On compact homogeneous Kähler manifolds

By ARMAND BOREL in Princeton N.J.
and REINHOLD REMMERT in Erlangen

1. Let V denote a connected compact complex manifold. V is called *homogeneous* if for any two points $v_1, v_2 \in V$ there exists a biholomorphic map g of V to itself with $g(v_1) = v_2$. A compact Kähler manifold is called *homogeneous* if it is homogeneous as a complex manifold. The aim of this note is to prove the following:

Theorem I *Every connected compact homogeneous Kähler manifold is the direct product of a complex torus and projective rational variety*[1)].

Special cases of this statement are known. For projective algebraic manifolds, it was already anounced in [4] §16; the proof, which is valid for a base field of arbitrary characteristic, shall be presented here. Under the additional assumption that a compact group of holomorphisms acts transitively on the given Kähler manifold, Matsushima [10] proved the theorem, and in the case that a complex Lie group of holomorphisms of the same dimension as V acts transitively, Wang [15] gave a proof (compare the corollary to Theorem 1).

The proof of Theorem I is prepared in Sections 2 to 5. We show that every compact homogeneous complex manifold V can be fibered in two ways: firstly (Theorem 1), *V is a holomorphic fiber bundle over the Albanese torus of V*, and secondly, (Theorem 7), in a generalization of a theorem by Goto, V can be canonically represented as a *holomorphic fiber bundle over a projective rational homogeneous manifold with a complex parallelisable fiber*. In the Kähler case, using the theory of complex Lie groups, we can make additional statements on these two fiber bundles that will yield a simple proof of Theorem I in Section 6.

Lie groups will mostly be denoted by G, H, M, N,... and its connected components of 1 by G°, H°, M°, N°,...; $G(V)$ denotes the group of all holomorphisms of V.

[1)] A compact complex manifold is called *projective rational* if it is projective algebraic and has a rational (over $\mathbb{C}$) field of meromorphic functions. Recall that by Chow's Theorem, an analytic subset of a projective algebraic variety is projective algebraic itself.

2. The Albanese bundle of a homogeneous compact complex manifold. We begin with the following well-known fact (compare [1], p. 163):

(a) *For every connected compact complex manifold V, there exists a complex torus $A(V)$ and a holomorphic map $\alpha : V \to A(V)$, such that every holomorphic map $\beta : V \to B$ from V to any complex torus B can be written in the form $\beta = \tau \circ \alpha$, where $\tau : A(V) \to B$ is a holomorphic affine map from $A(V)$ to B, uniquely determined by β. It holds that $\dim_{\mathbb{R}} A(V) \leq \mathrm{b}_1(V)$. If V is Kähler, then $\dim_{\mathbb{R}} A(V) = \mathrm{b}_1(V)$.*

The torus $A(V)$ and the map α are uniquely determined by V, that is, if A' is another torus and $\alpha' : V \to A'$ another holomorphic map with the above universal mapping property, then there is a biholomorphic affine map ι of A' to $A(V)$ such that $\alpha = \iota \circ \alpha'$.

We call $A(V)$ the *Albanese torus* of V and accordingly $\alpha : V \to A(V)$ the *Albanese map*. The number $a(V) := \dim_{\mathbb{C}} A(V)$ is called the *Albanese number* of V. If V is projective algebraic, then $a(V)$ is the irregularity of V.

The group $G(V)$ of all holomorphisms of V can by [2] be taken as a complex transformation group of V in a canonical way. If $G(V)$ acts transitively on V, then so does $G(V)^\circ$. By $T(V)$ we will denote the complex Lie group of translations of $A(V)$, and it holds that $T(V) = G(A(V))^\circ$. The torus group $T(V)$ is, as a complex manifold, biholomorphically equivalent to $A(V)$. We will make significant use of:

(a$'$) *There is a holomorphic homomorphism of groups $\gamma : G(V) \to G(A(V))$, such that$\alpha \circ g = \gamma(g) \circ \alpha$ for all $g \in G(V)$. Then $\gamma(G(V)^\circ) \subset T(V)$ holds. The kernel $\ker \gamma$ is a closed complex Lie subgroup of $G(V)$ that acts holomorphically on every α-fiber. A holomorphism $g \in G(V)^\circ$ belongs to $\ker \gamma$ if and only if there is a point $v_0 \in V$ such that v_0 and $g(v_0)$ lie in the same α-fiber. In particular, if H is a subgroup of $G(V)^\circ$ that acts transitively on V, then $H \cap \ker \gamma$ acts transitively on every α-fiber.*

The existence of γ was proven by Blanchard [1], Proposition I.2.1. That γ is holomorphic is not explicitely noted in [1], but if follows from the definition of γ (cf. [1], p. 165). The remaining statements of (a$'$) can be verified directly.

