On the Riemannian manifolds with holonomy group G_2 or Spin(7)

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1. Let $(1, e_i)$, $i = 1, \ldots, 7$, be a basis of Cayley's octave algebra: Each triplet

$$(e_i, e_{i+1}, e_{i+3})$$

forms a system of quaternions.

Let the group SO(7) act on $\mathbb{R}^7$,

$$\mathbb{R}^7 = \bigoplus_{i=1}^{7} \mathbb{R}e_i.$$

Let G_2 denote the maximal subgroup of SO(7) preserving the multiplication table T, which is in fact a vector-valued two-form,

$$x, y \in \mathbb{R}^7 \mapsto T(x, y) = x \cdot y \in \mathbb{R}^7, \quad x \cdot y = -y \cdot x.$$

On the other hand, $\mathbb{R}^7$ is equipped with a canonical scalar product

$$x, y \in \mathbb{R}^7 \mapsto (x, y) = \sum_{i=1}^{7} x^i y^i.$$

We then construct the trilinear form α,

$$x, y, z \in \mathbb{R}^7 \mapsto \alpha(x, y, z) = (x, y \cdot z),$$

which is clearly alternating.

If we let (ω^i) denote the dual basis of (e_i), then it follows immediately that

$$\alpha = \sum_{i=1}^{7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+3}. \tag{1}$$

We deduce the four-form β that is adjoint to the three-form α:

$$\beta = \star\alpha = \sum_{i=1}^{7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+2} \wedge \omega^{i+5}. \tag{2}$$

Proposition 1 G_2 *preserves* α *and* β.

We show by direct calculation that for all two-forms φ_2:

$$\alpha \wedge \varphi_2 = 0 \quad \Leftrightarrow \quad \varphi_2 = 0.$$

If we let φ_p denote any p-form, then:

Proposition 2 *Suppose* $p \leq 2$, *then* $\alpha \wedge \varphi_p = 0$ *implies* $\varphi_p = 0$.

Corollary *For* $p \leq 5$, *all* p*-forms* φ_p *have a unique decomposition of the form*

$$\varphi_p = (\alpha \wedge \mu_{p-3}) + \mu_p, \quad \text{where } \alpha \wedge (\star\mu) = 0.$$

The Lie algebra of G_2 can be realized as the set of 7×7-matrices $A = (a_{ij})$ satisfying the relations

$$a_{ij} + a_{ji} = 0, \quad a_{i+1,i+3} + a_{i+4,i+5} + a_{i+2,i+6} = 0. \tag{3}$$

We remark that the three triplets

$$(e_i, e_{i+1}, e_{i+3}), \quad (e_{i+4}, e_{i+5}, e_i), \quad (e_{i+6}, e_i, e_{i+2})$$

each form a system of quaternions.

2. Let $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ act on $\mathbb{R}^8$,

$$\mathbb{R}^8 = \mathbb{R}e_{7'} \oplus \bigoplus_{i=1}^{7} \mathbb{R}e_i.$$

The group G_2 is the isotropy group of the vector $e_{7'}$ and acts on the seven-plane of the equation $\omega^{7'} = 0$ if by (ω^k) we denote the dual basis of (e_k), where $k \in \{7'\} \cup \{1, \ldots, 7\}$.

The Lie algebra of $\mathrm{Spin}(7)$ can be realized as the set of matrices $A = (a_{ij})$ satisfying the relations

$$a_{ij} + a_{ji} = 0, \quad a_{7'i} + a_{i+1,i+3} + a_{i+4,i+5} + a_{i+2,i+6} = 0. \tag{4}$$

We construct the four-form γ,

$$\gamma = \omega^{7'} \wedge \alpha' + \beta',$$

where α' and β' are the evident extensions of α and β to $\mathbb{R}^8$,

$$\alpha' = \alpha \circ P, \quad \beta' = \beta \circ P,$$

where P denotes the orthogonal projection on $\mathbb{R}^7$. Then

$$\gamma = \sum_{i+1}^{7} \omega^{7'} \wedge \omega^i \wedge \omega^{i+1} \wedge \omega^{i+3} + \sum_{i=1}^{7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+2} \wedge \omega^{i+5}. \tag{5}$$

Proposition 3 Spin(7) *preserves* γ.

