Relative invariants and non-associative algebras

By JOSEF DORFMEISTER and MAX KOECHER

Contents

Introduction

The present work originates from the attempt to present some older results, partially available as lecture notes, in a unified form, as well as more recent results on regular cones. In doing so we found that many results could be generalized to arbitrary domains in a finite-dimensional real vector space V after a suitable modification.

In these notes we consider domains $\mathscr{G}$ in V together with a positive and infinitely often differentiable map $\eta : \mathscr{G} \to \mathbb{R}$. Of particular interest is the Lie group $\mathrm{Aut}(\mathscr{G}, \eta)$ of those $W \in \mathrm{GL}(V)$ that map $\mathscr{G}$ to itself and for which there exists an $\alpha(W) \in \mathbb{R}$ such that $\eta(Wx) = \alpha(W)\eta(x)$ for all $x \in \mathscr{G}$. Let $\eta'_x(u)$ denote the differential of $-\log(\eta(x))$ evaluated at $u \in V$, and $\mathfrak{L}(\mathscr{G}, \eta)$ the set of those $T \in \mathrm{End}(V)$ for which $\eta'_x(Tx)$ is independent of $x \in \mathscr{G}$. It turns out that $\mathfrak{L}(\mathscr{G}, \eta)$ is a Lie algebra of endomorphisms that is closely related to the Lie algebra of $\mathrm{Aut}(\mathscr{G}, \eta)$. For example, if η diverges to ∞ at the boundary of $\mathscr{G}$, then both Lie algebras coincide (see I, §1).

Of particular importance are the cases in which $\mathrm{Aut}(\mathscr{G}, \eta)$ acts transitively on $\mathscr{G}$, or in which $\mathrm{Aut}(\mathscr{G}, \eta)$ has an open orbit in $\mathscr{G}$. This situation can be described as $\mathscr{G}$ containing an open orbit of a linear group Γ and $\eta : \mathscr{G} \to \mathbb{R}^+$ being a relative invariant of Γ. In order to determine the properties of $\mathscr{G}$ and η it is therefore reasonable to choose Γ maximal, that is, to set $\Gamma = \mathrm{Aut}(\mathscr{G}, \eta)$. Note that we will not only consider this "maximal" situation. In I, §2 the vector space of relative invariants of a given group $\Gamma \subset \mathrm{Aut}(\mathscr{G})$ is studied and described via the Lie algebra of Γ, in case this Lie algeba is algebraic.

The fact that in many places we set $\Gamma = \mathrm{Aut}(\mathscr{G}, \eta)$ is one aspect which distinguishes the present work from the investigations of Sato and Kimura [14]. Another difference lies in the choice of the base field $\mathbb{R}$ rather than $\mathbb{C}$, and the fact that there are no assumptions made about $\mathrm{Aut}(\mathscr{G}, \eta)$.

Let us remark that in many applications the group $\mathrm{Aut}(\mathscr{G}, \eta)$ is not reductive. The reader can find a wealth of examples in I, §5.

The existence of an open orbit of $\mathrm{Aut}(\mathscr{G}, \eta)$ in $\mathscr{G}$ can be expressed by the existence of a linear injection $u \mapsto R(u)$ of V in the Lie algebra $\mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta))$ of $\mathrm{Aut}(\mathscr{G}, \eta)$. This well-known fact allows us to define an algebra $\mathfrak{R}$ on V via the product $(u, v) \mapsto uv = R(v)u$. If $\mathfrak{R}$ has an identity element $e \in \mathscr{G}$, then η is uniquely determined on $\mathfrak{R}$ by the linear form $\sigma = \eta'_e$ (II, Theorem 2.1). It seems remarkable that in this case certain algebraic relations hold for the pair $(\mathfrak{R}, \sigma)$ (II, Lemma 1.1). Therefore, we will study algebras $\mathfrak{R}$ with identity element e together with a "closed" linear form σ in Chapter II. Here, σ is called *closed* if $(\mathfrak{R}, \sigma)$ satisfied the aforementioned relations. Of particular interest is the case where σ is *exact*, that is, it can be represented in the form η'_e with a map η that is defined on a domain $\mathscr{G}(\mathfrak{R}, \sigma)$ determined by $\mathfrak{R}$ and σ alone (II, §3). These investigations are applied to an associative algebra in II, §5. It turns out that the associative algebras $\mathfrak{R}$ for which all closed linear forms are exact are precisely those for which $\mathfrak{R}/\operatorname{Rad}\mathfrak{R}$ only has central-simple summands.

Another way to define an algebra on V through η is described in I, §4. To any "non-degenerate" $\eta : \mathscr{G} \to \mathbb{R}$ and every $e \in \mathscr{G}$ we can associate in a canonical way a commutative (but in general not associative) algebra $\mathfrak{A}(\eta, e)$ on V. This construction can be found in several places in the literature, for example [8, 16]. It turns out $\mathfrak{A}(\eta, e)$ is a Jordan algebra if and only if the left-multiplication of $\mathfrak{A}(\eta, e)$ is contained in $\mathfrak{Lie}(\operatorname{Aut}(\mathscr{G}, \eta))$. In this case $\mathscr{G}(\mathfrak{A}, \eta'_x)$ is the connected component containing e of the group of invertible elements of $\mathfrak{A} = \mathfrak{A}(\eta, e)$. This is the case if the "dual" object $\mathscr{G}^{\eta,e}$ for a non-degenerate η equals $\mathscr{G}$, that is, if $\mathscr{G}$ is "self-dual" (II, §4).

Note that among the homogeneous regular cones $\mathscr{K}$ precisely those are self-dual for which the tube domain $V + \mathrm{i}\mathscr{K}$ is symmetric. In this case $\mathscr{K}$ is easily described by $\mathfrak{A}$ [1, 8]. Also the general case of a regular homogeneous cone or of a homogeneous Siegel domain can be described with the help of the algebras $\mathfrak{A}$, compare [3, 4].

Notation

Let V denote a real vector space of positive finite dimension. Let $\operatorname{End}(V)$ denote the algebra of endomorphisms of V, and $\operatorname{GL}(V)$ the general linear group of V. By $V^{(n)}$, $n \geq 1$, we mean the vector space of n-fold linear symmetric map from V^n to $\mathbb{R}$, and by $V^* = V^{(1)}$ we mean the dual space of V. In the natural topology of a finite-dimensional real vector space, let $\mathring{A}$ denote the (open) interior of a subset A, and let $\overline{A}$ denote the closed hull of A. $\mathbb{R}^+$ denotes the set of positive real numbers.

Let $\mathscr{G}$ always denote a non-empty open domain in V. By $\mathrm{C}^\infty(\mathscr{G})$ we mean the set of infinitely often differentiable maps $\eta : \mathscr{G} \to \mathbb{R}$, and by $\mathrm{C}^\infty_+(\mathscr{G})$ the set of positive maps from $\mathrm{C}^\infty(\mathscr{G})$.

For a closed subgroup $\varGamma$ of $\mathrm{GL}(V)$, we let $\mathfrak{Lie}(\varGamma)$ denote the Lie algebra of endomorphisms of V of the Lie group $\varGamma$, that is,

$$\mathfrak{Lie}(\varGamma) = \{T \in \mathrm{End}(V) \mid \exp(\tau T) \in \varGamma \text{ for all } \tau \in \mathbb{R}\}.$$

If σ is a symmetric and non-degenerate bilinear form on V, and $T \in \mathrm{End}(V)$, then T^σ denotes the adjoint endomorphism of T with respect to σ. For a subset $\mathfrak{T}$ of $\mathrm{End}(V)$ let $\mathfrak{T}^\sigma = \{T^\sigma \mid T \in \mathfrak{T}\}$.

Part I

Relative invariants of domains

§1 Certain subgroups of the automorphism group of a domain

1 Let V' be another finite-dimensional real vector space. For a continuously differentiable map $f : \mathscr{G} \to V'$ and $u \in V$ we define the *directional derivative* by

$$\Delta_x^u f(x) = \frac{\mathrm{d}}{\mathrm{d}\tau} f(x + \tau u)|_{\tau=0}, \quad x \in \mathscr{G}.$$

For fixed $x \in \mathscr{G}$ the map $u \mapsto \Delta_x^u f(x)$ is then linear. Under obvious assumptions, the *chain rule* holds,

$$\Delta_x^u (f \circ g)(x) = \Delta_y^{v(x)} f(y)|_{y=g(x)} \quad \text{with } v(x) = \Delta_x^u g(x).$$

Given $\eta \in \mathrm{C}_+^\infty(\mathscr{G})$, $n \geq 1$, define for η maps $\eta^{(n)} : G \to V^{(n)}$, $x \mapsto \eta_x^{(n)}$, via

$$\eta_x^{(n)}(u_1, \ldots, u_n) = (-1)^n \Delta_x^{u_1} \cdots \Delta_x^{u_n} \log(\eta(x)), \quad x \in \mathscr{G}.$$

For short, write $\eta' = \eta^{(1)}$, $\eta'' = \eta^{(2)}$, etc. Note that these are derivatives of $\log(\eta)$ and not of η.

From the definition,

$$\eta_x^{(n+1)}(u_1, \ldots, u_n, v) = -\Delta_x^v \eta_x^{(n)}(u_1, \ldots, u_n).$$

The domain $\mathscr{G}$ is called a *cone* of $x \in \mathscr{G}$ implies $\xi x \in \mathscr{G}$ for $\xi > 0$. The positive map $\eta : \mathscr{G} \to \mathbb{R}$ is called (*positively*) *homogeneous* if $\mathscr{G}$ is a cone and if there is $\kappa \in \mathbb{R}$ such that $\eta(\xi x) = \xi^\kappa \eta(x)$ for all $x \in \mathscr{G}$ and $\xi > 0$.

