If $\gamma_{ik} = 0$, then

$$\begin{aligned} D_1 &= -C_1 - (\varepsilon)(\varepsilon^{-\lambda+1}) + C_1, \\ &\vdots \\ D_n &= -C_n - \varepsilon^{-\lambda}T - (\varepsilon)(\varepsilon^{-\lambda+1}) + \varepsilon^{-\lambda}T + C_n, \end{aligned}$$

or

$$D_i = (\varepsilon^{-\lambda+1}).$$

If the γ_{ik} are not 0, then additional terms appear that behave like $\varepsilon^{-\lambda+2}$, since the γ_{ik} behave like ε^2. So be setting them to 0, we only lose terms that behave like $\varepsilon^{-\lambda+2}$. Hence the D_i behave like $\varepsilon^{-\lambda+2}$. In the same way we can conclude that the ε_{ik} for Γ_5 behave like ε^4, the E_i like $\varepsilon^{-\lambda+3}$. Finally, we arrive at the result that in Γ_n, the ν_{ik} behave like ε^{n-1} and the N_i like $\varepsilon^{-\lambda+n-2}$.

So if in $\Gamma_{\lambda+3}$ not all elements from a certain one onwards equal the identity, then $\Gamma_{\lambda+3}$ leads to infinitesimal substitutions.

But if the latter situation occurs for one Γ_i, then for all consecutive Γ_k, $k > i$, the same terms in the sequence equal the identity, so that our procedure does not lead to the desired outcome. This case will have to be treated seperately.

First, we will see that among the terms of Γ_2 and Γ_3, the identity cannot occur. Assume to the contrary that the identity occurs in Γ_2, say $\Gamma_2^{(h)}$ is an element for which this occurs, $\Gamma_2^{(h)} = 1$. Then, by definition of Γ_2,

$$\Gamma_1^{(h)} = \gamma_0 \Gamma_1^{(h)} \gamma_0^{-1}.$$

So the element γ_0 of Γ_0 would have to transform the element $\Gamma_1^{(h)}$ into itself. But we chose the γ_0 above precisely such that this is not the case ($ 5). Hence Γ_2 cannot contain the identity.

Now assume that Γ_3 contains the identity. Let $\Gamma_3^{(h)} = 1$. Then

$$\Gamma_2^{(h)} \Gamma_1^{(h)} (\Gamma_2^{(h)})^{-1} = \Gamma_1^{(h)}.$$

Then by §5,

$$\Gamma_2^{(h)} = \begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h'' \end{pmatrix}.$$

But from Theorem VIII in §7 it follows that $Q_h = 1$. Namely, $\Gamma_2^{(h)}$ and $\Gamma_1^{(h)}$ commute. Now suppose $Q_h \neq 1$. Then $\Gamma_2^{(h)} = \gamma_0 \Gamma_1^{(h)} \gamma_0^{-1} (\Gamma_1^{(h)})^{-1}$, hence $\Gamma_2^{(h)} \Gamma_1^{(h)}$ is a transform of $\Gamma_1^{(h)}$ and

$$\Gamma_2^{(h)} \Gamma_1^{(h)} = \begin{pmatrix} R_{2k} & 0 & 0 \\ 0 & Q_h & T_h''' \end{pmatrix}.$$

So if Q_h contained angles of rotation other than 0, then they would equal angles that appear for $\Gamma_1^{(h)}$, and would thus be irrational. In R_{2k}, the angle 0 would have to appear just as often as a non-zero one appears in Q_h. If

$$\Gamma_1^{(h)} = \begin{pmatrix} A_{2k} & 0 & 0 \\ 0 & 1 & T_h \end{pmatrix},$$

then A_{2k} and R_{2k} commute. Hence all prerequisites of Theorem VIII are satisfied. There would be an operation

$$\alpha = \begin{pmatrix} \alpha_{2k} & 0 & 0 \\ 0 & \beta_h & T_h \end{pmatrix}$$

in the group such that all angles of rotation of α_{2k} are irrational (and non-zero). Moreover, β_h, which is a power of Q_h, would contain non-zero irrational angles of rotation. Hence α would contain more irrational angles than the operation A from which Γ_1 is constructed. But above we chose A as that operation in the group with the maximal number of irrational rotation angles. Our assumption that Q_h is different from the identity thus leads to a contradiction. Hence $Q_h = 1$. But now

$$\Gamma_2^{(h)} \Gamma_1^{(h)} = \gamma_0 \Gamma_1^{(a)} \gamma_0^{-1}.$$

But then, again by Theorem VI (§5), γ_0 is of the following form:

$$\begin{pmatrix} F_{2k} & 0 & 0 \\ 0 & G_h & T_h \end{pmatrix}.$$

But we chose γ_0 precisely such that this is not the case. So our assumption that Γ_3 contains the identity leads to a contradiction. Hence Γ_3 cannot contain the identity.

In general, we cannot conclude in an analogous manner that Γ_n, $n > 3$, does not contain the identity. We have to slightly modify our reasoning.

First, assume that Γ_k, $k \geq 6$, leads to the identity, that is, from a certain term onward, all terms of Γ_k equal the identity. Further, assume no Γ_i with $i < k$ leads to the identity. To express that Γ_k leads to the identity, we shall write $\Gamma_k = 1$. We claim that *in this case, $\Gamma_{k-1}\Gamma_{k-2}^{-1}$ leads to infinitesimal operations.* Namely,

$$1 = \Gamma_k = \Gamma_{k-1}\Gamma_1\Gamma_{k-1}^{-1}\Gamma_1^{-1}.$$

This implies that all elements of Γ_{k-1} are of the following form:

$$\Gamma_{k-1} = \begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix},$$

and with Theorem VIII (§7) it follows by an argument similar to the one for Γ_3 above, that $Q_h = 1$. It also follows that Γ_{k-2} is of the form just given (where Q_h is not necessarily the identity). But now

$$\Gamma_{k-1} = \Gamma_{k-2}\Gamma_1\Gamma_{k-2}^{-1}\Gamma_1^{-1}$$

and

$$\Gamma_{k-2} = \Gamma_{k-3}\Gamma_1\Gamma_{k-3}^{-1}\Gamma_1^{-1}.$$

So $\Gamma_{k-1}\Gamma_1$ and $\Gamma_{k-2}\Gamma_1$ are transforms of Γ_1, and thus of each other. Namely,

$$\Gamma_{k-1}\Gamma_1 = \Gamma_{k-2}\Gamma_{k-3}^{-1}(\Gamma_{k-2}\Gamma_1)\Gamma_{k-3}\Gamma_{k-2}^{-1}. \qquad (*)$$

Now $\Gamma_{k-2}\Gamma_{k-3}^{-1}$ evidently leads to infinitesimal operations in its orthogonal part, because this is the case for Γ_{k-2} and Γ_{k-3}. But $\Gamma_{k-1}\Gamma_1$ is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T \end{pmatrix}.$$

Therefore, after omitting certain initial terms in our sequence, application of Theorem IX (§7) yields that $\Gamma_{k-2}\Gamma_1$ is also of the given form for $\Gamma_{k-1}\Gamma_1$. For, in regard to the given form of Γ_{k-2}, clearly $\Gamma_{k-2}\Gamma_1$ is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

If Q_h contained any angles of rotation other than 0, then they would have to occur in Γ_1 as well, in light of $(*)$. Accordingly, $\Gamma_{k-2}\Gamma_{k-3}^{-1}$ would transform certain points in the space $x_{2k+1} = \ldots = x_n = 0$ into points in the space $x_1 = \ldots = x_{2k} = 0$. But this is impossible by Theorem IX. So Q_h in Γ_{k-2} must equal the identity. Considering the definition of Γ_{k-2} it would follow that Γ_{k-3} is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