In this section we wish to prove:

Theorem 1 *If V is a connected compact homogeneous complex manifold, then:*

(a) *The Albanese map $\alpha : V \to A(V)$ and the homomorphism $\gamma : G(V)^\circ \to$*

$T(V)$ are surjective.

(b) *V is a holomorphic fiber bundle over $A(V)$ with respect to α. The typical fiber F is a connected compact complex manifold, and homogeneous with respect to* $\ker\gamma$. *There is an exact sequence*

$$\mathbf{0} \to \pi_1(F) \to \pi_1(V) \to \pi_1(A(V)) \to \mathbf{0}.$$

PROOF: (a) For short, we write $G^\circ := G(V)^\circ$. The group G° acts transitively on V. If $a_0 \in \alpha(V)$ is fixed, then by (a′) we have $\alpha(V) = \{t(a_0) \mid t \in \gamma(G^\circ)\}$. If we take $A(V)$ as a complex torus group with a_0 as identity, then $\alpha(V)$ becomes a subgroup of $A(V)$. Since $\alpha(V)$ is closed in $A(V)$ and is the orbit of a point under a complex transformation group of $A(V)$, $\alpha(V)$ is an analytic set in $A(V)$ (this also follows directly from [13], Satz 24). Hence $\alpha(V)$ is even a complex subtorus of $A(V)$. Now this torus $\alpha(V)$ together with the holomorphic map $\alpha' : V \to \alpha(V)$ induced by α evidently has the universal property of the Albanese torus. Hence $\alpha(V) = A(V)$. This further implies $\gamma(G^\circ) = T(V)$.

(b) Let M denote the kernel of $\gamma : G^\circ \to T(V)$. If H is the stabilizer subgroup in G° of a point $v_0 \in V$, then $H \subset M$ by (a′). Upon identifying V with G°/H and $A(V)$ with G°/M (the latter is possible by (a)), V appears as a holomorphic fiber bundle over $A(V)$ with respect to α, whose typical fiber is the homogeneous compact complex manifold M/H. In order to prove that M/H is connected, we consider the smallest open subgroup M' of M that contains H. Clearly, M' consists precisely of those connected components of M whose intersection with H is non-empty. Thus M'/H is connected. The holomorphic map $\alpha : V \to A(V)$ is now the composition of the holomorphic maps $\alpha_1 : V = G^\circ/H \to G^\circ/M'$ and $\alpha_2 : G^\circ/M' \to G^\circ/M = A(V)$. The compact complex manifold G°/M' is an unramified compact covering of $A(V)$ and hence a complex torus itself. From the universal property of $A(V)$ it follows that α_2 is biholomorphic. This implies $M = M'$, that is, the typical fiber $F = M/H$ of α is connected. We now have the following exact homotopy sequence

$$\cdots \to \pi_2(A(V)) \to \pi_1(F) \to \pi_1(V) \to \pi_1(A(V)) \to \mathbf{0}.$$

Since $A(V)$ is a torus, $\pi_2(A(V)) = \mathbf{0}$. Since moreover M acts transitively on each α-fiber by (a′), the proof of Theorem 1 is complete. ◇

From Theorem 1 we immediately obtain the following result due to Wang [15], Theorem 1 and Corollary 2:

Corollary 1 *A complex Lie group G acts holomorphically and transitively on a compact Kähler manifold V of the same dimension if and only if G is a complex torus.*