If η is homogeneous, then $\eta^{(n)} : \mathcal{G} \to V^{(n)}$, $n \geq 1$, is homogeneous of degree $-n$ and Euler's differential equations hold:

$$\eta_x^{(n+1)}(u_1, \ldots, u_n, v) = -\Delta_x^x \eta_x^{(n)}(u_1, \ldots, u_n) = n\eta_x^{(n)}(u_1, \ldots, u_n).$$

Let $\mathrm{Aut}(\mathcal{G})$ denote the group of those $W \in \mathrm{GL}(V)$ that map $\mathcal{G}$ to itself. A $W \in \mathrm{GL}(V)$ belongs to $\mathrm{Aut}(\mathcal{G})$ if and only if W maps both $\overline{\mathcal{G}}$ and the boundary of $\mathcal{G}$ to itself. Then $\mathrm{Aut}(\mathcal{G})$ is a closed subgroup of $\mathrm{GL}(V)$. The elements of $\mathrm{Aut}(\mathcal{G})$ are called *automorphisms of* $\mathcal{G}$. For $\eta \in \mathrm{C}_+^\infty(\mathcal{G})$ let $\mathrm{Aut}(\mathcal{G}, \eta)$ denote the set of those $W \in \mathrm{Aut}(\mathcal{G})$ for which there exists $\alpha(W) > 0$ with $\eta(Wx) = \alpha(W)\eta(x)$ for all $x \in \mathcal{G}$. Clearly $\alpha(W)$ is uniquely determined by W. $\mathrm{Aut}(\mathcal{G}, \eta)$ is a closed subgroup of $\mathrm{Aut}(\mathcal{G})$ and α a continuous homomorphism of $\mathrm{Aut}(\mathcal{G}, \eta)$ to the positive real numbers. We see directly that $\mathrm{Aut}(\mathcal{G}, \eta^\kappa) = \mathrm{Aut}(\mathcal{G}, \eta)$ for all real numbers $\kappa \neq 0$. From the chain rule we get

$$\eta'_{Wx}(Wu) = \eta'_x(u), \quad \eta''_{Wx}(Wu, Wv) = \eta''_x(u, v), \quad \text{etc.} \tag{1.1}$$

for all $W \in \mathrm{Aut}(\mathcal{G}, \eta)$, $x \in \mathcal{G}$, $u, v \in V$.

2 To study the Lie algebra $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$ of $\mathrm{Aut}(\mathcal{G}, \eta)$, we need

Lemma 1.1 *For $T \in \mathrm{End}(V)$, the following are equivalent:*

(a) $\eta'_x(Tx) = \eta'_y(Ty)$ *for* $x, y \in \mathcal{G}$.

(b) $\eta''_x(Tx, u) = \eta'_x(Tu)$ *for* $x \in \mathcal{G}$ *and* $u \in V$.

If η is homogeneous, then these are equivalent to:

(c) $\eta'''_x(Tx, u, v) = \eta''_x(Tu, v) + \eta''_x(u, Tv)$ *for* $x \in \mathcal{G}$ *and* $u, v \in V$.

PROOF: By definition of η' and η'',

$$\begin{aligned}\Delta_x^u \eta'_x(Tx) &= -\Delta_x^u \Delta_x^{Tx} \log(\eta(x)) \\ &= -\Delta_x^{Tu} \log(\eta(x)) - \Delta_x^{Tx} \Delta_x^u \log(\eta(x)) \\ &= \eta'_x(Tu) - \eta''_x(Tx, u).\end{aligned}$$

Then the equivalence of (a) and (b) follows. By applying Δ_x^v, (c) follows from (b), and (c) implies that $\eta''_x(Tx, u) - \eta'_x(Tu)$ does not depend on x and is homogeneous of degree -1. So (c) also follows from (b). ◇

The vector space of $T \in \mathrm{End}(V)$ that satisfy one of the conditions in Lemma 1.1 is denoted by $\mathfrak{L}(\mathcal{G}, \eta)$. Because of (1.1), $T \in \mathfrak{L}(\mathcal{G}, \eta)$ implies $W^{-1}TW \in \mathfrak{L}(\mathcal{G}, \eta)$ for $W \in \mathrm{Aut}(\mathcal{G}, \eta)$. Part (a) of Lemma 1.1 shows that

$$\lambda_\eta(T) = \lambda(T) = -\eta'_x(Tx), \quad T \in \mathfrak{L}(\mathcal{G}, \eta), \tag{1.2}$$

does not depend on $x \in \mathcal{G}$. Hence λ is a linear form on $\mathfrak{L}(\mathcal{G}, \eta)$. From (1.1),

$$\lambda(W^{-1}TW) = \lambda(T) \quad \text{for } T \in \mathfrak{L}(\mathcal{G}, \eta), W \in \mathrm{Aut}(\mathcal{G}, \eta). \tag{1.3}$$

Lemma 1.2 *$\mathfrak{L}(\mathcal{G}, \eta)$ is a Lie algebra of endomorphisms of V, and $\lambda([T, S]) = 0$ for all $T, S \in \mathfrak{L}(\mathcal{G}, \eta)$.*

PROOF: For $T, S \in \mathfrak{L}(\mathcal{G}, \eta)$, set $u = Sx$ in part (b) of Lemma 1.1 and obtain $\eta''_x(Tx, Sx) = \eta'_x(TSx)$. Exchange T and S, subtract and obtain $\eta'_x([T, S]x) = 0$. Now part (a) of Lemma 1.1 shows that $[T, S] \in \mathfrak{L}(\mathcal{G}, \eta)$ and that $\lambda([T, S]) = 0$ holds. ◇

3 By definition, η is associated to a continuous homomorphism α of $\mathrm{Aut}(\mathcal{G}, \eta)$ to the positive real numbers $\mathbb{R}^+$. For $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$, $\exp(\tau T)$, $\tau \in \mathbb{R}$, is a one-parameter subgroup of $\mathrm{Aut}(\mathcal{G}, \eta)$ such that $\tau \mapsto \alpha(\exp(\tau T))$ is a continuous homomorphism to $\mathbb{R}^+$. Thus there is a $\lambda \in \mathbb{R}$ with $\alpha(\exp(\tau T)) = \mathrm{e}^{\lambda}$. First, we study the analogous situation for $T \in \mathfrak{L}(\mathcal{G}, \eta)$ and indicate the relation to $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$.

Lemma 1.3

(a) $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta)) = \mathfrak{Lie}(\mathrm{Aut}(\mathcal{G})) \cap \mathfrak{L}(\mathcal{G}, \eta)$.

(b) *Let $x \in \mathcal{G}$, $T \in \mathfrak{L}(\mathcal{G}, \eta)$ and I an open interval in $\mathbb{R}$. If also $\exp(\tau T)x \in \mathcal{G}$ for $\tau \in I$, then $\eta(\exp(\tau T)x) = \mathrm{e}^{\tau\lambda(T)}\eta(x)$ for all $\tau \in I$.*

(c) *For $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$ and $x \in \mathcal{G}$, we have $\eta(\exp(\tau T)x) = \mathrm{e}^{\lambda(T)}\eta(x)$.*

PROOF: For $T \in \mathrm{End}(V)$, $x \in \mathcal{G}$ and $\tau \in \mathbb{R}$ set $W = \exp(\tau T)$. If I is an open interval of $\mathbb{R}$ for which $Wx \in \mathcal{G}$ for $\tau \in I$ holds, then set $\varphi(\tau) = \log(\eta(Wx))$. The chain rule then gives

$$\dot\varphi(\tau) = -\eta'_{Wx}(\dot W x) = -\eta'_{Wx}(TWx), \quad \tau \in I. \tag{1.4}$$

If $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta))$, then $\dot\varphi(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\log(\alpha(W))$ is independent of x, and (1.4) shows for $\tau = 0$ that T belongs to $\mathfrak{L}(\mathscr{G},\eta)$ by Lemma 1.1 (a). For $T \in \mathfrak{L}(\mathscr{G},\eta)$ we have for the same reason $\dot\varphi(\tau) = \lambda(T)$ such that $\eta(Wx) = \mathrm{e}^{\tau\lambda(T)}\eta(x)$ for $\tau \in I$. If also $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}))$, this statement holds for all τ. This proves the lemma. ◇

4 By (b) of Lemma 1.3, we clearly obtain for $\mathscr{G} = V\backslash\{0\}$ that $\mathfrak{L}(\mathscr{G},\eta) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta))$. For arbitray domains and arbitrary η one cannot expect such a general statement. The previous arguments remain valid if $\mathscr{G}$ is replaced by a subset of $\mathscr{G}$. In doing so, $\mathfrak{L}(\mathscr{G},\eta)$ need not necessarily change, while $\mathrm{Aut}(\mathscr{G},\eta)$ can degenerate to the trivial group. But one can expect the same result if the function η defines the domain $\mathscr{G}$.

The map $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ is called *exploding* if $\eta(x)$ diverges to $+\infty$ as x converges to a boundary point of $\mathscr{G}$.

Theorem 1.4 *If η is exploding, then:*

(a) $\mathfrak{L}(\mathscr{G},\eta) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta))$.

(b) $\eta(\exp(\tau T)x) = \mathrm{e}^{\lambda(T)}$ *for all* $x \in \mathscr{G}$, $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta))$.