If $k \geq 6$, we can apply the same reasoning here and prove that $Q_h = 1$. In summary: It has been shown that $\Gamma_{k-1}\Gamma_1$, $\Gamma_{k-2}\Gamma_1$, Γ_{k-2}, Γ_{k-3}, and thus $\Gamma_{k-2}\Gamma_{k-3}^{-1}$ are all of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T \end{pmatrix}.$$

But this implies that the translational components of $\Gamma_{k-1}\Gamma_1$ and $\Gamma_{k-2}\Gamma_1$ must coincide by $(*)$. Hence the sequence

$$\Gamma_{k-1}\Gamma_1 \cdot \Gamma_1^{-1}\Gamma_{k-2}^{-1} = \Gamma_{k-1}\Gamma_{k-2}^{-1}$$

consists of pure rotations. It leads to an infinitesimal operation as soon as we know that $\Gamma_{k-1} \neq \Gamma_{k-2}$. Now,

$$\Gamma_{k-1}\Gamma_1 = \Gamma_{k-2}\Gamma_1\Gamma_{k-2}^{-1}.$$

So if we assume $\Gamma_{k-1} = \Gamma_{k-2}$, then $\Gamma_{k-2} = 1$ follows. But we made the assumption that no $\Gamma_i = 1$ for $i < k$. Hence $\Gamma_{k-1} = \Gamma_{k-2}$ is impossible, and thus $\Gamma_{k-1}\Gamma_{k-2}^{-1}$ leads to infinitesimal operations.

Now it remains to investigate the two case $\Gamma_4 = 1$ and $\Gamma_5 = 1$. First, consider $\Gamma_4 = 1$. Then form

$$\Gamma_4' = \Gamma_3\Gamma_1\Gamma_2 \cdot \Gamma_3^{-1} \cdot \Gamma_3^{-1}\Gamma_1^{-1}\Gamma_2^{-1}.$$

We claim that *if Γ_2 and Γ_3 do not commute, then $\Gamma_4'\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_1^{-1}$ is infinitesimal.* Γ_1 and Γ_3 are of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 0 & T_h \end{pmatrix},$$

and Γ_2 is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

Hence

$$\Gamma_4'\Gamma_2\Gamma_1 = \begin{pmatrix} P_{2k}' & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}$$

and

$$\Gamma_1\Gamma_2 = \begin{pmatrix} P_{2k}'' & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix},$$

as both arise from transformations of Γ_3 from one another and have identical Q_h and T_h. Hence

$$\Gamma_4'\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_1^{-1} = \begin{pmatrix} P_{2k}''' & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is a pure rotation and infinitesimal if it is not the identity. But if we assume

$$\Gamma_4'\Gamma_2\Gamma_1 = \Gamma_1\Gamma_2,$$

then by

$$\Gamma_4'\Gamma_2\Gamma_1 = \Gamma_3\Gamma_1\Gamma_2\Gamma_3^{-1},$$

Γ_3 would transform the sequence $\Gamma_1\Gamma_2$ into itself, so that

$$\Gamma_3\Gamma_1\Gamma_2\Gamma_3^{-1} = \Gamma_1\Gamma_3\Gamma_2\Gamma_3^{-1} = \Gamma_1\Gamma_2,$$

as Γ_1 and Γ_3 commute due to $\Gamma_4 = 1$. But then Γ_2 and Γ_3 commute, as claimed. But if Γ_2 and Γ_3, as well as Γ_1 and Γ_3, commute, then Γ_3, which never equals the identity, is already infinitesimal. To demonstrate this, first transform the whole group, in particular Γ_2, Γ_3, Γ_1, such that Γ_2 assumes normal form. If we only write the last h columns that only contain translations and we denote the sequences of the thus truncated substitutions by Γ_2', Γ_3', Γ_1', then:

$$\Gamma_2' = A_1|A_2|\ldots|A_\nu|\begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & A \end{pmatrix}$$

$$\Gamma_3' = \begin{pmatrix} 1 & \cdots & 0 & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & B_h \end{pmatrix},$$

$$\Gamma_1' = \begin{pmatrix} 1 & \cdots & 0 & C_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & C_h \end{pmatrix}.$$

Now Γ_2' is assumed to commute with Γ_3'. The A_i merely contain irrational angles of rotation, as $\Gamma_2\Gamma_1$ is a transform of Γ_1. This implies (Theorem VI, §5)

$$B_1 = \ldots = B_{2\nu} = 0.$$

Now assume further that Γ_2' transforms the sequence Γ_1' into the sequence $\Gamma_3'\Gamma_1'$. But $\Gamma_3'\Gamma_1'$ is a translation with components

$$C_1, \ldots, C_{2\nu}, C_{2\nu+1} + B_{2\nu+1}, \ldots, C_h + B_h,$$

whereas the translation components of Γ_1' are

$$C_1, \ldots, C_{2\nu}, C_{2\nu+1}, \ldots, C_h.$$

Upon forming $\Gamma_2'\Gamma_1'(\Gamma_2')^{-1}$ and denoting the angles of rotation of the A_i by ϑ_i, the translation components become

$$C_1\cos(\vartheta_i 2\pi) - C_1\sin(\vartheta_i 2\pi), \ldots, C_{2\nu-1}\sin(\vartheta_i 2\pi) + C_\nu\cos(\vartheta_i 2\pi), C_{2\nu+1}, \ldots, C_h.$$

If these components are supposed to equal those of Γ_1', then it follows that

$$C_1 = \ldots = C_{2\nu} = 0.$$

Then Γ_3 is a rotation and infinitesimal, as it cannot be the identity. This settles the case $\Gamma_4 = 1$.

We now consider the case $\Gamma_5 = 1$. So suppose $\Gamma_5 = 1$. We have

$$\Gamma_4 = \Gamma_3 \Gamma_1 \Gamma_3^{-1} \Gamma_1^{-1},$$
$$\Gamma_3 = \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_1^{-1},$$
$$\Gamma_2 = \Gamma_0 \Gamma_1 \Gamma_0^{-1} \Gamma_1^{-1}.$$

If we form

$$\Gamma_5' = \Gamma_3 \cdot \Gamma_4 \Gamma_1 \Gamma_3^{-1} \Gamma_1^{-1} \Gamma_4^{-1},$$

then this sequence is infinitesimal: It cannot be that $\Gamma_5' = 1$, for then

$$\Gamma_3 \Gamma_4 \Gamma_1 = \Gamma_4 \Gamma_1 \Gamma_3.$$

Now

$$\Gamma_4 \Gamma_1 \Gamma_3 = \Gamma_3 \Gamma_1,$$

so

$$\Gamma_3 \Gamma_4 \Gamma_1 = \Gamma_3 \Gamma_1$$

and therefore

$$\Gamma_4 = 1.$$

But this case has already been settled. We may thus assume $\Gamma_5' \neq 1$. Now $\Gamma_5' \Gamma_4 \Gamma_1$ is a transform of $\Gamma_4 \Gamma_1$ via Γ_3. But $\Gamma_5' \Gamma_4 \Gamma_1$ and $\Gamma_4 \Gamma_1$ as well as Γ_3 are of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T_h \end{pmatrix}.$$

Namely, $\Gamma_4 \Gamma_1$ is a transform of Γ_1 via Γ_3. As Γ_4 commutes with Γ_1, it is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

Hence $\Gamma_4 \Gamma_1$ is of this form as well. By this is a transform of Γ_1 by Γ_3. Therefore, by Theorem IX (§7), since Γ_3 is infinitesimal in its orthogonal part, $\Gamma_4 \Gamma_1$ is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T_h \end{pmatrix}$$

and thus Γ_3 (Theorem VI, §5) is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

But $\Gamma_3 \Gamma_1$ is a transform of Γ_1 by Γ_2. As Γ_2 is infinitesimal in its orthogonal part, again by Theorem IX, Γ_3 is of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T_h \end{pmatrix}.$$

Then, by definition of Γ_5', the sequence $\Gamma_5' \Gamma_4 \Gamma_1$ and the sequence $\Gamma_4 \Gamma_1$ are both of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & 1 & T_h \end{pmatrix}.$$

As Γ_3 is of this form as well, and as $\Gamma_5' \Gamma_4 \Gamma_1$ arises by transformation with Γ_3 from $\Gamma_4 \Gamma_1$, the translation parts of $\Gamma_5' \Gamma_4 \Gamma_1$ and $\Gamma_4 \Gamma_1$ coincide. Therefore,

$$\Gamma_5' \Gamma_4 \Gamma_1 \Gamma_1^{-1} \Gamma_4^{-1} = \Gamma_5'$$

is infinitesimal, since, as we just saw, $\Gamma_5 \neq 1$.