PROOF: Clearly, a complex torus has this property. Conversely, let V be a compact Kähler manifold and G a complex Lie group of the same dimension as V that acts holomorphically and transitively on V. Without loss of generality we may assume G is connected. Then V is equivalent to a complex quotient variety G/H, where H is a discrete subgroup of G. If now $n := \dim_{\mathbb{C}} G$, then there are n linearly independent right-invariant holomorphic differential forms of degree 1 on G (the Maurer-Cartan forms). These forms induce n linear independent holomorphic differential forms on V, as the natural projection $G \to G/H$ is locally biholomorphic. Since V is Kählerian, $\mathrm{b}_1(V) \leq 2n$, so that $A(V)$ has at least the same dimension as V. Then $\alpha : V \to A(V)$ is biholomorphic by Theorem 1. $\diamondsuit$

3. Homogeneous projective rational manifolds. The aim of this section is to prove Theorem 4. To prepare, we show:

Theorem 2 *Every holomorphic map of a homogeneous projective rational manifold Q has a fixed point.*

PROOF: By [3], Q admits a "analytic cell decomposition": The $2s$-dimensional closed cells are all s-dimensional irreducible algebraic sets in Q and form a basis $\{\gamma_1^{2s}, \ldots, \gamma_{j_s}^{2s}\}$ of the $2s$th homology group of Q, for $s = 1, \ldots, \dim_{\mathbb{C}} Q$. From results by Chow [7] it follows that every irreducible s-dimensional analytic cycle in Q is homologous to a cycle $\sum_{i=1}^{j_s} n_i \cdot \gamma_i^{2s}$, where all $n_i \geq 0$. Since every holomorphic map τ from Q to itself maps irreducible analytic sets to sets of the same type (compare [13], Satz 24), with respect to the bases $\{\gamma_i^{2s}\}$, the traces of all homomorphisms induced by τ in the homology groups are non-negative. As the trace in dimension 0 is 1 and all homology groups of odd dimension vanish, the alternating sum over their traces is positive. But then the holomorphic map τ has a fixed point in Q by the Lefschetz-Hopf formula. $\diamondsuit$

Every complex Lie group of complex dimension n can be interpreted in a natural way as a $2n$-dimensional real Lie group. We will denote this real Lie group determined by G by G_{r}. Then:

Lemma 3 *Let X be a projective algebraic manifold with* $\mathrm{b}_1(X) = 0$, *and M an* *analytic subset of X. Moreover, let H be a solvable connected real Lie subgroup of $G(X)_{\mathrm{r}}$, such that $h(M) = M$ for all $h \in H$. Then there exists at least one point $x_0 \in M$ such that $h(x_0) = x_0$ for all $h \in H$.*

PROOF: As $\mathrm{b}_1(X) = 0$, by the Théorème Principal I from [1], there exists a projective embedding of X into a $\mathbb{P}_s$, such that $G(X)^\circ$ can be identified with the maximal connected subgroup of $G(\mathbb{P}_s)$ that leaves X invariant. The group H is then a solvable connected real Lie subgroup of $G(\mathbb{P}_s)_{\mathrm{r}}$. Let H_{c} denote the connected complex Lie subgroup of $G(\mathbb{P}_s)$ whose Lie algebra is the complexification of the real Lie algebra of H. Then H_{c} contains H and is solvable. By a classical theorem of Lie[2)] there exists a flag $T_0 \subset T_1 \subset \ldots \subset T_{s-1}$, where T_i is an i-dimensional analytic plane in $\mathbb{P}_s$, such that $h(T_i) = T_i$ for all $h \in H$ and $i = 0, \ldots, s-1$. As M is analytic and as such algebraic in $\mathbb{P}_s$, we can choose the index j, $0 \leq j \leq s-1$, such that $M \cap T_j$ contains at least on isolated point x_0. Since $H \subset H_{\mathrm{c}}$ is connected and preserves $M \cap T_j$, we obtain $h(x_0) = x_0$ for all $h \in H$. ◇

Remark Instead of H_{c}, we also consider the smallest algebraic subgroup H' of $G(\mathbb{P}_s)$ containing H in the preceding proof. It is solvable, connected, leaves M invariant and thus has a fixed point in M by [5], p. 64.