PROOF: By continuity, for $T \in \mathfrak{L}(\mathscr{G},\eta)$ and $x \in \mathscr{G}$ there exists $\varepsilon > 0$ with

$$\exp(\tau T)x \in \mathscr{G} \qquad \text{for } |\tau| < \varepsilon. \tag{1.5}$$

With Lemma 1.3 (b) it follows

$$\eta(\exp(\tau T)x)\mathrm{e}^{\tau\lambda(T)}\eta(x) \qquad \text{for } |\tau| < \varepsilon. \tag{1.6}$$

Let ω be the supremum of those ε for which (1.5) and (1.6) hold. If ω is finite, define points $y^\pm$ by $y^\pm = \exp(\pm\omega T)x$. Since η remains finite for $\tau \to \pm\omega$ by (1.6), neither y^+ nor y^- is a boundary point of $\mathscr{G}$. Thus we can again apply Lemma 1.3 (b) again and obtain $\varepsilon > 0$, such that (1.5) and (1.6) hold for $x = y^\pm$. But then (1.5) and (1.6) hold for all τ with $|\tau| < \omega + \varepsilon$, contradicting our assumption. Hence $\omega = +\infty$ and (1.6) holds for all $\tau \in \mathbb{R}$. So $T \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta))$ and the remaining assertion follows from Lemma 1.3 (a). ◇

5 A map f from $\mathcal{G}$ to another real vector space V' is called *rational map* (or *polynomial map*), if the components of f with respect to a basis of V' are rational (polynomial, respectively) in the components of a basis of V. In I.§3 we will see that for a large class of maps η the associated map $\eta' : \mathcal{G} \to V$ is rational.

More generally, consider a rational map $\chi : \mathcal{G} \to V^*$. Then there exist polynomials $\nu : \mathcal{G} \to \mathbb{R}$ and $\pi : V \to V^*$ such that

$$\chi_x = \frac{1}{\nu(x)} \pi_x \quad \text{for all } x \in \mathcal{G}. \tag{1.7}$$

Here, ν can be chosen such that ν is an *exact denominator* of χ, that is, no non-constant divisor of ν is contained in π. In this case the representation (1.7) is called *reduced*. For polynomials $\nu : V \to \mathbb{R}$ we define

$$\mathcal{D}(\nu) = \{x \in V \mid \nu(x) \neq 0\}. \tag{1.8}$$

If ν is an exact divisor of χ, then χ is defined on $\mathcal{D}(\nu)$ and real analytic.

If χ is given by a reduced representation (1.7), then $\Gamma(\chi)$ denotes the set of those $W \in \mathrm{GL}(V)$ that satisfy the two conditions

$$\nu(x)\nu(Wx) = \nu(Wx)\nu(x) \quad \text{for } x, y \in V, \tag{1.9}$$

$$\nu(y)\pi_x(Wu) = \nu(Wy)\pi_x(u) \quad \text{for } x, y \in V. \tag{1.10}$$

One easily verifies:

Lemma 1.5 *If $\chi : \mathcal{G} \to V^*$ is rational and ν is an exact denominator of χ, then:*

(a) *$\Gamma(\chi)$ is a linear algebraic subgroup of* $\mathrm{GL}(V)$.

(b) *$W \in \mathrm{GL}(V)$ belongs to $\Gamma(\chi)$ if and only if $W\mathcal{D}(\nu) = \mathcal{D}(\nu)$ and $\chi_{Wx}(Wu) = \chi_x(u)$ hold for $x \in \mathcal{D}(\nu)$, $u \in V$.*

6 Let once more $\eta \in C^\infty_+(\mathcal{G})$. If $\eta' : \mathcal{G} \to V^*$ is rational, then the arguments of 5. can be applied to $\chi = \eta'$: As in (1.7) we write

$$\eta'_x = \frac{1}{\nu(x)} \pi_x \tag{1.11}$$

with exact denominator $\nu = \nu_\eta$ of η'. With the notation from (1.8) we obtain:

Lemma 1.6 *Let* U *be a neighborhood of* 0 *in* $\mathbb{R}^n$ *and* $\varphi \in C^\infty(U)$*. If there exist coprime polynomials* π *and* ν *such that* $\varphi(x) = \frac{\pi(x)}{\nu(x)}$ *holds for all* $x = (x_1, \ldots, x_n) \in U$ *with* $\nu(x) \neq 0$*, then* $\nu(0) \neq 0$.

PROOF: Without loss of generality we may assume that $\varphi(0) = 0$. Set

$$\psi(x_1, \ldots, x_{n+1}) = \pi(x_1, \ldots, x_n) - x_{n+1}\nu(x_1, \ldots, x_n).$$

Then $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ is real analytic and $\psi(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n)) = 0$ holds. Then φ is also real analytic at 0 (cf. Malgrange [10], VI, Example 3.11.2).

Now consider a holomorphic extension of φ on a connected neighborhood $\tilde{U}$ of 0 in $\mathbb{C}^n$ and obtain $\varphi(z) = \frac{\pi(z)}{\nu(z)}$ for all $z = (z_1, \ldots, z_n) \in \tilde{U}$. We may further assume without loss of generality that ν is irreducible over $\mathbb{C}$. If also $\nu(0) = 0$, then π vanishes on $M \cap \tilde{U}$, where $M = \{z \in \mathbb{C}^n \mid \nu(z) = 0\}$. The π vanishes on the irreducible variety M and ν is a divisor of π. ◇

Lemma 1.7 *If for* $\eta \in C_+^\infty(\mathcal{G})$ *the map* $\eta' : \mathcal{G} \to V$ *is rational and* ν *is an exact denominator of* η'*, then:*

(a) $\mathcal{G} \subset \mathcal{D}(\nu)$.

(b) $\mathrm{Aut}(\mathcal{G}, \eta)$ *is a closed subgroup of the linear algebraic group* $\Gamma(\eta')$.

PROOF: For fixed $u \in V$ and $a \in \mathcal{G}$ apply Lemma 1.6 to $\varphi(x) = \eta'_{a+x}(u)$. It follows that $\nu(a) \neq 0$, that is, part (a).

To prove (b), apply (1.1) and Lemma 1.6. ◇

Theorem 1.8 *If* $\eta \in C_+^\infty(\mathcal{G})$ *is exploding,* $\eta' : \mathcal{G} \to V^*$ *and* ν *an exact denominator of* η'*, then:*

(a) $\mathcal{G}$ *is a connected component of* $\mathcal{D}(\nu)$.

(b) $\mathrm{Aut}(\mathcal{G}, \eta)$ *is a closed subgroup of finite index in* $\Gamma(\eta')$.

(c) $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta)) = \mathfrak{Lie}(\Gamma(\eta')) = \mathfrak{L}(\mathcal{G}, \eta)$ *is the Lie algebra of a linear algebraic group.*

PROOF: (a) Suppose there exists a on the boundary of $\mathcal{G}$ with $\nu(a) \neq 0$. On the open set $\mathcal{D}(\nu)$ there exists a rational differential form μ that coincides with η' on $\mathcal{G} \cap \mathcal{D}(\nu)$. From the closedness of η' we obtain the closedness of μ. Since

$a \in \mathcal{D}(\nu)$, there exists (after a choice of norm on V) an open ball U centered at a and a differentiable map $\psi : U \to \mathbb{R}$ with $\Delta_x^u \psi(x) = -\eta'_x(u)$ for $x \in U$ and $u \in V$. But on $\mathcal{G} \cap U$, $\log(\eta)$ is a solution to this differntial equation, so that ψ and $\log(\eta)$ differ only by a constant on the connected component of $\mathcal{G} \cap U$. Now let $x \in \mathcal{G} \cap U$. Consider the line $g(\tau) = x + \tau(a - x)$ and define

$$\tau_0 = \sup\{\varepsilon > 0 \mid g(\tau) \in \mathcal{G} \text{ for all } \tau \text{ with } 0 \leq \tau \leq \varepsilon\}.$$

Clearly $\tau_0 > 0$, $g(\tau_0)$ lies in the boundary of $\mathcal{G}$, and the line segment S between x and $g(\tau_0)$ is contained in a connected component of $\mathcal{G} \cap U$. Thus on the one hand the maps ψ and $\log(\eta)$ differ along S only by a constant, on the other hand $\log(\eta(g(\tau)))$ diverges for $\tau \to \tau_0$, while $\psi(g(\tau))$ remains finite. Therefore, $\nu(a) = 0$ for a on the boundary of $\mathcal{G}$. The claim now follows from part (a) of Lemma 1.6.

(b) By Whitney's Theorem (cf. [17], Theorem 3) the algebraic set $\mathcal{D}(\nu)$ has only finitely many connected components in the natural topology of V, which are permuted by the $W \in \Gamma(\eta')$. Then Γ has finite index in $\Gamma(\eta')$. By part (a), $\mathcal{G}$ is a connected component of $\mathcal{D}(\nu)$. Hence $W\mathcal{G} = \mathcal{G}$ for all $W \in \Gamma$, hence $\Gamma \subset \mathrm{Aut}(\mathcal{G})$. By Lemma 1.5 (b), $\eta'_{Wx}(Wu) = \eta'_x(u)$ for all $x \in \mathcal{G}$, $u \in V$, $W \in \Gamma$, and therefore $\eta(Wx) = \alpha(W)\eta(x)$ for $x \in \mathcal{G}$, $W \in \Gamma$ with suitable $\alpha : \Gamma \to \mathbb{R}^+$. It follows that $\Gamma \subset \mathrm{Aut}(\mathcal{G}, \eta)$. Together with Lemma 1.7 (b) the claim follows.