Proposition 4 *Suppose* $p \leq 2$, *then* $\gamma \wedge \varphi_p = 0$ *implies* $\varphi_p = 0$.

Corollary *For* $p \leq 6$, *all* p*-forms* φ_p *have a unique decomposition*

$$\varphi_p = (\gamma \wedge \mu_{p-6}) + \mu_p, \quad \textit{where } \gamma \wedge (\star\mu) = 0.$$

3. Let G be a closed subgroup of the orthogonal group $\mathrm{O}(d)$.

Definition A Riemannian manifold V_d of dimension d is called a *G-manifold* if the homogeneous holonomy group is a subgroup of G. The fiber space of orthonormal tangent frames $\mathrm{E}[V_d, \mathrm{O}(d)]$ admits a subfibration $\mathrm{E}^{\mathrm{a}}[V_d, \mathrm{O}(d)]$.

The preceding results then apply at every point $x \in V_d$; in particular:

Proposition 5 *All* G_2*-manifolds* V_7 *admit a global three-form and a global four-form with vanishing covariant derivative.*

Proposition 6 *All* Spin(7)*-manifolds* V_8 *admit a global four-form with vanishing covariant derivative.*

The decompositions established in Propositions 2 and 4 are clearly global.

By manipulations of the Betti numbers $b_k(V_d)$, we find that under the assumptions of Propositions 5 and 6, and assuming that V_d is compact:

$$b_3(V_7) \neq 0, \quad b_3(V_7) \geq b_1(V_7), \quad b_4(V_8) \neq 0, \quad b_4(V_8) \geq b_1(V_8).$$

Let now R_{ijkl} denote the curvature tensor of the Riemannian connection of V_7. With respect to the frames in $\mathrm{E}^{\mathrm{a}}[V_7, \mathrm{G}_2]$, it satisfies the following relations, among others:

$$R_{i+1,i+3,kl} + R_{i+4,i+5,kl} + R_{i+2,i+6,kl} = 0 \tag{6}$$

for all $k, l, i \in \{1, \ldots, 7\}$.

We compute the Ricci tensor,

$$R_{ij} = \sum_{l=1}^{7} R_{iljl}.$$

It is enough to have R_{00} and R_{10}:

$$R_{00} = R_{0101} + R_{0202} + R_{0303} + R_{0404} + R_{0505} + R_{0606},$$
$$R_{01} = R_{0212} + R_{0343} + R_{0414} + R_{0515} + R_{0616}.$$

Using relation (6), we obtain

$$
\begin{aligned}
R_{00} &= R_{0125} + R_{0164} + R_{0243} + R_{0251} + R_{0324} + R_{0356} \\
&\quad + R_{0416} + R_{0432} + R_{0512} + R_{0563} + R_{0635} + R_{0641}, \\
R_{01} &= R_{0205} + R_{0236} + R_{0354} + R_{0362} + R_{0435} \\
&\quad + R_{0460} + R_{0543} + R_{0520} + R_{0623} + R_{0604}.
\end{aligned}
$$

These expressions are zero by virtue of the Ricci identities.

Proposition 7 *Every* G_2*-manifold* V_7 *has zero Ricci curvature.*

Proposition 8 *Every* $\mathrm{Spin}(7)$*-manifold* V_8 *has zero Ricci curvature.*

4. Remarks:

- From Kostant's results in [4], one obtains the existence of a four-form for, in particular, the groups G_2 and Spin(7).
- The three-form α has been previously constructed by Chevalley in [3].
- The present work extends [7], [2] and [6].

References

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