It now readily follows:

Theorem 4 *If Q is a homogeneous projective rational manifold, then every connected real Lie subgroup U of $G(Q)_{\mathrm{r}}$ that acts transitively on Q is semisimple. Moreover, the centralizer of U in $G(Q)$ contains only the identity.*

PROOF: Let $\mathrm{R}(U)$ denote the radical[3)] of U. As $\mathrm{b}_1(Q) = 0$[4)], the group $\mathrm{R}(U)$ has a fixed point on Q by Lemma 3. Since $\mathrm{R}(U)$ is a normal subgroup of U and U acts transitively on Q, every point of Q is a fixed point of $\mathrm{R}(U)$. As U acts effectively, this implies $\mathrm{R}(U) = \{1\}$. So $\mathrm{R}(U)$ is semisimple.

[2)] Lie's Theorem is stated as follows: *Let H be a solvable connected complex Lie group and $\varrho : H \to \mathrm{GL}(n, \mathbb{C})$ a holomorphic homomorphism. Then there is an element $a \in \mathrm{GL}(n, \mathbb{C})$ such that all matrices $a \cdot \varrho(H) \cdot a^{-1}$ are triangular matrices (all coefficients below the diagonal are zero).* In our case, this theorem is applied to the complex subgroup of $\mathrm{GL}(s+1, \mathbb{C})$ that is the connected component of 1 of the preimage of H_{c} under the canonical homomorphism $\mathrm{GL}(s+1, \mathbb{C}) \to G(\mathbb{P}_s)$.

[3)] The *radical* $\mathrm{R}(G)$ of a Lie group G is by definition the largest connected solvabe normal Lie subgroup of G. It is always closed in G.

[4)] It is well-known that every projective rational manifold is simply connected, see for example [8] and [14].

It remains to prove that the identity is the only holomorphism of Q that commutes with all $u \in U$. Let $g_0 \in G(Q)$ such that $g_0 \cdot u = u \cdot g_0$ for all $u \in U$. Then U preserves the set F_0 of fixed points of g_0. By Theorem 2, F_0 is not empty, and U acts transitively on Q. It follows that $F_0 = Q$. As $G(Q)$ acts effectively, g_0 is the identity, which proves Theorem 4. $\diamond$

4. Proof of the triviality of certain fiber bundles. In this section we prove:

Theorem 5 *Let Q be a connected projective rational manifold and E a holomorphic vector bundle over a complex torus A with Q as typical fiber. Then E is a homogeneous complex manifold if and only of Q is homogeneous and E is equivalent as a bundle over A to the direct product $A \times Q$.*

The proof is based on the following

Lemma 6 *Let G be a connected real or complex Lie group and S a connected closed real or complex normal subgroup of G that is semisimple and has trivial center. Suppose the quotient group G/S is solvable. Then G is the direct product of S and the radical $\mathrm{R}(G)$. In particular, if G/S is abelian, then $\mathrm{R}(G)$ coincides with the center $\mathrm{Z}(G)$ of G.*

PROOF: By [6], Corollaire 3, p. 76, as well as [12], Theorem 84, p. 278, the radical $\mathrm{R}(G)$ is mapped to the radical of G/S by the canonical projection $G \to G/S$, and hence to itself. Therefore, $G = \mathrm{R}(G) \cdot S$. The intersection $\mathrm{R}(G) \cap S$ is a closed solvable normal subgroup of S and hence, as S is semisimple, discrete. As a discrete normal subgroup, $\mathrm{R}(G) \cap S$ is contained in the center of S. It follows that $\mathrm{R}(G) \cap S = \{1\}$ and thus $G = \mathrm{R}(G) \times S$. If G/S is abelian, it further follows that $\mathrm{R}(G) = \mathrm{Z}(G)$, since every element of $\mathrm{R}(G)$ commutes with every element of S and $\mathrm{R}(G)$ is isomorphic to G/S. $\diamond$

PROOF OF THEOREM 5: If Q is homogeneous and $E = A \times Q$, then E is homogeneous as well. Conversely, suppose E is homogeneous. We write E in the form G°/H, where $G^\circ := G(E)^\circ$ and H is the stabilizer subgroup in G° of a point in E. We let $\pi : E \to A$ denote the holomorphic bundle projection and $\alpha : E \to A(E)$ the Albanese map. Since every π-fiber F is projective rational and therefore has a one-pointed Albanese torus, $\alpha|_F$ is always constant. So there exists a holomorphic map $\iota : A \to A(E)$ such that $\alpha = \iota \circ \pi$. By the universal property of $A(E)$, the map ι is necessarily biholomorphic and affine, so that we may identify A with $A(E)$ and π with α. By Theorem 1 (b), we may further set