(c) By part (b), the Lie algebras of $\mathrm{Aut}(\mathcal{G}, \eta)$ and $\Gamma(\eta')$ coincide. The claim now follows from Theorem 1.4 (a) and Lemma 1.5. ◇

§2 Relative invariants

1 Given a closed connected subgroup Γ of $\mathrm{Aut}(\mathcal{G})$, we are interested in the set $R(\mathcal{G}, \Gamma)$ of those $\eta \in \mathrm{C}^\infty_+(\mathcal{G})$ with $\Gamma \subset \mathrm{Aut}(\mathcal{G}, \eta)$. The elements of $R(\mathcal{G}, \Gamma)$ are called *relative invariants* of $\mathcal{G}$ with respect to Γ. By definition there exists for any $\eta \in R(\mathcal{G}, \Gamma)$ a uniquely defined homomorphism $\alpha_\eta : \Gamma \to \mathbb{R}^+$ such that

$$\eta(Wx) = \alpha_\eta(W)\eta(x) \quad \text{for } x \in \mathcal{G}, W \in \Gamma. \tag{2.1}$$

Clearly $R(\mathcal{G}, \Gamma)$ is a real vector space with pointwise multiplication taken for addition and $(\lambda, \eta) \mapsto \eta^\lambda$, $\lambda \in \mathbb{R}$, taken as scalar multiplication. It contains the constant maps $\mathbb{R}^+$ as a subspace.

Consider also the set $H(\Gamma)$ of continuous homomorphisms $\beta : \Gamma \to \mathbb{R}^+$, which can be considered a real vector space in the same manner. Every $\beta \in H(\Gamma)$ is real analytic on the Lie group Γ (cf. Helgason [6, Chapter II]). By construction,

$$\Phi : R(\mathcal{G}, \Gamma) \to H(\Gamma), \quad \eta \mapsto \alpha_\eta,$$

defines a homomorphism of vector spaces. If we write

$$I(\mathcal{G}, \Gamma) = \{\eta \in R(\mathcal{G}, \Gamma) \mid \eta(Wx) = \eta(x) \text{ for } W \in \Gamma\},$$

then $\ker \Phi = I(\mathcal{G}, \Gamma)$.

2 After a choice of $a \in \mathcal{G}$ let Γ_a denote the subgroup of those $W \in \Gamma$ with $Wa = a$, and let $H(\Gamma, a)$ denote the subspace of those $\beta \in H(\Gamma)$ with $\beta(W) = 1$ for all $W \in \Gamma_a$. Clearly $H(\Gamma, a)$ is independent of a for transitive Γ. By (2.1), Φ induces an injective homomorphism

$$\Phi^* : R(\mathcal{G}, \Gamma)/I(\mathcal{G}, \Gamma) \to H(\Gamma, a), \quad \eta \ker \Phi \mapsto \Phi(\eta) = \alpha_\eta,$$

of vector spaces.

Lemma 2.1 *If Γ acts transitively on $\mathcal{G}$, then Φ^* is an isomorphism.*

PROOF: Given $\beta \in H(\Gamma, a)$, define $\eta(x) = \beta(W)$ for $x = Wa$ with $W \in \Gamma$. Thus η is well-defined on $\mathcal{G}$, real analytic and $\Phi^*(\eta) = \alpha_\eta = \beta$ holds.

For $\eta \in R(\mathcal{G}, \Gamma)$, Γ is contained in $\mathrm{Aut}(\mathcal{G}, \eta)$. By Lemma 1.3 (a) it then follows that $\mathfrak{Lie}(\Gamma) \subset \mathfrak{L}(\mathcal{G}, \eta)$. By (1.2) η is then associated to a linear form λ_η of $\mathfrak{Lie}(\Gamma)$. It is easy to verify that

$$\Psi : R(\mathcal{G}, \Gamma) \to \mathfrak{Lie}(\Gamma)^*, \quad \eta \mapsto \lambda_\eta,$$

is a homomorphism of vector spaces.

Now set

$$\Lambda(\mathcal{G}, \eta) = \bigcap_{\eta \in R(\mathcal{G}, \Gamma)} \ker \lambda_\eta, \tag{2.2}$$

and obtain $[\mathfrak{Lie}(\Gamma), \mathfrak{Lie}(\Gamma)] \subset \Lambda(\mathcal{G}, \Gamma)$ by Lemma 1.2. Moreover, use Lemma 1.3 to check that $\Lambda(\mathcal{G}, \Gamma)$ contains the Lie algebras of all stabilizer subgroups Γ_a, $a \in \mathcal{G}$, as well as the Lie algebras of all compact subgroups of Γ. ◇

By definition λ_η for $\eta \in R(\mathcal{G}, \Gamma)$ vanishes on $\Lambda(\mathcal{G}, \Gamma)$ and thus induces a linear form $\lambda_\eta^* : \mathfrak{Lie}(\Gamma)/\Lambda(\mathcal{G}, \Gamma) \to \mathbb{R}$ defined by $\lambda_\eta^*(T + \Lambda(\mathcal{G}, \Gamma)) = \lambda_\eta(T)$.

Lemma 2.2

(a) $\ker \Psi = I(\mathcal{G}, \Gamma)$.

(b) Ψ *induces an isomorphism of vector spaces via*

$$\Psi^* : R(\mathcal{G}, \Gamma)/I(\mathcal{G}, \Gamma) \to (\mathfrak{Lie}(\Gamma)/\Lambda(\mathcal{G}, \Gamma))^*, \quad \eta I(\mathcal{G}, \Gamma) \mapsto \lambda_\eta^*.$$

PROOF: (a) $\eta \in R(\mathcal{G}, \Gamma)$ lies in $\ker \Psi$ if and only if $\lambda_\eta(T) = 0$ for all $T \in \mathfrak{Lie}(\Gamma)$. By Lemma 1.3 (c) this is equivalent to $\eta(\exp(T)x) = \eta(x)$ for $x \in \mathcal{G}$ and $T \in \mathfrak{Lie}(\Gamma)$. As Γ was assumed to be connected, η is Γ-invariant.

(b) Ψ^* is well-defined and a homomorphism of vector spaces. If $\eta I(\mathcal{G}, \Gamma)$ is in $\ker \Psi^*$, then $\lambda_\eta(T) = 0$ for all $T \in \mathfrak{Lie}(\Gamma)$, which implies $\eta \in \ker \Psi = I(\mathcal{G}, \Gamma)$. Thus Ψ^* is injective. To prove surjectivity of Ψ^*, note first that there are $\eta_1, \ldots, \eta_r \in R(\mathcal{G}, \Gamma)$ with $\Lambda(\mathcal{G}, \Gamma) = \ker \lambda_1 \cap \ldots \cap \ker \lambda_r$, where $\lambda_i = \lambda_{\eta_i}$. Choose r minimal such that the $\lambda_1^*, \ldots, \lambda_r^*$ are a basis of $(\mathfrak{Lie}(\Gamma)/\Lambda(\mathcal{G}, \Gamma))^*$. For an arbitrary $\mu = \gamma_1\lambda_1^* + \ldots + \gamma_r\lambda_r^*$ of $(\mathfrak{Lie}(\Gamma)/I(\mathcal{G}, \Gamma))^*$, set $\eta = \eta_1^{\gamma_1} \cdots \eta_r^{\gamma_r}$ and verify that $\lambda_\eta^* = \mu$. Thus Ψ^* is surjective. ◇

Corollary 1 *If* $\Lambda(\mathcal{G}, \Gamma) = \mathfrak{Lie}(\Gamma)$, *then* $R(\mathcal{G}, \Gamma) = I(\mathcal{G}, \Gamma)$.

Corollary 2 $R(\mathcal{G}, \Gamma)/I(\mathcal{G}, \Gamma)$ *is finite-dimensional.*

3 Now we wish to study $\Lambda(\mathcal{G}, \Gamma)$ more closely. We need the following:

Lemma 2.3 *Let* $\Omega \subset \mathrm{GL}(V)$ *be a connected Lie subgroup of* $\mathrm{GL}(V)$. *Let* $\lambda : \mathfrak{Lie}(\Omega) \to \mathbb{R}$ *be a homomorphism of Lie algebras. The the following are equivalent:*

(a) *There is a continuous homomorphism* $\alpha : \Omega \to \mathbb{R}^+$ *with* $\mathrm{d}\alpha = \lambda$.

(b) $\ker \lambda$ *contains the Lie algebras of all compact subgroups of* Ω.

(c) $\ker \lambda$ *contains the Lie algebras of all compact subgroups of the radical of* Ω.

PROOF: (1) Clearly is is sufficient to prove (c) $\Rightarrow$ (a). To this end, we write (following Hochschild [7, XVIII, Theorem 4.3]) $\Omega = \Omega_1 \cdot \Omega_2$ as a semidirect product of a reductive group Ω_1 and a simply connected solvable normal subgroup

Ω_2 of Ω. Clearly, Ω_1 and Ω_2 are connected. Moreover, from the proof of [7, XVIII, Theorem 4.2] we infer that $\Omega_1 = \Omega_{11}\Omega_{12}$ for a connected semisimple subgroup Ω_{11} and a connected compact subgroup Ω_{12} which is contained in the radical of Ω.

(2) Now let $\tilde{\alpha} : \Omega_2 \to \mathbb{R}^+$ the uniquely determined Lie group homomorphism that satisfies $\mathrm{d}\tilde{\alpha} = \lambda|_{\mathfrak{Lie}(\Omega_2)}$.