In this way we have also shown that in case $\Gamma_5 \neq$, the group contains infinitesimal operations if it is indecomposable. We have thus proved the theorem *that a group of motions with fundamental domain that contains operations with irrational angles of rotation is necessarily decomposable.*

9 Groups of orthogonal substitutions

Our next goal is to prove the theorem stating that infinite groups with infinite fundamental domain are alwys decomposable, and vice versa. To get there, we will first prove an auxiliary theorem here. We wish to investigate those groups of motions containing only orthogonal substitutions, that is, motions with a fixed point. If there is a point fixed by all operations in the group, then this group can be written as a homogeneous group of orthogonal substitutions. If it is finite, then it is easy to see that it contains infinitesimal operations. This follows readily from the fact that in this situation all coefficients lie between -1 and $+1$ and thus the group must contain two arbitrarily close operations. Their composition leads to infinitesimal operations. Now the next theorem is:

XI *Every finite group of motions can be written as a homogeneous group, that is, it can be transformed into this form by suitable linear subsitutions.*

There is no problem is adding an $n+1$st column to the operations in our group, so that it can be written as

$$\begin{pmatrix} \alpha_{11}^{(h)} & \cdots & \alpha_{1n}^{(h)} & A_1^{(h)} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1}^{(h)} & \cdots & \alpha_{nn}^{(h)} & A_n^{(h)} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad (h = 1, \dots, H)$$

where H is the order of the group. By a theorem of Maschke,[8)] our group can be brought into the form

$$\begin{pmatrix} \alpha_{11}^{(h)} & \cdots & \alpha_{1n}^{(h)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1}^{(h)} & \cdots & \alpha_{nn}^{(h)} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

by a transformation of the form

$$\begin{pmatrix} B_n & T_n \\ 0 & 1 \end{pmatrix}.$$

So now there are points $x_1 = \ldots = x_n = 0$ mapped to themselves by all operations in the group. Therefore, the operations in the original group also map certain points to themselves. But then this group can be written homogeneously. This proves Theorem XI.

Now we come to the next theorem:

XII *An infinite group G consisting of orthogonal substitutions contains infinitesimal operations.*

The case of a homogeneous group was treated before. To prove the theorem for an inhomogeneous group, we first assume the following auxiliary theorem. *Let*

$$A = \begin{pmatrix} 1+\alpha_{11} & \cdots & \alpha_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1} & \cdots & 1+\alpha_{nn} & 0 \end{pmatrix}$$

be an orthogonal substitution and

$$B = \begin{pmatrix} 1+\beta_{11} & \cdots & \beta_{1n} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ \beta_{n1} & \cdots & 1+\beta_{nn} & B_n \end{pmatrix}$$

[8)]Maschke, Math. Ann. 52, p. 363.

a motion with $\sum B_i^2 = b$*. Let*

$$C_1, \ldots, C_n$$

denote the components of the translation part of $C = BAB^{-1}A^{-1}$*. Then there exists a number* $\lambda < 1$ *such that* $\sum_{i=1}^{n} C_i^2 < b$ *as soon as all* $|\alpha_{ik}| < \lambda$*, and a number* η *such that the orthogonal part of* C *is* λ *times closer to the identity than the orthogonal part of* A *as soon as* $|\beta_{ik}| < \eta$*, that is, that the largest* $|\gamma_{ik}|$ *of* C *is smaller than the* λ*-fold multiple of the largest* $|\alpha_{ik}|$*.*

Namely,

$$\sum_{i=1}^{n} C_i^2 = \sum_{h=1}^{n}\Big(\sum_{i=1}^{n} \alpha_{ih} \sum_{k=1}^{n} \beta_{ik} B_k\Big)^2.$$

Let all $|\alpha_{ik}| < \varepsilon$, where a suitable ε will be chosen later. Then

$$\sum_{i=1}^{n} C_i^2 < n\varepsilon^2\Big(\sum_{i=1}^{n}\sum_{k=1}^{n} \beta_{ik} B_k\Big)^2$$

If m is the number of terms in this square, then $\sum C_i^2 < nm\varepsilon^2 b$. If we choose $\varepsilon < \frac{1}{\sqrt{nmb}}$, then $\sum C_i^2 < b$.

The second part of the auxiliary theorem has a very similar proof, which we omit here.

Using this auxiliary theorem, we will prove Theorem XII. As the group is infinite, we can find at least one operation in it whose coefficients in the orthogonal part all differ by less than λ above from the identity substitution. Consider the totality Γ of operations of this kind. Different cases are possible now:

I. The points fixed by A are invariant under all operations in the group. Then all operations are of the form

$$\begin{pmatrix} P_{2k} & 0 & 0 \\ 0 & Q_h & T_h \end{pmatrix}.$$

If the group is supposed to be infinite, then either the group formed by the first k rows is infinite and the group formed by the last h rows is finite, or the first one is finite and the second one is infinite, or both are infinite. In the first case we can immediately deduce the existence of infinitesimal operations, in the second and third case we can do so if we assume our theorem proved for the case of less than n variables. For binary groups it can be proved by repeating our argument and if at last there are only translation parts in the last column left, we can immediately deduce the existence of translations of screw-motions. This contradicts our assumptions.

II. There are operations in Γ that do not preserve the points fixed by A.

1. For every B with sum of squares $\sum B_i^2 = 1$ there exists an operation with smaller sum of squares. Then there is a sequence of operations such that the sums of squares of its translation parts converge to a certain value. But then there must exist pairs of operations in the group all of whose coefficients differ by an arbitrarily small quantity. This means the group contains infinitesimal transformations.

2. Among all operations of Γ that do not preserve the points fixed by A, there is one, say B, for which $\sum B_i^2$ assumes the smallest possible value b. By our auxiliary theorem there exists an operation $C^{(1)}$ with $\sum C_i^2 < b$. But this can only be 0. If now (a) $C^{(1)} = 1$, then it follows from Theorem VI (§5) that B preserves the points fixed by A, in contradiction to our assumption. Hence (b) $C^{(1)} \neq 1$ and it is λ times as close to the identity as A. We proceed with $C^{(1)}$ exactly as with A above and return to case I or II 1. or to an operation $C^{(2)}$ that is λ^2 times as close to the identity as A. Continuing this way eventually leads to infinitesimal operations.

From Theorem XII it follows immediately:

XIII *An infinite group of motions with fundamental domain that does not contain any operation with an irrational angle of rotation necessarily contains translations.*

For all angles of rotations are rational and the group cannot consist of rotations alone, so that by taking powers of suitable operations we obtain pure translations.

This Theorem XIII forms the foundation of all further investigations and of the second treatise.

10 Distinction of the groups with infinite fundamental domain from those with a finite one

By Theorem X (§8) it is proved that every group containing operations with irrational angles of rotations is decomposable. By Theorems XI, XII and XIII (§9) it is proved that every group of motions without infinitesimal operations that consists only of orthogonal substitutions is finite and homogeneous, that is, every infinite group of motions without irrational angles of rotations has translations among its operations.