$Q = M/H$, where M is the kernel of $\gamma : G^\circ \to T(E)$. The groups M and M° are both closed complex normal Lie subgroups of G, both act transitively on each α-fiber. For every $a \in A$ let $M_a := \{g \in M \mid g|_{\pi^{-1}(a)} = \mathrm{id}\}$ and $M'_a := M_a \cap M^\circ$. The groups M_a and M'_a are normal subgroups of M and M°, respectively, and it holds that $\bigcap_{a\in A} M_a = \{1\}$, as M acts effectively on E. The connected complex Lie group M°/M'_a now acts holomorphically, transitively and effectively on the fiber $\pi^{-1}(a)$. By Theorem 4, it therefore follows that M°/M'_a is always semisimple with trivial center. But then M° itself is semisimple with trivial center: The radical and the center of M° are mapped by any homomorphism $M^\circ \to M^\circ/M'_a$, $a \in A(E)$, to the radical and the center of M°/M'_a, respectively, that is, to $\{1\}$. Radical and center are thus contained in every group M'_a and thus consist only of the identity.

As G°/M° is a connected covering group of the abelian group $T(E)$, it is abelian itself, and it follows from Lemma 6 that $G^\circ = Z \times M^\circ$, where Z is the center of G°. In particular, $M = (M \cap Z) \times M^\circ$. As $M \cap Z$ is a central complex Lie subgroup of M and M/M_a can be considered as a complex Lie subgroup of $G(\pi^{-1}(a))$ acting transitively and effectively on $\pi^{-1}(a)$, $M \cap Z$ is mapped by $M \to M/M_a$ into the centralizer of M/M_a in $G(\pi^{-1}(a))$, and thus maps onto $\{1\}$ by Theorem 4. It follows that $M \cap Z \subset M_a$ for every $a \in A$ and thus $M \cap Z = \{1\}$. This means $M = M^\circ$, that is, $G = Z \times M$. As the stabilizer H is contained in M and $Z = G/M$ can be identified with A, it follows that $E = G^\circ/H = Z \times (M/H) = A \times Q$. ◇

5. Generalization of a theorem by Goto. In [9], Proposition 3, it is proved that every compact homogeneous complex manifold V with finite fundamental group is a holomorphic fiber bundle over a projective rational manifold with a complex torus as fiber. We show here:

Theorem 7 *Every compact homogenous complex manifold V is in a canonical way a holomorphic fiber bundle over a projective rational (simply connected) homogeneous manifold B with a connected complex-parallelizable fiber P.*

Here, an r-dimensional complex manifold X is called *complex-parallelizable* if there are r holomorphic vector fields on X that are linearly independent at every point of X. It is clear that X is always complex-parallelizable if there exists an r-dimensional complex Lie group that acts holomorphically and transitively on X. By Wang [15], the converse also holds if X is connected and compact.

The proof of Theorem 7 makes use of:

Theorem* *Every connected compact homogeneous Kähler manifold with finite fundamental group is projective rational and in particular simply connected.*

Compare [3] and [9]. The proof of the first mentioned author for simple connectedness has been expounded in the exposé of a talk by J.P. Serre (Séminaire Bourbaki, May 1954).

For the proof of Theorem 7, we let H denote the stabilizer subgroup in $G(V)^\circ$ of a point in V, and N the normalizer of H° in $G(V)^\circ$. Then N is a closed complex Lie subgroup of G containing H. Thus there exists a natural holomorphic fiber bundle map from $V = G/H$ onto the compact homogeneous complex manifold $B := G/N$ with typical fiber $P := N/H$. By definition, H° is normal in N, so that furthermore $P = (N/H^\circ)/(H/H^\circ)$ holds. As H/H° is discrete in N/H°, the fiber P is complex-parallelisable. To prove that this fiber bundle also has the other properties claimed in Theorem 7, it is therefore sufficient to prove that N is connected and B is projective rational. But this follows immediately from

Theorem 7′ *Let G be a connected complex Lie group and H a closed complex Lie subgroup of G, such that G/H is compact. Then:*

(1) *The normalizer N of H° in G contains the radical* $\mathrm{R}(G)$ *of G.*[5)]