(3) According to [7, XII, Theorem 2.2, 3.1], the normal subgroup $[\Omega, \Omega_2]$ generated by $\{WAW^{-1}A^{-1} \mid W \in \Omega, A \in \Omega_2\} \subset \Omega_2$ is a connected closed subgroup of Ω_2 with Lie algebra $\mathfrak{Lie}([\Omega, \Omega_2] = [\mathfrak{Lie}(\Omega), \mathfrak{Lie}(\Omega_2)]$. Then $[\mathfrak{Lie}(\Omega), \mathfrak{Lie}(\Omega_2)] \subset \ker\lambda \cap \mathfrak{Lie}(\Omega_2)$, so that $[\mathfrak{Lie}(\Omega), \mathfrak{Lie}(\Omega_2)] \subset \ker \mathrm{d}\tilde{\alpha}$. Now $[\Omega, \Omega_2]$ is generated by $[\mathfrak{Lie}(\Omega), \mathfrak{Lie}(\Omega_2)]$, hence $[\Omega, \Omega_2] \subset \ker\tilde{\alpha}$.

(4) Now define $\alpha : \Omega_1 \cdot \Omega_2 \to \mathbb{R}^+$, $WA \mapsto \tilde{\alpha}(A)$. Then α is continuous and for all W, $\check{W} \in \Omega_1$, A, $\check{A} \in \Omega_2$, we have

$$\begin{aligned}\alpha(WA\check{W}\check{A}) &= \alpha(W\check{W}\check{W}^{-1}A\check{W}\check{A}) = \tilde{\alpha}(\check{W}^{-1}A\check{W}\check{A}) = \tilde{\alpha}(\check{W}^{-1}A\check{W}A^{-1}A\check{A}) \\ &= \tilde{\alpha}(A\check{A}) = \tilde{\alpha}(A)\tilde{\alpha}(\check{A}) = \alpha(WA)\alpha(\check{W}\check{A}).\end{aligned}$$

Thus α is a homomorphism of groups.

(5) Verify $\mathrm{d}\alpha(T_1 + T_2) = \mathrm{d}\tilde{\alpha}(T_2) = \lambda(T_2)$ for all $T_1 \in \mathfrak{Lie}(\Omega_1)$, $T_2 \in \mathfrak{Lie}(\Omega_2)$. Taking into account (1), we obtain $\mathfrak{Lie}(\Omega_1) = \mathfrak{Lie}(\Omega_{11}) + \mathfrak{Lie}(\Omega_{12})$. As Ω_{11} is semisimple, $\mathfrak{Lie}(\Omega_{11}) \subset \ker\lambda$ holds. Furthermore Ω_{12} is compact and contained in the radical of Ω. By assumption it follows that $\mathfrak{Lie}(\Omega_{12}) \subset \ker\lambda$ and we obtain $\mathfrak{Lie}(\Omega_1) \subset \ker\lambda$. Thus $\mathrm{d}\alpha(T_1 + T_2) = \lambda(T_2) = \lambda(T_1 + T_2)$ and the lemma is proven. ◇

The above lemma is now applied to $\Omega = \Gamma$. Note that in this paragraph Γ is assumed to be connected and closed in $\mathrm{Aut}(G)$.

Theorem 2.4 *If Γ acts transitively on $\mathscr{G}$ and the stabilizer Γ_e at $e \in \mathscr{G}$, has only finitely many connected components, then for every maximal compact Lie subalgebra $\mathfrak{m}$ of $\mathfrak{Lie}(\Gamma)$*

$$\Lambda(\mathscr{G}, \Gamma) = [\mathfrak{Lie}(\Gamma), \mathfrak{Lie}(\Gamma)] + \mathfrak{Lie}(\Gamma_e) + \mathfrak{m}$$

holds.

PROOF: For short, write $\mathfrak{a} = [\mathfrak{Lie}(\Gamma), \mathfrak{Lie}(\Gamma)] + \mathfrak{Lie}(\Gamma_e) + \mathfrak{m}$. Note that the map $\eta \mapsto \hat{\lambda}_\eta$ with $\hat{\lambda}_\eta(T + \mathfrak{a}) = \lambda_\eta(T)$ defines an injective homomorphism

$R(\mathcal{G},\Gamma) \to (\mathfrak{Lie}(\Gamma)/\mathfrak{a})^*$. By $\mathfrak{a} \subset \Lambda(\mathcal{G},\Gamma)$ and Lemma 2.2 it is for dimension reasons sufficient to show that this map is surjective. Let $\lambda : \mathfrak{Lie}(\Gamma) \to \mathbb{R}$ be a linear map whose kernel contains the ideal $\mathfrak{a}$. Then λ is a homomorphism of Lie algebras and $\mathfrak{m} \subset \ker\lambda$ holds. If $\mathfrak{m}'$ is another maximal compact Lie subalgebra of $\mathfrak{Lie}(\Gamma)$, then there is a $W \in \Gamma$ with $W\mathfrak{m}'W^{-1} = \mathfrak{m}$ (cf. [11]). Now, Γ is connected and $\exp(\mathrm{ad}(T)) = \mathrm{Ad}(\exp(T))$, so that $\lambda(\mathfrak{m}') = \lambda(W\mathfrak{m}'W^{-1}) = \lambda(\mathfrak{m}) = 0$ follows.

Thus $\ker\lambda$ contains the Lie algebras of all compact subgroups of Γ. By Lemma 2.3 there exists a homomorphism $\alpha : \Gamma \to \mathbb{R}^+$ that satisfies $\mathrm{d}\alpha = \lambda$.

Since $\mathfrak{Lie}(\Gamma_e) \subset \ker \mathrm{d}\alpha$, the identity component Γ_e° of Γ_e is contained in $\ker\alpha$. As Γ_e° has finite index in Γ_e, this implies $\Gamma_e \subset \ker\alpha$. By Lemma 2.1 there exists $\eta \in R(\mathcal{G},\Gamma)$ with $\alpha_\eta = \alpha$ and $\eta(e) = 1$. With this we verify that $\lambda_\eta = \mathrm{d}\alpha = \lambda$ holds (see Lemma 1.3), and the claim is proved. ◇

With regard to Lemma 3.1, the following corollary is of interest. It shows that the prerequisites for Theorem 2.4 hold at least for the identity component $\mathrm{Aut}(\mathcal{G},\eta)^\circ$ in the case of transitive $\mathrm{Aut}(\mathcal{G},\eta)$. As usual, a Lie algebra $\mathfrak{a} \subset \mathrm{End}(V)$ of endomorphisms is called *algebraic* if there exists an algebraic subgroup of $\mathrm{GL}(V)$ with Lie algebra $\mathfrak{a}$.

Corollary 1 *If Γ acts transitively on $\mathcal{G}$ and $\mathfrak{Lie}(\Gamma)$ is algebraic, then*

$$\Lambda(\mathcal{G},\Gamma) = [\mathfrak{Lie}(\Gamma),\mathfrak{Lie}(\Gamma)] + \mathfrak{Lie}(\Gamma_e) = \mathfrak{m}$$

holds for every maximal compact Lie subalgebra $\mathfrak{m}$ of $\mathfrak{Lie}(\Gamma)$.

PROOF: Let $\hat{\Gamma}$ be an algebraic subgroup of $\mathrm{GL}(V)$ with $\mathfrak{Lie}(\hat{\Gamma}) = \mathfrak{Lie}(\Gamma)$. As Γ is assumed connected, the connected component $\hat{\Gamma}^\circ$ of id in $\hat{\Gamma}$ coincides with Γ. Now set $\hat{\Gamma}_e = \{W \in \hat{\Gamma} \mid We = e\}$. Then $\hat{\Gamma}_e$ is an algebraic group and $\hat{\Gamma}_e \cap \Gamma = \Gamma_e$. Thus we can easily see that the connected components Γ_e° and $\hat{\Gamma}_e^\circ$ of id coincide. By [17], $\hat{\Gamma}_e^\circ$ has finite index in $\hat{\Gamma}_e$, hence Γ_e° has finite index in Γ_e, since $\Gamma_e^\circ = \hat{\Gamma}_e^\circ \subset \Gamma_e \subset \hat{\Gamma}_e$. Now the claim follows from Theorem 2.4. ◇

4 An $\eta \in \mathrm{C}_+^\infty(\mathcal{G})$ is called *non-degenerate* if the bilinear form η_x'' is non-degenerate. If $\hat{\eta}(x)$, for $x \in \mathcal{G}$, denotes the absolute value of the functional determinant of the map $\eta' : \mathcal{G} \to V^*$, then η is non-degenerate if and only if $\hat{\eta}$ is positive on $\mathcal{G}$.