The translations contained in a group of motions form a distinguished subgroup. If all translations transform the space $x_1 = \dots x_k = 0$ into itself, then it readily follows that the group formed by the orthogonal parts of our group also transforms this space into itself. All operations in the group are thus of the form

$$\begin{pmatrix} A_k & 0 & T_k \\ 0 & B_h & T_h \end{pmatrix}.$$

But the following theorem holds, by which the T_k are zero:

XIV *If all translations in a group of motions of n-dimensional space are contained in a linear subspace R_k, $k < n$, then the group is decomposable.*

If operations with irrational angles of rotation appear in the group, then by Theorem X (§8) the proof of this theorem can be reduced to the proof of the analogous theorem for groups in less than n variables consisting only of operations with rational angles of rotation. We may thus restrict ourselves to groups of the latter type to begin with.

If the group of the first k rows is finite, then Theorem XIV follows from Theorem X. We may thus assume that it is infinite.

To prove the theorem, we distinguish several cases. Consider the totality of operations that equal the identity in their last h rows. Those form a group Γ. It thus contains no translations. If it was infinite, then it would contain infinitesimal operations by Theorem XII. But then our original group would contain infinitesimal operations. Therefore, Γ has to be finite. By a suitable transformation of the whole group, we may thus achieve, by Theorem XI, that its translational components are all 0. But the group Γ is a distinguished subgroup in our original group. Depending on its type we distinguish several cases. Either, the only point fixed by Γ in its first k rows is the origin. Then all other operations in the group must fix this point, and Theorem XIV is proved. Or there are other points fixed by Γ. Then our theorem is reduced to the analogous theorem in less than n dimensions, where Γ does not consist of the identity alone (Theorem VI, §5). Namely, in the case just considered, all operations in the group are of the form

$$\begin{pmatrix} A_{k-\mu} & 0 & 0 & 0 \\ 0 & A_\mu & 0 & T_\mu \\ 0 & 0 & A_h & T_h \end{pmatrix}.$$

The last $\mu + h$ rows form a group. If we show, that the impossibility to write this latter group such that all T_μ vanish implies the existence of infinitesimal operations, then clearly our original group contains infinitesimal operations. We may thus apply the same reasoning to the group formed by the last $h+\mu$ rows as before for the group in n variables. This works until we find a group for which it holds that if one element equals the identity in the last h rows, all other rows equal the identity. This last case still needs to be investigated.

If then $A^{(1)}$ is an element of the first kind in the group whose last h rows equal the identity, and $A^{(0)}$ is an arbitrary element in the group, then

$$A^{(3)} = A^{(2)}A^{(1)}A^{(2)-1}A^{(1)-1},$$

with $A^{(2)} = A^{(0)}A^{(1)}A^{(0)-1}A^{(1)-1}$, equals the identity in the last h rows. Therefore, in the case under consideration, $A^{(3)} = 1$ for every element $A^{(0)}$ in the group. We will now draw a conclusion from this. Let

$$A^{(1)} = \begin{pmatrix} A_{k-\nu} & 0 & 0 & 0 \\ 0 & 1_\nu & 0 & 0 \\ 0 & 0 & 1_h & T_h \end{pmatrix},$$

where $A_{k-\nu}$ has no fixed points other than $x_1 = \ldots = x_{k-\nu} = 0$. Then since $A^{(3)} = 1$ by Theorem VI (§5), $A^{(2)}$ is of the form

$$A^{(2)} = \begin{pmatrix} B_{k-\nu} & 0 & 0 & 0 \\ 0 & C_\nu & 0 & T'_\nu \\ 0 & 0 & D_h & T''_h \end{pmatrix}. \qquad (*)$$

But now $A^{(2)}A^{(1)}$ is a transform of $A^{(1)}$. Hence the angles of rotation appearing in C_ν are angles that also appear in $A^{(2)}$. If we choose $A^{(0)}$ such that the coefficients of its orthogonal part in the first k rows differs from the those of the identity by less than the quantity stated in Theorem IX (§7), then $C_\nu = 1$ by this theorem, and hence $T'_\nu = 0$, since otherwise the rationality of the angles of rotation would imply the existence of translations not contained in the subspace $x_1 = \ldots = x_k = 0$. But then $A^{(0)}$ must be of the form for $A^{(2)}$ given in $(*)$. So we see that there is a number λ such that all operations in the group whose orthogonal parts in the first k rows differ from the identity by less than λ must be of the given form. Consider now all operations of the group that have this form. These form a subgroup Φ. Consider the group Φ' formed by the last $h+\nu$ rows. If infinitesimal transformations appear here, then the original group also contains infinitesimal transformations. Now we can apply the same reasoning to Φ'. This leads to the result that all operations in the group whose coefficients in the orthogonal parts in the first k rows differ from the identity by less than λ have the following form:

$$\begin{pmatrix} A_k & 0 & 0 \\ 0 & R_h & T_h \end{pmatrix}.$$

Now we can choose a second quantity $\lambda' < \lambda$ small enough such that all operations in the group obtained by transformations from an operation in the group, whose coefficients in the orthogonal part all differ from the identity by less than λ', are themselves contained in the set of substitutions belonging to λ, that is, all of their cofficients differ by less than λ from the identity matrix. We observe that such a quantity λ' exists and depends only on λ and n. Now consider the substitutions formed by the first k rows of the set belonging to λ'. If all of these substitutions fix only the point $x_1 = \ldots = x_k = 0$, then all operations in the group have form given above for the set belonging to λ. Then Theorem XIV is proved. But if all of them fix additional points, then by a common argument and due to Theorem VI (§5) the proof of the theorem is reduced to the proof of the analogous theorem for groups in less than n variables. For if the manifold $x_1 = \ldots = x_{k-\nu} = 0$ is fixed pointwise, then we only need to consider the group formed by the last $h + \nu$ rows. If in the latter we can identify a sequence leading to infinitesimal operations, then also the original group contains infinitesimal operations, since the first h rows do not have a translational component and all coefficients lie between -1 and $+1$, so that all operations whose coefficients in the first h rows differ by arbitrarily small amounts appear in the sequence. Thus follows the existence of infinitesimal operations.

We can the apply the same reasoning to the group in less than n variables, until either the first k rows are exhausted or until a finite group appears, which can be transformed into a homogeneous one by Theorem XI (§8). This concludes the proof of Theorem XIV.

From Theorem XIV we draw an important conclusion:

XV *Given a group without infinitesimal operations, a necessary and sufficient condition for the existence of a finite fundamental domain is that the group is not decomposable. By Theorem XIV a group with finite fundamental domain always contains a translation subgroup whose operations do not all transform a linear subspace of dimension less than n into itself.*

First, we show that the fundamental domain of a decomposable group necessarily extends to infinity. Namely, the projection of a point in our space onto the linear space $x_{k+1} = \ldots = x_n = 0$ has constant distance from the origin for all operations in our group (the length of this projection is $\sum_{i=1}^{k} x_i^2$). But since by §6, the fundamental domain must contain an equivalent point for each point in space, the fundamental domain contains points whose projection onto said space can have arbitrarily large distance from the origin. Hence the fundamental domain must extend to infinity. So the condition in the theorem is necessary. It is also sufficient: If a group has a finite fundamental domain, then, firstly, it cannot contain irrational angles of rotation, for it is decomposable (Theorem X, §8). Secondly, it cannot consist only of orthogonal subsititions, since then it is finite and homogeneous and thus has infinite fundamental domain. Thirdly, it cannot contain a subgroup of translations with fewer than n linearly independent translations, for then it would be decomposable (Theorem XIV). Hence it must contain a subgroup of translations with n linearly independent translations. But such a subgroup has finite fundamental domain. This we see immediately if we think of n linearly independent directions of translations as coordinate axes. For then every point in space has an equivalent one contained in the parallelepiped whose edge lengths are the lengths of the shortest translations in the directions of these coordinate axes. This proves Theorem XV.