(2) *N is connected and G/N is projective rational.*

PROOF: It is clearly enough to prove the theorem for the case that G is simply connected. Let $n := \dim_{\mathbb{C}} G$ and $k := \dim_{\mathbb{C}} H$. We consider the Lie algebra $\mathfrak{h}$ of H as a point in the Grassmann manifold $M_{n,k}$ of k-dimensional subspaces of the Lie algebra $\mathfrak{g}$ of G. Via its adjoint representation, G has a natural holomorphic action on $M_{n,k}$, and the stabilzier subgroup of $\mathfrak{h}$ in G is precisely N. Since $H \subset N$, there exists a holomorphic map from G/H onto the orbit $B \approx G/N$ of $\mathfrak{h} \in M_{n,k}$ under G. As G/H is compact by assumption, B is an analytic subset in $M_{n,k}$ on which G acts transitively. As $M_{n,k}$ is a projective algebraic manifold with vanishing first Betti number, by Lemma 3, $\mathrm{R}(G)$ has a fixed point in B. As $\mathrm{R}(G)$ is normal in G and G acts transitively on B, it follows that every point of B, in particular $\mathfrak{h}$, is a fixed point of $\mathrm{R}(G)$. This implies $\mathrm{R}(G) \subset N$, which proves claim (1).

By the theorem of Levi-Malcev (compare [6], Théorème 5, p. 89, as well as [12], Theorem 84, p. 278), G is the semidirect product $\mathrm{R}(G) \cdot S$ of $\mathrm{R}(G)$ by a closed

[5)]The first statement including its proof was communicated to us personally by J. Tits.

semisimple complex (in general not normal) subgroup S of G. If G is simply connected, so are $\mathrm{R}(G)$ and S. As $\mathrm{R}(G) \subset N^\circ$,

$$N = \mathrm{R}(G) \cdot L, \quad N^\circ = \mathrm{R}(G) \cdot L^\circ,$$

where $L := N \cap S$. From this, we obtain the quotient representation $B = S/L$ for $B = G/N$.

As a complex semisimple Lie group, the group S is algebraic, and the representation $x \mapsto \mathrm{Ad}_{\mathfrak{g}}(x)$ of S induced by the adjoint representation of G is rational (see the remark at the end of the proof). As $L = \{x \in S \mid \mathrm{Ad}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}\}$ by definition of L, it follows that L is an algebraic subgroup of S. As such, L decomposes into only finitely many connected components, that is, L/L° is a finite group. As $B = S/L$ and S is connected and simply connected, we have $\pi_1(B) \cong L/L^\circ$ by the homotopy sequence. Hence the fundamental group of the homogeneous complex manifold B is finite. As was proved above, B lies in the Grassmann manifold $M_{n,k}$ as an analytic subset, and thus B is also projective algebraic. It follows from Theorem* that B is even projective rational. In particular, $L/L^\circ = \pi_1(B) = \{1\}$, that is, $L = L^\circ$. Since this implies $N = N^\circ$, Theorem $7'$ is proved. ◇

Remark The property of S used above is well-known; it can be proved as follows. Every connected semisimple complex Lie group G has a faithful holomorphic representation (see Séminaire Sophus Lie, Paris 1954; Exp. 22). As a connected linear complex Lie group whose Lie algebra coincides with its commutator algebra, it is always algebraic (cf. Chevalley: Théorie des groupes de Lie, Vol. 2, Groupes algébriques, Paris 1951, p. 177), and hence G may be identified with an algebraic subgroup of $\mathrm{GL}(m, \mathbb{C})$ if m is large enough. Now let $\varrho : G \to \mathrm{GL}(n, \mathbb{C})$ be any holomorphic representation of G. The graph $\Gamma_\varrho := \{(g, \varrho(g)) \mid g \in G\}$ of ϱ is a semisimple complex Lie subgroup of $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$, and therefore algebraic by the remarks above. The projection of $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ onto the first factor induces a bijective rational map from Γ_ϱ onto G. As G and Γ_ϱ are free of singularities, this map $\Gamma_\varrho \to G$ is even biholomorphic and birational (for example due to an elementary special case of Zariski's "Main Theorem", cf. Lang: Introduction to algebraic geometry, Interscience Publishers, New York 1958, Chap. V). This implies that ϱ itself is a rational map.