Now suppose η is non-degenerate. As the absolute value of the functional determinant of $\eta' : \mathscr{G} \to V^*$ coincides with $\hat{\eta}$, $\eta' : \mathscr{G} \to V^*$ is an open map and $\mathscr{G}^\eta = \{\eta'_x \mid x \in \mathscr{G}\}$ is a domain in V^*. From (1.1) it follows that

$$\hat{\eta}(Wx) = \det(W)^{-2}\hat{\eta}(x) \quad \text{for } x \in \mathscr{G}, W \in \mathrm{Aut}(\mathscr{G}, \eta), \tag{2.3}$$

so in particular,

$$\mathrm{Aut}(\mathscr{G}, \eta) \subset \mathrm{Aut}(\mathscr{G}, \hat{\eta}). \tag{2.4}$$

Thus the corresponding inclusion holds for the respective Lie algebras. Analogously,

Lemma 2.5 *If η is non-degenerate, then $\mathfrak{L}(\mathscr{G}, \eta) \subset \mathfrak{L}(\mathscr{G}, \hat{\eta})$.*

PROOF: Again let $\lambda = \lambda_\eta$ be the linear form of $\mathfrak{L}(\mathscr{G}, \eta)$ defined by (1.2). For $x \in \mathscr{G}$ choose a compact neighborhood U of x in $\mathscr{G}$ and $\varepsilon > 0$ such that $\exp(\tau T)y \in \mathscr{G}$ for $y \in U$ and $|\tau| < \varepsilon$. By Lemma 1.3 (b), $\eta(Wy) = \mathrm{e}^{\tau\lambda(T)}\eta(y)$ for $y \in U$, $|\tau| < \varepsilon$ and $W = \exp(\tau T)$. After taking the logarithm and differentiating twice with respect to y, we obtain $\eta''_{Wy}(Wu, Wv) = \eta''_y(u, v)$. Analogously to (2.3) this implies

$$\hat{\eta}(Wy) = \det(W)^{-2}\hat{\eta}(y) \quad \text{for } W = \exp(\tau T), y \in U, |\tau| < \varepsilon.$$

By taking the logarithm again, differentiating again with respect to τ and setting $\tau = 0$, we obtain $\hat{\eta}'_x(Tx) = -2\,\mathrm{tr}(T)$, because $\log(\det(W)) = -\tau\,\mathrm{tr}(T)$. As $x \in \mathscr{G}$ is arbitrary, it follows that $T \in \mathfrak{L}(\mathscr{G}, \hat{\eta})$ from Lemma 1.1. ◇

5 As in 1. we consider the vector space $R(\mathscr{G}, \Gamma)$, which contains the constant maps $\mathbb{R}^+$ as a vector subspace. Clearly the notion of "non-degeneracy" carries over to elements of $R(\mathscr{G}, \Gamma)/\mathbb{R}^+$.

Lemma 2.6 *If Γ acts transitively on $\mathscr{G}$, then the set of non-degenerate elements of $R(\mathscr{G}, \Gamma)/\mathbb{R}^+$ is either empty or is dense in $R(\mathscr{G}, \Gamma)/\mathbb{R}^+$.*

In the latter case, $R(\mathscr{G}, \Gamma)$ contains a basis of non-degenerate elements.

PROOF: After choosing $a \in \mathscr{G}$, the elements of $R(\mathscr{G}, \Gamma)/\mathbb{R}^+$ are represented by $\eta \in R(\mathscr{G}, \Gamma)$ with $\eta(a) = 1$ in the following. Given non-degenerate η and arbitrary ζ, consider the element $\xi = \zeta\eta^\tau$ of $R(\mathscr{G}, \Gamma)$, for $\tau \in \mathbb{R}$. We obtain $\xi''_x = \zeta''_x + \tau\eta''_x$, so that $\hat{\xi}(x)$ is positive for all sufficiently small non-zero τ. As Γ acts transitively, the claim follows from (2.3). ◇

§3 Homogeneous domains

1 The domain $\mathcal{G}$ is called *homogeneous* if the group $\mathrm{Aut}(\mathcal{G})$ (cf. §1.1) acts transitively on $\mathcal{G}$. For a subgroup Γ of $\mathrm{GL}(V)$ and $a \in V$, we call $\Gamma a = \{Wa \mid W \in \Gamma\}$ the *orbit of Γ through a*. If $\mathcal{G}$ is homogeneous, then $\mathcal{G}$ coincides with the orbit through $a \in \mathcal{G}$ of $\mathrm{Aut}(\mathcal{G})$ for every a.

In an obvious generalization of the results of Vinberg [16, Chapter I, §6, Propositions 13, 14], we have

Lemma 3.1 *If Γ is a connected subgroup of* $\mathrm{Aut}(\mathcal{G})$ *that acts transitively on $\mathcal{G}$, then:*

(a) *The identity component $N_0(\Gamma)$ of the normalizer $N(\Gamma)$ of Γ in* $\mathrm{GL}(V)$ *is contained in the identity component* $\mathrm{Aut}(\mathcal{G})_0$ *of* $\mathrm{Aut}(\mathcal{G})$.

(b) *$N(\Gamma)$ is a linear algebraic group.*

PROOF: For $a \in \mathcal{G}$ choose a connected neighborhood U of the identity in $\mathrm{GL}(V)$ with $Wa \in \mathcal{G}$ for all $W \in U$. For $W \in N(\Gamma) \cap U$, we then have $W\mathcal{G} = W\Gamma a = W\Gamma W^{-1}Wa = \Gamma Wa = \mathcal{G}$. We obtain $N(\Gamma) \cap U \subset \mathrm{Aut}(\mathcal{G})_0$, that is, part (a) holds.

For (b), $W \in \mathrm{GL}(V)$ is contained in $N(\Gamma)$ if and only if $W\mathfrak{Lie}(\Gamma)W^{-1} \subset \mathfrak{Lie}(\Gamma)$. $\diamond$

For $\Gamma = \mathrm{Aut}(\mathcal{G})_0$ we obtain:

Corollary 1 *If $\mathcal{G}$ is homogeneous, then:*

(a) *The positive multiples of the identity belong to* $\mathrm{Aut}(\mathcal{G})$.

(b) *$\mathcal{G}$ is a cone.*

(c) $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}))$ *is a Lie algebra of a linear algebraic group.*

Compare with part (c) of Theorem 1.8.

In the following we study maps $\eta \in \mathrm{C}^{\infty}_{+}(\mathcal{G})$ for which the subgroup $\mathrm{Aut}(\mathcal{G}, \eta)$ of $\mathrm{Aut}(\mathcal{G})$ acts transitively on $\mathcal{G}$. In this case $\mathcal{G}$ is a cone. The multiples of the identity belong to $\mathrm{Aut}(\mathcal{G}, \eta)$ if η is homogeneous.

2 In the following, algebras on V will play an important role. A pair $(V, \cdot)$ is called an *algebra* on V, if $(u, v) \mapsto uv$ defines a *product*, that is, if $V \times V \to V$, $(u, v) \mapsto uv$ is bilinear. The algebra $(V, \cdot)$ defines two linear maps L and R of V to $\mathrm{End}(V)$, via

$$uv = L(u)v = R(v)u.$$

We call L the *left-* and R the *right-multiplication* of $(V, \cdot)$.

Let $\mathfrak{T}$ be a Lie algebra of endomorphisms of V. Let $(\mathcal{G}, \mathfrak{T}, e)$ denote the set of those algebras $\mathfrak{R} = (V, \cdot)$ with the following properties:

(A) $\mathfrak{R}$ has e as a left-identity element, and $e \in \mathcal{G}$.

(B) Let R denote the right-multiplication on $\mathfrak{R}$. Then $R(u) \in \mathfrak{T}$ for all $u \in V$.

For $\mathfrak{T} = \mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$ write $(\mathcal{G}, \eta, e) = (\mathcal{G}, \mathfrak{T}, e)$ for short.

Lemma 3.2 *If Γ is a closed subgroup of* $\mathrm{Aut}(\mathcal{G})$ *with an open orbit through $e \in \mathcal{G}$, then:*

(a) *For all $S \in \mathfrak{Lie}(\Gamma)$ with $Se \neq 0$ there exists $\mathfrak{R} = (V, \cdot)$ in $(\mathcal{G}, \mathfrak{Lie}(\Gamma), e)$ with $u \cdot (Se) = Su$ for all $u \in V$.*

(b) *If the identity belongs to $\mathfrak{Lie}(\Gamma)$, then there exists $\mathfrak{R}$ in $(\mathcal{G}, \mathfrak{Lie}(\Gamma), e)$ that has e as identity element.*

PROOF: (a) Set $H_0 = \{T \in \mathfrak{Lie}(\Gamma) \mid Te = 0\}$ and write $\mathfrak{Lie}(\Gamma) = H_0 \oplus H_1$ as a direct sum of vector subspaces, with $S \in H_1$. Since the orbit of Γ through e is open, it follows, for example from Helgason [6, Chapter II, Theorem 3.2, Proposition 4.3], that $T \mapsto Te$ is a surjection from $\mathfrak{Lie}(\Gamma)$ onto V. Then $T \mapsto Te$ is an isomorphism from H_1 to V. Now choose a linear injection $F : V \to H_1$ with $F(e) = S$ and set $Au = F(u)e$. After a choice of F, A is linear and injective, so that $A \in \mathrm{GL}(V)$. Setting $R(u) = F(A^{-1}u)$ it follows now that $R(u)e = F(A^{-1}u)e = u$ for all $u \in V$ and $R(Se) = R(F(e)e) = R(Ae) = F(e) = S$. Now define the algebra $\mathfrak{R}$ on V via the product $uv = R(v)u$ and obtain (a).