11 Concluding remarks on groups with infinite fundamental domain

The result of our investigations can be summarized as the infinite groups with infinite fundamental domain being necessarily decomposable, and the decomposable groups, if they do not contain infinitesimal operations, having infinite fundamental domain. We wish to further characterize this decomposability. First, consider the groups whose operations only have rational angles of rotation. We saw that these groups always contain a subgroup of translations with i linearly independent translations, and that they can be brought into the form

$$\begin{pmatrix} A_l & 0 & 0 \\ 0 & A_i & T_i \end{pmatrix}.$$

Those groups whose operations also contain irrational angles of rotation do not always contain a subgroup of translations. But if the maximal number of irrational angles of rotation appearing in one operation is $\frac{k}{2}$, then, as we have seen, the group can be brought into the following form:

$$\begin{pmatrix} A_k & 0 & 0 \\ 0 & A_h & T_h \end{pmatrix}.$$

The group generated by the last h rows is necessarily infinite. Otherwise, it could be transformed into a homogeneous group. Then the whole group would be of the form

$$\begin{pmatrix} A_l & 0 & 0 \\ 0 & A_h & 0 \end{pmatrix}.$$

But then it would contain infinitesimal operations due to the existence of irrational angles of rotation. Thus the group formed by the last h rows must be infinite. But it cannot contain infinitesimal operations, since then we could once more deduce the existence of infinitesimal operations in the original group. We may assume that it does not contain any operations with irrational angles of rotations; otherwise, we can apply Theorem X repeatedly until this is the case. The group formed by the last h rows is thus a group of translations of, say, i linearly independent translations. So the whole group can be brought into the following form:

$$\begin{pmatrix} A_l & 0 & 0 \\ 0 & A_i & T_i \end{pmatrix}.$$

This is as much as we wish to say on the form of decomposability. Now we turn our attention to the necessary and sufficient conditions for the existence of a fundamental domain. In addition to the above form that the group can be transformed into, these are the following:

1. The group formed by the last i rows is a group with finite fundamental domain.

The second condition is obtained if we take into account that the two groups respectively formed by the first $n-i$ rows and the last i rows are isomorphic. Then:

2. The subgroup of the first group corresponding to the identity element of the second group is a finite group.

In the introductory remarks in this paragraph we already explained that the first condition is necessary. That the second condition is necessary follows from the fact that the group mentioned in it has a fundamental domain, being a subgroup of a group with fundamental domain, and as the group is homogeneous, this is only possible if the group is finite.

We will now show that the given conditions are sufficient. For this, we only need to show that under these conditions, no infinitesimal operations can appear in the group. This can be seen as follows: Consider an arbitrary point. Then accumulations will occur among the coordinates $x_1, \ldots, x_l$ of the equivalent points. But only a finite number among them correspond to the same $x_{l+1}, \ldots, x_n$. On the other hand, among the $x_{l+1}, \ldots, x_n$ alone there are no accumulations, as the group has finite fundamental domain here. So the projection of the distance of two equivalent points to the linear space $x_2 = \ldots = x_l = 0$ has a lower bound greater than 0. So accumulations can only occur among points whose $x_{l+1}, \ldots, x_n$ coincide. But this is impossible, since to any given $x_{l+1}, \ldots, x_n$ correspond only finitely many $x_1, \ldots, x_l$. This concludes our investigations.

Part II

Groups with finite fundamental domain

1 Introduction

This article continues the investigations on groups of motions in Euclidean space, part I, that appeared in volume 70 of the Mathematische Annalen. In the meantime, those results were derived in a simplified manner by Frobenius in a work on indecomposable discrete groups of motions that appeared in the Sitzungsberichte der Berliner Akademie, 1911. The following pages are concerned exclusively with groups of motions with finite fundamental domain. As already suggested in part I of this treatise and explained in Frobenius's work, only finitely many of such groups exist.

By a *motion* of n-dimensional Euclidean space we mean a real linear subsitution of n variables,

$$x_i' = \sum_{k=1}^{n} a_{ik} x_k + A_i \quad (i = 1, \ldots, n),$$

where the homogeneous linear substitution obtained by setting A_i to 0 is an orthogonal substitution, which, following Frobenius, we shall call the *rotational part* of the motion. The substitution

$$x_i' = x_i + A_i \quad (i = 1, \ldots, n)$$

represents the *translational part* of the motion. If $\mathfrak{R}$ denotes the rotational and $\mathfrak{T}$ the translational part of a motion $\mathfrak{B}$, then in matrix calculus we can represent the latter as the product $\mathfrak{B} = \mathfrak{T}\mathfrak{R}$. A motion that coincides with its rotational part is called a *rotation*, a motion that coincides with its translational part is called a *translation*. A group composed of such motions is called a *group of motions*. The requirement that this group has a fundamental domain is equivalent to the requirement that it does not contain infinitesimal operations. We observed this in §6 in the first part of this treatise. Depending on whether the fundamental domain of the group extends to infinity or not, two types of groups of motions can be distinguished. The groups with infinite fundamental domain are always decomposable, and all decomposable groups have infinite fundamental domain, provided they have an at all. Here, we called a group *decomposable* if for a suitable choice of coordinates its elements take the form

$$x_i' = \sum_{k=1}^{h} a_{ik}^{(\nu)} x_k \quad (i = 1, \ldots, h),$$
$$x_\mu' = \sum_{\tau=h+1}^{n} a_{\mu\tau}^{(\nu)} x_\tau + A_\mu \quad (\mu = h+1, \ldots, n).$$

We further observed that the groups with finite fundamental domain always contain translations. It is easy to see that these translations form a distinguished subgroup of the group of motions. Namely, of $\mathfrak{T}$ is a translation and $\mathfrak{B}$ an arbitrary motion, then $\mathfrak{B}\mathfrak{T}\mathfrak{B}^{-1}$ is a translation. If the fundamental domain is finite, then the group contains n linearly independent translations, that is, there are n translations

$$\mathfrak{T}_i \equiv x_i' = x_i + A_i^{(\nu)} \quad (i, \nu = 1, \ldots, n)$$

in the group, such that there are no numbers $a_1, \ldots, a_n$ (other than 0) satisfying the relations (Math. Ann. 70, p. 333)

$$\sum_{\nu=1}^{n} a_\nu A_i^{(\nu)} \quad (i = 1, \ldots, n).$$

These results from the first treatise form the basis for the following.

Two groups of motions are considered distinct if they are not equivalent. Two groups are called *equivalent*, if they arise from one another by a linear change of coordinates. Then, as we will show, the following theorem holds:

There are only finitely many distinct groups of motions with finite fundamental domain.

This notion of equivalence is alreadu suggested by crystallography and the theorem was also proved in this form by Frobenius. In my own note in the Göttinger Nachrichten I considered two groups identical if they are isomorphic. In fact, this amounts to the same. The theorem holds (which was not stated back then) that in the sense of the above definition, *two isomorphic groups are equivalent.*

Our proof makes use of methods that extend ideas used by Gauß and Dirichlet in the theory of ternary forms, and by Minkowski in the theory of positive quadratic forms in n variables. In order to give a self-contained presentation and to emphasize the relation to Minkowski's theory, we will in some places discuss some well-known facts anew. The idea of the proof was sketched in the Göttinger Nachrichten 1910.