Corollary 2 (Goto) *If $\pi_1(V)$ is finite, then P is a complex torus.*

PROOF: Let G be a universal covering group of $G(V)^\circ$. We keep the notation from the proof of Theorem $7'$. Then the fundamental group of V is isomorphic to the group H/H°. By assumption, H/H° is finite. From $P = (N/H^\circ)/(H/H^\circ)$ it thus follows that P is compact, that N/H° is a compact complex Lie group and also a complex torus group. But then P is necessarily a complex torus itself. ◇

Corollary 3 *Every connected compact complex manifold V on which a solvable complex Lie group G acts holomorphically and transitively, is complex parallelizable.*

PROOF: Without loss of generality we may assume G to be connected and to act effectively on V. If we again set $V = G/H$, then it follows from Theorem 7′ that, since G is solvable, H° is a normal subgroup of G. As G acts effectively on V, this implies $H^\circ = \{1\}$. Thus V is a quotient of G by a discrete subgroup H and therefore complex-parallelizable. ◇

6. Proof of Theorem I and corollaries With the preparations made in Sections 2 to 5, we can now prove Theorem I. For the given compact homogeneous Kähler manifold V, we first consider the natural holomorphic fibration described in Theorem 7. As the fiber P is also a Kähler manifold, P is a complex torus according to Corollary 1 of Theorem 1. In particular, $\pi_1(P)$ is abelian. From the exact homotopy sequence

$$\cdots \to \pi_1(P) \to \pi_1(V) \to \pi_1(B) = \mathbf{0}$$

it then follows that $\pi_1(V)$ is abelian as well. By Theorem 1, V is a holomorphic fiber bundle with respect to the Albanese map α over $A(V)$ with a connected typical fiber F. From the exact sequence

$$\mathbf{0} \to \pi_1(F) \to \pi_1(V) \to \pi_1(A(V)) \to \mathbf{0}$$

it follows that F also has an abelian fundamental group, and moreover, that the first Betti numbers of V, $A(V)$ and F satisfy $\mathrm{b}_1(V) = \mathrm{b}_1(A(V)) + \mathrm{b}_1(F)$. As V is a Kähler manifold, we also have $\mathrm{b}_1(V) = \dim_{\mathbb{R}} A(V) = \mathrm{b}_1(A(V))$. Hence $\mathrm{b}_1(F) = 0$, which shows that the fundamental group of F is finite. As F is also a homogeneous compact Kähler manifold, F is projective rational by Theorem*. By Theorem 5, it follows that $V = A(V) \times F$, which proves Theorem I. ◇

Finally, we note three consequence of Theorem I.

Corollary 1 *If V is a connected compact homogeneous Kähler manifold, then $G(V)^\circ$ is reductive and every maximal compact subgroup of $G(V)^\circ$ acts transitively on V. In particular, V can be endowed with a Kähler metric such that the groups of all holomorphic isometries acts transitively on V.*

The proof is immediate from Theorem I and the fact that on a connected and simply connected compact manifold the maximal compact subgroups of a transitive Lie group of homeomorphisms act transitively themselves (cf. [11]).

Corollary 2 *Let V be a connected compact Kähler manifold and $H \subset G(V)_{\mathrm{r}}$ a real Lie group that acts transitively on V. Then:*

(a) *If H is solvable, then V is a complex torus.*

(b) *If H is semisimple, then V is projective rational.*

(c) *If $\dim_{\mathbb{R}} H = \dim_{\mathbb{R}} V$, then V is a complex torus.*

PROOF: According to Theorem I we write $V = A \times F$, where A is the Albenese torus of V and F is projective rational. Then: $G(V)^\circ = G(A)^\circ \times G(F)^\circ$ (cf. [1], Corollaire, p. 161). Here, $G(A)^\circ$ is a complex (abelian) torus group and by Theorem 4, $G(F)^\circ$ is a semisimple complex Lie group. If $\gamma : G(V)^\circ \to G(A)^\circ$ and $\delta : G(V)^\circ \to G(F)^\circ$ are the canonical holomorphic epimorphisms, then we consider the groups $\gamma(H^\circ)$ and $\delta(H^\circ)$. Since H° acts transitively on V as well, $\gamma(H^\circ)$ and $\delta(H^\circ)$ act transitively on A and F, respectively. In particular, $\gamma(H^\circ) = G(A)^\circ$.