(b) In part (a) set $S = \mathrm{id}$. $\diamond$

If there is an $\mathfrak{R}$ in $(\mathcal{G}, \eta, e)$, then $\exp(R(u), u \in V$, belongs to $\mathrm{Aut}(\mathcal{G}, \eta)$ and e is an inner point of the orbit of $\mathrm{Aut}(\mathcal{G}, \eta)$ through e. We thus obtain:

Corollary 1 $(\mathcal{G}, \eta, e)$ *is non-empty if and only if the orbit of* $\mathrm{Aut}(\mathcal{G}, \eta, e)$ *has an open orbit through e.*

3 Suppose $\mathfrak{R}$ is an algebra in $(\mathscr{G}, \eta, e)$ with product $uv = L(u)v = R(v)u$. Note that the existence of such an algebra $\mathfrak{R}$ is guaranteed by the above corollary if $\mathrm{Aut}(\mathscr{G}, \eta)$ acts transitively on $\mathscr{G}$. For the left-identity e of $\mathfrak{R}$ we have $L(e) = \mathrm{id}$, so that $\det(L(x))$ is not the zero-polynomial. In the following let $u, v \in V$ and $x \in \mathscr{G}$ with $\det(L(x)) \neq 0$ be chosen. In Lemma 1.1 set $T = R(v)$ and obtain $\eta'_x(xv) = \eta'_e(v)$ and $\eta''_x(xv, u) = \eta'_x(uv)$. This means

$$\eta'_x(u) = \eta'_e(L^{-1}(x)u) \tag{3.1}$$

and

$$\eta''_x(u, b) = \eta'_e(L^{-1}(x)L(u)L^{-1}(x)v). \tag{3.2}$$

From (3.1) and Lemma 1.6 we deduce:

Theorem 3.3 *If $(\mathscr{G}, \eta, e)$ is non-empty, then η is real analytic and the map $\eta' : \mathscr{G} \to V^*, x \mapsto \eta'_x$ is rational.*

This result was proven in special cases by Rothaus [13] and Vinberg [16, Chapter III, §4].

By (3.1), the function η is determined by the linear form η'_e of V and any algebra in $(\mathscr{G}, \eta, e)$ up to a factor. In II, §2 we will investigate this relationship further.

For another application consider a map $\eta : \mathscr{G} \to \mathbb{R}$ that is non-degenerate in the sense of §2.4. We saw there that $\eta' : \mathscr{G} \to V^*, x \mapsto \eta'_x$ is an open map from $\mathscr{G}$ to the domain $\mathscr{G}^\eta = \{\eta'_x \mid x \in \mathscr{G}\}$.

Theorem 3.4 *If η is non-degenerate and if* $\mathrm{Aut}(\mathscr{G}, \eta)$ *acts transitively on $\mathscr{G}$, then the map $\eta' : \mathscr{G} \to \mathscr{G}^\eta$ is bijective.*

PROOF: By definition of $\mathscr{G}^\eta$ we only need to show the injectivity of this map. Let $x, y \in \mathscr{G}$ with $\eta'_x = \eta'_y$ be given. By assumption, there exists $W \in \mathrm{Aut}(\mathscr{G}, \eta)$ with $x = We$. For $y = Wz$, $\eta'_e = \eta'_z$ follows from (1.1). The corollary after Lemma 3.2 allows to pick an algebra $\mathfrak{R}$ in $(\mathscr{G}, \eta, e)$ and evaluate $\eta'_e = \eta'_z$ at $zu = L(z)u$. By (3.1) it follows that $\eta'_x(zu) = \eta'_z(zu) = \eta'_e(u)$ for all $u \in V$. By (3.2), $\eta''_e(u, v) = \eta'_e(uv)$, so that $\eta''_e(z, u) = \eta''_e(e, u)$ follows. As the symmetric bilinear form $(u, v) \mapsto \eta''_e(u, v)$ was assumed to be non-degenerate, it follows that $z = e$ and thus $x = y$. ◇

A more precise analysis of the map $\eta' : \mathscr{G} \to \mathscr{G}$ shows that η' is even birational. This is explored in II, §4.

4 If Γ is a subgroup of $\mathrm{Aut}(\mathscr{G})$, then ζ is called a *Γ-invariant* if

(I.1) $\zeta \in \mathrm{C}^\infty_+(\mathscr{G})$.

(I.2) ζ is exploding.

(I.3) $\zeta(Wx) = \det(W)^{-2}\zeta(x)$ for all $x \in \mathscr{G}$ and $W \in \Gamma$.

The last condition implies $\Gamma \subset \mathrm{Aut}(\mathscr{G}, \zeta)$. It is known that $\mathrm{Aut}(\mathscr{G})$-invariants exist in important cases. If Γ acts transitively on $\mathscr{G}$, the (I.3) shows that a Γ-invariant is uniquely determined up to a constant factor.

A summary of the results so far:

Theorem 3.5 *If Γ is a closed subgroup of $\mathrm{Aut}(\mathscr{G})$, ζ a Γ-invariant and if $(\mathscr{G}, \mathfrak{Lie}(\Gamma), e)$, $e \in \mathscr{G}$, is non-empty, then:*

(a) *$\zeta : \mathscr{G} \to \mathbb{R}$ is real-analytic.*

(b) *$\zeta' : \mathscr{G} \to V^*$, $x \mapsto \zeta'_x$, is rational.*

(c) $\mathfrak{L}(\mathscr{G}, \zeta) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \zeta))$.

(d) *$\zeta(\exp(T)x) = \mathrm{e}^{-2\,\mathrm{tr}(T)}\zeta(x)$ for $x \in \mathscr{G}$ and $T \in \mathfrak{Lie}(\Gamma)$.*

(e) *If $\eta : \mathscr{G} \to \mathbb{R}$ is differentiable and if $\eta(Wx) = \det(W)^{-2}\eta(x)$ for $x \in \mathscr{G}$, $W \in \Gamma$, then η is a constant multiple of ζ.*

PROOF: Because of $\Gamma \subset \mathrm{Aut}(\mathscr{G}, \zeta)$ we have $(\mathscr{G}, \mathfrak{Lie}(\Gamma), e) \subset (\mathscr{G}, \zeta, e)$. So we can apply Theorem 3.3 and obtain statements (a) and (b). Because of (I.2), part (c) follows from Theorem 1.4.

Now consider the differentiable map $\omega : \mathscr{G} \to \mathbb{R}$ with $\omega(Wx) = \det(W)^{-\kappa}\omega(x)$ for $W \in \Gamma$. For $T \in \mathfrak{Lie}(\Gamma)$ and $\tau \in \mathbb{R}$, set $W = \exp(\tau T)$, differentiate $\log(\omega(Wx))$ with respect to τ and set $\tau = 0$. As $\log(\det(W)) = \tau\,\mathrm{tr}(T)$, it follows that $\omega'_x(Tx) = \kappa\,\mathrm{tr}(T)$. For $\omega = \zeta$ we thus obtain (d).

To prove (e), define $\omega(x) = \frac{\eta(x)}{\zeta(x)}$ and obtain $\omega(Wx) = \omega(x)$ for $W \in \Gamma$, so that $\omega'_x(Tx) = 0$ for all $T \in \mathfrak{Lie}(\Gamma)$. Choose an algebra $\mathfrak{R}$ in $(\mathscr{G}, \mathfrak{Lie}(\Gamma), e)$ with product $(u, v) \mapsto uv = L(u)v = R(v)u$ and set $T = R(v)$. It follows that $\omega'_x(xv) = 0$ for $v \in V$, that is, $\omega'_x = 0$ for all x from a dense subset of $\mathscr{G}$. Thus ω is constant, which is statement (e). ◇

5 In §2.4 we assigned to every non-degenerate map $\eta : \mathscr{G} \to \mathbb{R}$ a map $\hat{\eta} : \mathscr{G} \to \mathbb{R}$ that due to (2.3) transforms as

$$\hat{\eta}(Wx) = \det(W)^{-2}\hat{\eta}(x) \tag{3.3}$$

for $x \in \mathscr{G}$ and $W \in \mathrm{Aut}(\mathscr{G}, \eta)$. In analogy to Theorem 1.4 we obtain:

Theorem 3.6 *Let Γ be a closed subgroup of* $\mathrm{Aut}(\mathscr{G})$, *$\zeta$ a Γ-invariant and assume $(\mathscr{G}, \mathfrak{Lie}(\Gamma), e)$ is not empty. For every non-degenerate map $\eta \in R(\mathscr{G}, \Gamma)$ it holds that:*

(a) $\mathfrak{L}(\mathscr{G}, \eta) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta))$.

(b) $\eta(\exp(T)x) = \mathrm{e}^{\lambda(T)}\eta(x)$ *for $x \in \mathscr{G}$ and $T \in \mathfrak{L}(\mathscr{G}, \eta)$.*

Here, λ denotes the linear form of $\mathfrak{L}(\mathscr{G}, \eta)$ assigned to η according to (1.2).

PROOF: Because of (3.3), Theorem 3.5 (e) shows that $\hat{\eta}$ and ζ differ only by a constant factor. Together with Theorem 3.5 (c) it follows that $\mathfrak{L}(\mathscr{G}, \hat{\eta}) = \mathfrak{L}(\mathscr{G}, \zeta) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \zeta))$. By Lemma 2.5, $\mathfrak{L}(\mathscr{G}, \eta) \subset \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \zeta))$. For $T \in \mathfrak{L}(\mathscr{G}, \eta)$ and $\tau \in \mathbb{R}$, $W = \exp(\tau T)$ thus belongs to $\mathrm{Aut}(\mathscr{G}, \zeta)$, hence to $\mathrm{Aut}(\mathscr{G})$. Lemma 1.3 (b) now gives part (b) and thus part (a). ◇

§4 A connection to Jordan algebras

1 Let $\eta \in \mathrm{C}^{\infty}_{+}(\mathscr{G})$ be non-degenerate and homogeneous. After choosing $e \in \mathscr{G}$, one defines the non-degenerate symmetric bilinear form $\sigma : V \times V \to \mathbb{R}$ by $\sigma = \eta''_e$. Now an algebra $\mathfrak{A}(\eta, e)$ is defined on V whose product $(u, v) = A(u)v = u \cdot v$ is defined by

$$\sigma(u \cdot v, w) = \frac{1}{2}\eta'''_e(u, v, w). \tag{4.1}$$

Clearly $\mathfrak{A}(\eta, e)$ is defined by the behavior of η in a neighborhood of e.