2 The translation subgroup

As we saw before, the distinguished subgroup of translations always contains n linearly independent translations. Let these be

$$\mathfrak{A}_i \;\equiv\; x'_k = x_k + A_k^{(k)} \quad (k, i = 1, \ldots, n),$$

then there are no non-zero numbers $a_1, \ldots, a_n$ such that the following relations hold:

$$\sum_{i=1}^{n} a_i A_k^{(i)} = 0 \quad (k = 1, \ldots, n).$$

Translation groups of this type are evidently obtained by choosing any n linearly independent translations $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$ and forming a group by taking any linear combination of them. But now we wish to prove that in any translation group, we can find n linearly independent translations $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$ such that the group can be generated by them in this manner, so that every other translation $\mathfrak{T}$ is of the form

$$\mathfrak{T} = \mathfrak{T}_1^{t_1} \cdots \mathfrak{T}_n^{t_n},$$

where the $t_1, \ldots, t_n$ are integer numbers. This can be seen as follows (cf. for example Minkowski, *Geometrie der Zahlen*, p. 172): Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be any n linearly independent translations in the group. Then the components $T_1, \ldots, T_n$ of any translation $\mathfrak{T}$ can be represented as

$$T_i = t_1 A_i^{(1)} + \ldots + t_n A_i^{(n)} \quad (i = 1, \ldots, n).$$

Symbolically, we write

$$\mathfrak{T} = \mathfrak{A}_1^{t_1} \cdots \mathfrak{A}_n^{t_n}.$$

Here, in general the t_i are not integers. This is only the case if the $\mathfrak{A}_i$ are generators of the group. Of the two translations

$$\mathfrak{T}_1 = \mathfrak{A}_1^{a_1} \cdots \mathfrak{A}_n^{a_n}$$

and

$$\mathfrak{T}_2 = \mathfrak{A}_1^{b_1} \cdots \mathfrak{A}_n^{b_n}$$

we call the first one *smaller* than the the second one if the first non-vanishing difference among $a_n - b_n$, $a_{n-1} - b_{n-1}$, …, $a_1 - b_1$ is negative. Moreover, we consider the totality of all translations $\mathfrak{T}$ in the group for which in the representation

$$\mathfrak{T} = \mathfrak{A}_1^{t_1} \cdots \mathfrak{A}_n^{t_n}$$

all numbers $t_1, \ldots, t_n$ are positive and less or equal to 1. There are at least n such translations, as the $\mathfrak{A}_i$ themselves belong to them. But there are also only finitely many such translations, since otherwise the group would contain infinitesimal operations. So there is a smallest one among these translations. We will call it $\mathfrak{T}_1$. It is necessarily of the form $\mathfrak{A}_1^{t_{11}}$. Among all translations that are not of the form $\mathfrak{A}_1^{a_1}$, there again exists a smallest one. We will call it $\mathfrak{T}_2$, and it is necessarily of the form $\mathfrak{A}_1^{t_{12}}\mathfrak{A}_2^{t_{22}}$. Among all translations that are not of the form $\mathfrak{A}_1^{a_1}\mathfrak{A}_2^{a_2}$, there again exists a smallest one. We will call it $\mathfrak{T}_3$, and it is necessarily of the form $\mathfrak{A}_1^{t_{13}}\mathfrak{A}_2^{t_{23}}\mathfrak{A}_3^{t_{33}}$. Continuing in this way, we evidently obtain n linearly independent translations:

$$\mathfrak{T}_1 = \mathfrak{A}_1^{t_{11}},\ \mathfrak{T}_2 = \mathfrak{A}_1^{t_{12}}\mathfrak{A}_2^{t_{22}},\ \ldots,\ \mathfrak{T}_n = \mathfrak{A}_1^{t_{1n}} \cdots \mathfrak{A}_n^{t_{nn}}.$$

With these we can now represent any other translation

$$\mathfrak{B} = \mathfrak{A}_1^{b_1} \cdots \mathfrak{A}_n^{b_n}$$

in the form $\mathfrak{T}_1^{\beta_1} \cdots \mathfrak{T}_n^{\beta_n}$, where the $\beta_1, \ldots, \beta_n$ are integers: The integers $\beta_1, \ldots, \beta_n$ can be determined such that

$$\mathfrak{B} \cdot \mathfrak{T}_1^{-\beta_1} \cdots \mathfrak{T}_n^{-\beta_n} = \mathfrak{C} = \mathfrak{A}_1^{c_1} \cdots \mathfrak{A}_n^{c_n},$$

where $c_1 = b_1 - \beta_1 t_{11}, c_2 = b_2 - \beta_1 t_{12} - \beta_2 t_{22}, \ldots, c_n = b_n - \beta_1 t_{1n} - \ldots - \beta_n t_{nn}$. The number c_1 is smaller than the corresponding number t_{11} in $\mathfrak{T}_1$, the number c_2 is smaller than the corresponding number t_{22} in $\mathfrak{T}_2$, and so on, and finally the number c_n is smaller than the corresponding number t_{nn} in $\mathfrak{T}_n$. This translation $\mathfrak{C}$ is therefore smaller than $\mathfrak{T}_n$ and thus $c_n = 0$, it is smaller than $\mathfrak{T}_{n-1}$ and thus $c_{n-1} = 0$, and finally it is smaller than $\mathfrak{T}_1$ and thus $c_1 = 0$. Hence $\mathfrak{C}$ is the identity and so every translation $\mathfrak{B}$ in the group os of the form

$$\mathfrak{B} = \mathfrak{T}_1^{\beta_1} \cdots \mathfrak{T}_n^{\beta_n}$$

with integers $\beta_1, \ldots, \beta_n$.

We have thereby identified n generating translations in the group. These are not the only ones of this kind. For if $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$ are generating translations, so are the n translations

$$\mathfrak{T}_i' = \mathfrak{T}_1^{c_{1i}} \cdots \mathfrak{T}_n^{c_{ni}},$$

if the determinant of the integer (c_{ik}) equals ± 1. Conversely, any system of generating translations can be obtained in this way from any other.

One can think of the n translations $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$ as vectors drawn from the origin, and these vectors as unit segments of a new skew coordinate system. Analytically, if

$$\mathfrak{T}_i \;\equiv\; x_k' = x_k + A_k^{(i)} \quad (i, k = 1, \ldots, n)$$

are the n translations, we can introduce new variables ξ_i via

$$x_i = \sum_{k=1}^{n} A_i^{(k)} \xi_k \quad (i = 1, \ldots, n).$$

If all translations in the group are taken as vectors in this new coordinate system, then the totality of their endpoints comprises all points with integer coordinates in the new coordinate system. Analytically, if we introduce new coordinates via the given substitution, then in the new coordinate system the translations are written again in the form

$$\xi_k' = \xi_k + A_k^{(i)} \quad (i, k = 1, \ldots, n).$$

But now the coefficients $A_k^{(i)}$ are integer numbers. In particular, the n generating translations used to introduce the ξ_i become

$$\xi_k' = \xi_k + A_k^{(i)}, \quad A_k^{(i)} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad (i, k = 1, \ldots, n).$$

The aforementioned unimodular integer substitutions (integer substitutions with determinant ± 1) connecting the different systems of generating translations now also provide the transition between different coordinate systems, in which the translations can be expressed in this manner with integer coefficients.

The totality of integral points in a certain coordinate system is called a *lattice*. It is the concept of a lattice that gives Minkowski's theory of quadratic forms its transparent form. We are also lead to this concept via the translation groups. The close connection is geometrically based on this. The only thing required is to establish the analytic expression for the squared distance with respect to one of our skew coordinate systems in which the translation group is represented by integers.

Then we obtain one of the infinitely many positive quadratic forms associated to our lattice. As the transitions between the different coordinate systems are effected by unimodular matrices, all these quadratic forms arise from one another by unimodular transformations of their variables.