If H is solvable, then $\delta(H^\circ)$ is a connected solvable real Lie subgroup of $G(F)_{\mathrm{r}}$. But by Theorem 4, $\delta(H^\circ)$ is also semisimple. Hence $\delta(H^\circ) = \{1\}$, that is, F consists of a single point.

If H is semisimple, then $\gamma(H^\circ) = G(A)^\circ$ is an abelian semisimple real Lie group, which means $G(A)^\circ = \{1\}$, that is, A consists of a single point.

Now assume $\dim_{\mathbb{R}} H = \dim_{\mathbb{R}} V$. The real Lie group $N := G(F)^\circ \cap H^\circ$ is the kernel of $\gamma|_{H^\circ} : H^\circ \to G(A)^\circ$ and hence acts transitively on F. Moreover,

$$\dim_{\mathbb{R}} N + \dim_{\mathbb{R}} G(A)^\circ = \dim_{\mathbb{R}} H^\circ,$$

and together with

$$\dim_{\mathbb{R}} H^\circ = \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} A + \dim_{\mathbb{R}} F, \quad \dim_{\mathbb{R}} A = \dim_{\mathbb{R}} G(A)^\circ$$

it follows that N has the same dimension as F. Therefore, F is the quotient of N° by a discrete subgroup. Since F is simply connected and N° is connected, F and N° are topologically equivalent; in particular, N° is a compact real Lie group.

Since F is homogeneous and projective rational, it now follows that the Euler-Poincaré characteristic of F is positive, $\chi(F) > 0$ (compare [3, 9]). On the other hand, χ vanishes for every connected non-trivial compact real Lie group, since there are fixed point-free homeomorphisms that are homotopic to the identity. From $\chi(F) = \chi(N^\circ)$ it thus follows that F consists of a single point. $\diamond$

Corollary 3 *Every compact homogeneous Kähler manifold, whose universal covering is a cell, is a complex torus.*

PROOF: The universal covering of V is the product of the universal coverings of $A(V)$ and F by Theorem I. $\diamond$

References

[1] A. Blanchard: *Sur les variétés analytiques complexes*, Annales Scientifiques de l'Ecole Normale Supérieure 73, 157-202 (1956)

[2] S. Bochner, D. Montgomery: *Groups on analytic manifolds*, Annals of Mathematics 48, 659-669 (1947)

[3] A. Borel: *Kählerian coset spaces of semisimple Lie groups*, Proceedings of the National Academy of Sciences U.S. 40, 1147-1151 (1954)

[4] A. Borel: *Topology of Lie groups and characteristic classes*, Bulletin of the American Mathematical Society 61, 397-432 (1955)

[5] A. Borel: *Groupes linéaires algébriques*, Annales of Mathematis 64, 20-82 (1956)

[6] N. Bourbaki: *Groupes et Algèbres de Lie*, Chapitre I, Paris: Hermann 1960

[7] W.L. Chow: *Algebraic varieties with rational dissections*, Proceedings of the National Academy of Sciences U.S. 42, 116-119 (1956)

[8] C. Ehresmann: *Sur la topologie de certains espaces homogènes*, Annals of Mathematics 35, 396-443 (1934)

[9] M. Goto: *On algebraic homogeneous spaces*, American Journal of Mathematics 76, 811-818 (1954)

[10] Y. Matsushima: *Sur les espaces homogènes Kähleriens d'un groupe de Lie reductif*, Nagoya Mathematical Journal 11, 53-60 (1957)

[11] D. Montgomery: *Simply connected homogeneous spaces*, Proceedings of the American Mathematical Society 1, 467-469 (1950)

[12] L. Pontrjagin: *Topological Groups*, Princeton University Press 1946

[13] R. Remmert: *Holomorphe und meromorphe Abbildungen komplexer Räume*, Mathematische Annalen 133, 328-370 (1957)

[14] J.P. Serre: *On the fundamental group of a unirational variety*, Journal of the London Mathematical Society 34, 481-484 (1959)

[15] H.C. Wang: *Complex parallelisable manifolds*, Proceedings of the American Mathematical Society 5, 771-776 (1954)

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