3 The group of rotational parts

The *rotational part* of a motion

$$x_i' = \sum_{k=1}^{n} b_{ik} x_k + B_i \quad (i = 1, \dots, n)$$

is the orthogonal substitution

$$x_i' = \sum_{k=1}^{n} b_{ik} x_k \quad (i = 1, \dots, n).$$

It is easy to see that the rotational parts of the motions in a group of motions form a group themselves. As we will show now, this group of rotational parts is a finite group. To see this, we represent the whole group of motions, and thus also the group of rotational parts, in a coordinate system of the previous paragraph, which is characterized by the translations being integer transformations. Once such variables have been introduced, we find that *the rotational parts also have integer coefficients*. This follows from the fact that for every motion $\mathfrak{B}$ with rotational part B and any translation $\mathfrak{T}$ in the group, $\mathfrak{B}\mathfrak{T}\mathfrak{B}^{-1} = B\mathfrak{T}B^{-1}$ is also a translation in the group. If we apply this, in particular, to the generating translations $\mathfrak{T}_i$, in which only one component is different from 0, namely 1, say

$$\mathfrak{T}_i \equiv \begin{array}{l} x_i' = x_i + 1, \\ x_k' = x_k \quad (k \neq i), \end{array}$$

then for the rotational part

$$B \equiv x_i' = \sum_{k=1}^{n} b_{ik} x_k \quad (i = 1, \dots, n)$$

the translation becomes

$$B\mathfrak{T}B^{-1} \equiv x_k' = x_k + b_{ki} \quad (k = 1, \dots, n).$$

The components of this translation must be integer numbers. But this means that the coefficients in the i-th column of the rotational part are integers, and thus also in all other columns. The determinant of these integer subsitutions is ± 1, as it was obtained from changing the variables of an orthogonal substitution. The integral group of rotational parts preserves a positive quadratic form, for the very reason that it was obtained from a group of orthogonal substitutions for which the sum of squares of the variables is invariant. (Geometrically we would say: As the rotational parts are motions they must preserve the quadratic form that is the analytic expression of distance with respect to the skew coordinate system.) From this fact, that the group of rotational parts preserves a positive quadratic form, its finiteness follows from a well-known theorem. It is even possible to give an upper bound for the order of this group, depending on the number of variables.

For completeness, we will sketch the basic idea of a proof for this theorem in a geometric guise (although we could also refer to theorem used in §4), cf. Minkowski, *Geometrie der Zahlen*, p. 176. From the rotations in the group of rotational parts, we obtain from a first system of n generating translations of the translation subgroup further such systems of generating translations. Let l be the largest lengths among the $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$. Then all translations obtained in this way via rotations from the $\mathfrak{T}_i$ are shorter than l. But there are only finitely many translations shorter than l (since otherwise the group would contain infinitesimal operations). These finitely many translations can be combined in only finitely many ways to systems of n generators. As the substitution that effects the coordinate transformation between the two such generating systems is completely determined by the two systems, only finitely many substitutions can appear as rotational parts. Hence the group of rotational parts is finite.

In order to establish the existence of an upper bound for the order only depending on n, it is convenient to use a system of n linearly independent translations other than the aforementioned system of n linearly independent translations. Let $\mathfrak{T}_1$ denote the shortest translation in the group, $\mathfrak{T}_2$ the shortest one that cannot be represented in the form $\mathfrak{T}_1^a$, $\mathfrak{T}_3$ the shortest one that cannot be represented in the form $\mathfrak{T}_1^a \mathfrak{T}_2^b$, and so on, and let $\mathfrak{T}_n$ be the shortest translation that cannot be represented in the form $\mathfrak{T}_1^{a_1} \cdots \mathfrak{T}_{n-1}^{a_{n-1}}$ (here, the numbers a_i need not be integers, as the thus obtained system of translations is not necessarily a system of generators). Through rotations by the rotational parts of the group we only obtain finitely many new ones from these, and as before we can conclude the finiteness of the group of rotational parts. But now we can also show that the number of translations that can be obtained from via rotations from these n is bounded by a number depending only on n, and thus prove this fact also for the group of rotational parts. Namely, from the shortest $\mathfrak{T}_1$ we only get finitely many translations whose number cannot surpass a certain bound depending on n, for if l_1 is the length of $\mathfrak{T}_1$, then all translations obtained by rotations from it lie on a sphere of radius l_1. They form a system of points in which no two points have distance less than l_1 from one another. So their number is less than a certain number $s = (2\sqrt{n}+1)^n$ depending only on n. Now let l_2 be the length of $\mathfrak{T}_2$. Then all translations obtained from $\mathfrak{T}_2$ by rotations lie on a sphere of radius l_2, so they form a system of points in which at a distance l_1 from any point, there are at most s other points of the system. It follows that the number of these points is at most s^2. This reasoning is repeated for all translations and thus the theorem is established. This is the basic idea used by implicitely Minkowski, which has the advantage of yielding sharper results. He considers the remainders modulo 2 of the coordinates of the lattice points just constructed with respect to a suitable coordinate system. It turns out that these systems of remainders are all distinct. This yields the known bound $(2^{n+1}-2)^n$ for the order of the group of rotational parts.

4 Preparing the proof of finiteness

Based in the results of the two preceding paragraphs, we now wish to show that two isomorphic groups of motions always arise from one another by a change of variables. So if $A_1, A_2, \ldots$ are the elements of one group of motions and $B_1, B_2, \ldots$ are the elements of another group of motions, such that any two motions with the same index correspond to one another under an isomorphism, then we will show that there exists a substitution S, independent of the index i, such that $SA_iS^{-1} = B_i$ for all indices i. To see this, we first note that in under an isomorphism, the translations of one group necessarily correspond to the translations of the other group. A translation A in the first group corresponds to a motion B in the second group such that all $B_iBB_i^{-1}$ commute with B. If we take n linearly independent translations for the B_i, then by arguments we used several times in the first part of this treatise (e.g. part I, §5) it follows that B is a translation. Since now the translation subgroups correspond to one another under the isomorphism, on the basis of the results of the preceding paragraphs, we choose the coordinate, we choose for both groups variables such that the translations are integral substitutions, and such that the corresponding translations in both groups coincide.

All we need to do is to introduce in both groups the corresponding systems of n generating translations as unit lengths in the new coordinate systems. Once we do this, the by now integral rotational parts of two corresponding motions in the two groups become identical. Namely, if

$$A \;\equiv\; x_i' = \sum_{k=1}^{n} a_{ik} x_k + A_i \quad (i - 1, \ldots, n)$$

and

$$B \;\equiv\; x_i' = \sum_{k=1}^{n} b_{ik} x_k + B_i \quad (i = 1, \ldots, n)$$

are two corresponding motions in the two groups, and $\mathfrak{T}_i$ the generating translations occuring in both groups,

$$\mathfrak{T}_i \;\equiv\; x_i' = x_i + 1, \quad x_k' = x_k \quad (k \neq i),$$

then

$$x_k' = x_k + a_{ik} \quad (i, k = 1, \ldots, n)$$

and

$$x_k' = x_k + b_{ik} \quad (i, k = 1, \ldots, n)$$

are now corresponding translations and must thus coincide. But this also means that the i-th columns of the rotational parts coincide. As this holds for all columns, it follows that the rotational parts of any two corresponding motions coincide. So now let

$$\mathfrak{A}_h \;\equiv\; x_i' = \sum_{k=1}^{n} a_{ik}^{(h)} x_k + A_i^{(h)}$$

and

$$\mathfrak{B}_h \;\equiv\; x_i' = \sum_{k=1}^{n} a_{ik}^{(h)} x_k + B_i^{(h)}$$

be the motions in both groups, then we form the motion

$$\mathfrak{C}_h \;\equiv\; x_i' = \sum_{k=1}^{n} a_{ik}^{(h)} x_k + A_i^{(h)} - B_i^{(h)}.$$

Evidently, these form a group themselves. This group of motion is finite, as only finitely many rotational parts appear and it contains no translations (for, if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are two motions in this group with coinciding rotational parts, then $\mathfrak{C}_1^{-1}\mathfrak{C}_2$ is a translation and thus the identity, as the group contains no translations). But then by a theorem of Maschke (compare also §9 in part I) there is a translation

$$\mathfrak{T} \equiv x_i' = x_i + T_i,$$

such that $\mathfrak{T}\mathfrak{C}_h\mathfrak{T}^{-1} = \mathfrak{D}_h$ becomes the following homogeneous substitution:

$$\mathfrak{D}_h \equiv x_i' = \sum_{k=1}^{n} a_{ik}^{(h)} x_k.$$

If we apply this translation to the group of motions $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$, so that we obtain a group of motions $\mathfrak{T}\mathfrak{A}_1\mathfrak{T}^{-1}, \mathfrak{T}\mathfrak{A}_2\mathfrak{T}^{-1}, \ldots$, then the corresponding motions $\mathfrak{T}\mathfrak{A}_h\mathfrak{T}^{-1} = \mathfrak{A}_h'$ and $\mathfrak{B}_h$ are fully identical. Namely,

$$\mathfrak{A}_h' \equiv x_i' = \sum_{k=1} a_{ik}^{(h)} x_k + A_i',$$

where

$$A_i' = T_i - \sum_{k=1}^{n} a_{ik} T_k + A_i,$$

and we obtain

$$\mathfrak{D}_h \equiv x_i' = \sum_{k=1}^{n} a_{ik}^{(h)} x_k + D_i,$$

where

$$D_i = T_i = \sum_{k=1}^{n} a_{ik} T_k + A_i - B_i.$$

But here $D_i = 0$. Hence $A_i' = B_i$, and this was just our claim.

So isomorphic groups of motions arise from one another by suitable changes of variables. To prove that there only finitely many groups of motions that do not arise from another by changes of variables, as we set out to do in §1, we just need to prove that *there are only finitely many non-isomorphic groups of motions.*

To this end, we henceforth choose the variables such that the translations have integral coefficients. This can be achieved in different ways for each group, depending on the system of n linearly independent translation that is used to introduce the skew coordinate system. For this different ways, the groups of rotational parts are transformed into each other by introducing new variables via a suitable unimodular substitution (compare §§2, 3). So if two groups of motions are isomorphic, then the groups of rotational parts arise from one another via an integral unimodular substitution, since, as we just saw, they become identical for a suitable choice of a skew coordinate system. So if we wish to prove that there only finitely many non-isomorphic groups of motions in n variables, we have to show at first that there are only finitely many distinct finite groups of integral substitutions in n variables that do not arise from one another by integral unimodular transformations. (One readily checks that every finite group of integral substitutions can appear as the group of rotational parts for a suitable group of motions.) This is the content of a deep theorem proved in the reduction theory of positive quadratic forms.

Said theorem has been discoverd by Jordan[9] and he proved it by using a method of reduction due to Korkine and Zolotarev.[10] Another proof similar to Jordan's was given by Minkowski based on his improved version of Hermite's method of reduction.[11] A third proof was published by myself in the Göttinger Nachrichten, also based in Minkowski's theory of reduction.[12] It would lead too far to study these proofs here. Therefore, we refer to the aforementioned sources, as well as to a soon to be published work on the reduction of quadratic forms published jointly by the author and Schur.

5 Proof of finiteness

Now we first summarize how the current state of our proof of finiteness. If we write the groups of motions with respect to a suitable skew coordinate system, we fix the translation subgroup. It it is the group generated by the n translations

$$\mathfrak{T}_i \equiv x_i' = x_i + 1, \quad x_k' = x_k \quad (i \neq k).$$

Furthermore, we know from the last paragraph that there are only finitely many possibilities for the group of rotational parts. But so far we do not know anything about the translational parts of those motions that are not translations themselves. So now we will collect all groups of motions that coincide in their group of rotational parts into one class (out of finitely many classes) and then show that also

[9] In the first part of this treatise I erroneously ascribed this theorem to Minkowski.

[10] C. Jordan, Journal de l'École Polytechnique cah. 48.

[11] H. Minkowski, *Diskontinuitätsbereich für arithmetische Äquivalenz*, Journal für Mathematik 129.

[12] L. Bieberbach, *Über die Minkowskische Reduktion der positiven quadratischen Formen*, Göttinger Nachrichten 1912.

for the translational parts of these motions there only finitely many possibilities. By §4, two groups are to be considered identical if they are isomorphic. Then we may argue as follows.[13)]

Let $\mathfrak{B}$ be a motion and $\mathfrak{T}$ a translation in the group. Then all motions $\mathfrak{T}\mathfrak{B}$ have the same rotational part, and conversely all motions with the same rotational part arise from one another in this way. We collect them in one out of finitely many classes $\mathfrak{T} \cdot \mathfrak{B}$ of motions. By a suitable choice of $\mathfrak{T}$, we can always ensure that the motion

$$\mathfrak{T}\mathfrak{B} \equiv x_i' = \sum_{k=1}^{n} b_{ik} x_k + B_i$$

satisfies $0 \leq B_i < 1$. In each class of motions $\mathfrak{T}\mathfrak{B}$ there is precisely one such motion. We call it the *reduced motion* in the class. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are reduced motions with product $\mathfrak{B}_1\mathfrak{B}_2$, and if $\mathfrak{B}_3$ is the reduced motion in the class $\mathfrak{T} \cdot \mathfrak{B}_1\mathfrak{B}_2$, then for a suitable choice of $\mathfrak{T}$ we have $\mathfrak{B}_1\mathfrak{B}_2 = \mathfrak{T}\mathfrak{B}_3$. We now show that for given $\mathfrak{B}_1$, $\mathfrak{B}_2$, only finitely many possibilities for $\mathfrak{T}$ occur. Let b_1, b_2 denote the rotational parts of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, that is, $\mathfrak{B}_1 = t_1 b_1$ and $\mathfrak{B}_2 = t_2 b_2$. Let further b_3 denote the rotational part of $\mathfrak{B}_3$, that is, $\mathfrak{B}_3 = t_3 b_3$, and hence

$$\mathfrak{B}_1\mathfrak{B}_2 = \mathfrak{T} \cdot t_3 b_3 = \mathfrak{T} \cdot t_3 b_1 b_2.$$

Then

$$\mathfrak{B}_1\mathfrak{B}_2 = t_1 b_1 t_2 b_2 = t_1 \cdot b_1 t_2 b_1^{-1} \cdot b_1 b_2$$

and as $b_3 = b_1 b_2$,

$$\mathfrak{T} t_3 = t_1 \cdot b_1 t_2 b_1^{-1}.$$

The components of t_1, t_2, t_3 are all less than 1, the coefficients of b_1 are fixed, and thus the components of $b_1 t_2 b_1^{-1}$ are bounded by some identifiable number. Hence all components of $\mathfrak{T}$ must be bounded by some identifiable number and it follows that there are only finitely many possibilities for $\mathfrak{T}$, as it is a translation in the group (with integer components). We can repeat this argument for all finitely many products of reduced motions, and if for now we consider two groups as equal if for any two products of reduced motions those translations coincide, by which the product differs from the reduced motion of the product, then we obtain only finitely many of such groups. The point is now that two such groups, for

[13)]Frobenius gives a different, perhaps somewhat simpler argument in the aforementioned work. The main difference is that Frobenius removes the remaining arbitrariness in the translational parts by the choice of a suitable coordinate system, and then, other than our argument here, uses a computation involving congruences.

which all these translations coincide, are isomorphic. To see this, we only need to assign in both groups the identical translations and the respective reduced motions with identical rotational parts to each other, in order to obtain an isomorphism of the groups. This proves our theorem.

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