Algebras that are defined in this manner play an important role in the theory of self-dual cones (cf. [1, VI, §8], [8, Chapter II]). A similar construction is found in Harrison [5].

Lemma 4.1

(a) *The algebra $\mathfrak{A}(\eta, e)$ is commutative and has e as identity element.*

(b) $\sigma(u,v) = \eta'_e(u \cdot v)$ *and* $\sigma(u \cdot v, w) = \sigma(u, v \cdot w)$.

(c) *For* $W \in \mathrm{Aut}(\mathcal{G}, \eta)$, $W : \mathfrak{A}(\eta, e) \to \mathfrak{A}(\eta, We)$ *is an isomorphism of Lie algebras.*

Part (b) states that $A(v)$, $v \in V$, is self-adjoint with respect to σ.

PROOF: As η'''_e is symmetric in all three arguments, commutativity and the second identity in part (b) follow directly from (4.1). For homogeneous η, η''_x is homogeneous of degree -2 in x, so that $\eta'''_e(u,v,e) = 2\eta''_e(u,v)$ (cf. §1.1). Thus e is the identity element. Since η'_x is homogeneous of degree -1, it follows from the second identity in (b) that $\sigma(u,v) = \sigma(e, u \cdot v) = \eta''_e(e, u \cdot v) = \eta'_e(u \cdot v)$.

For prove part (c), let $(u,v) \mapsto u \cdot_W v$ denote the product in $\mathfrak{A}(\eta, We)$. It follows from (1.1) that

$$\begin{aligned}\eta''_{We}(Wu \cdot_W v, Ww) &= \frac{1}{2}\eta'''_{We}(Wu, Wv, Ww) = \frac{1}{2}\eta'''_e(u,v,w)\\ &= \eta''_e(u \cdot v, w) = \eta''_{We}(W(u \cdot v), Ww),\end{aligned}$$

which is the claim. ◇

Corollary 1 *If* $\mathrm{Aut}(\mathcal{G}, \eta)$ *acts transitively on* $\mathcal{G}$*, then all algebras* $\mathfrak{A}(\eta, e)$*,* $e \in \mathcal{G}$*, are isomorphic.*

It follows from Lemma 1.1 (c) that $2\sigma(Te \cdot u, v) = \sigma(Tu, v) + \sigma(u, Tv)$ for $T \in \mathfrak{L}(\mathcal{G}, \eta)$. This means for $T \in \mathfrak{L}(\mathcal{G}, \eta)$,

$$T + T^\sigma = 2A(Te) \tag{4.2}$$

and in particular

$$T^\sigma e = Te. \tag{4.3}$$

For the following it is convenient to introduce the *quadratic representation*

$$P(u) = 2A(u)^2 - A(u \cdot u) \tag{4.4}$$

well-known from the theory of Jordan algebras. Finally, let

$$\mathrm{Inv}(\mathfrak{A}(\eta, e)) = \{x \in V \mid \det(P(x)) \neq 0\}$$

and set for $x \in \mathrm{Inv}(\mathfrak{A}(\eta, e))$,

$$x^{-1} = P^{-1}(x)x. \tag{4.5}$$

2 Particularly interesting are those pairs $(\mathscr{G}, \eta)$ for which there is a commutative algebra in $(\mathscr{G}, \eta, e)$. For an algebra $\mathfrak{K}$ in $(\mathscr{G}, \eta, e)$ with product $(u, v) \mapsto uv = L(u)v = R(v)u$ it follows immediately from (4.2) that

$$R(u) + R(u)^{\sigma} = 2A(u). \tag{4.6}$$

Theorem 4.2 *For non-degenerate and homogeneous $\eta \in C^{\infty}_{+}(\mathscr{G})$ the following are equivalent:*

(a) *The exists a commutative algebra in $(\mathscr{G}, \eta, e)$.*

(b) $\mathfrak{A}(\eta, e) \in (\mathscr{G}, \eta, e)$.

If this is the case, then:

(1) $\mathfrak{A}(\eta, e)$ *is the only commutative algebra in $(\mathscr{G}, \eta, e)$.*

(2) $\mathfrak{A}(\eta, e)$ *is a Jordan algebra.*

(3) $\mathscr{G}$ *is the connected component of* $\mathrm{Inv}(\mathfrak{A}(\eta, e))$ *that contains e.*

(4) $\eta'_x(u) = \sigma(x^{-1}, u)$, $\eta''_x(u, v) = \sigma(P^{-1}(x)u, v)$ *for $x \in \mathscr{G}$.*

PROOF: Let $\mathfrak{K}$ be a commutative algebra in $(\mathscr{G}, \eta, e)$, that is, $R(u) = L(u)$ for all $u \in V$. With (3.2) we compute $\eta''_e(u, v) = \eta'_e(uv)$, $\eta'''_e(u, v, w) = \eta'_e(w(uv)) + \eta'_e(u(wv))$, that is, $\eta'''_e(u, v, w) = \sigma(w, uv) + \sigma(u, wv)$. From the symmetry in u, v, it follows that $\sigma(u, wv) = \sigma(v, wu)$, so that $L(w) = R(w)$ is self-adjoint with respect to σ. From (4.6) it thus follows that $\mathfrak{K} = \mathfrak{A}(\eta, e)$, and the equivalence of (a) and (b), as well as claim (1), is established.

Now let $\mathfrak{A}(\eta, e) \in (\mathscr{G}, \eta, e)$, that is, $A(u) \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta))$ for $u \in V$. Define $A(u, v) = A(u \cdot v) + [A(u), A(v)]$ and verify that $A(x, v)x = P(x)v$ and $A(x, v)u = A(u, v)x$. As $A(u, v) \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta))$, we easily obtain from Lemma 1.1 with $T = A(x, v)$ and Lemma 4.1 that

$$\eta''_x(P(x)v, u) = \eta'_x(A(u, v)x) = \eta'_e(A(u, v)e) = \eta'_e(u \cdot v) = \sigma(u, v)$$

for $x \in \mathscr{G}$. In particular, $\mathscr{G} \subset \mathrm{Inv}(\mathfrak{A}(\eta, e))$ and

$$\eta''_x(u, v) = \sigma(P^{-1}(x)u, v), \quad x \in \mathscr{G}. \tag{$*$}$$

For $T = A(v)$ it follows from Lemma 1.1 that $\eta''_x(v \cdot x, u) = \eta'_x(v \cdot u)$. Thus $\eta''_x(v \cdot x, u)$ is symmetric in u, v and therefore by $(*)$, $P^{-1}(x)A(x)$ is self-adjoint with respect to σ. Since $A(x)$ and $P(x)$ are self-adjoint, $A(x)$ and $P(x)$ commute, and thus $A(x)$ and $A(x \cdot x)$ commute. This proves claim (2).

Let Π denote the subgroup of $\mathrm{GL}(V)$ generated by $P(x)$ for $x \in \mathrm{Inv}(\mathfrak{A}(\eta, e))$, and let Π_0 denote its identity component. By [1, XI, Satz 2.2 and Satz 2.4], $\Pi_0 \subset \mathrm{Aut}(\mathcal{G}, \eta)$ and the orbit of Π_0 through e is the connected component $\mathcal{K}$ of $\mathrm{Inv}(\mathfrak{A}(\eta, e))$ that contains e. Thus $\mathcal{K} \subset \mathcal{G}$ and since $\mathcal{G} \subset \mathrm{Inv}(\mathfrak{A}(\eta, e))$, claim (3) holds.

Now claim (4) follows from $(*)$ with $u = x$. $\diamond$

Remark 1 If we define $H : \mathcal{G} \to \mathrm{End}(V)$ by $\eta''_x(u, v) = \sigma(H(x)u, v)$, then (1.1) shows that $W^\sigma H(Wx)W = H(x)$ holds for all $W \in \mathrm{Aut}(\mathcal{G}, \eta)$. If $\mathfrak{A}(\eta, e) \in (\mathcal{G}, \eta, e)$, then we can choose $W = \exp(A(u))$, $u \in V$, and find that for all x in the orbit Y of $\mathrm{Aut}(\mathcal{G}, \eta)$ through e, $H(x)$ belongs to $\mathrm{Aut}(\mathcal{G}, \eta)$. Thus Y is essentially an *ω-domain* in the sense of [8, Chapter II]. There, claims (2) to (4) were proved by different methods.

Compare also Shima [15].

3 Let $\mathfrak{A}$ be a Jordan algebra on V with identity element e and product $(u, v) \mapsto uv$. Let $\mathrm{Inv}(\mathfrak{A})$ denote the set of elements in V that are invertible in $\mathfrak{A}$, and let $\mathcal{G}(\mathfrak{A})$ be the connected component of $\mathrm{Inv}(\mathfrak{A})$ that contains e. As a converse of Theorem 4.2, the following holds:

Theorem 4.3 *Let $\mathfrak{A}$ be a Jordan algebra with identity element e and σ a non-degenerate symmetric bilinear form on V with $\sigma(u, vw) = \sigma(uv, w)$. If there exists $\eta \in \mathrm{C}^\infty_+(\mathcal{G}(\mathfrak{A}))$ with $\eta'_x(u) = \sigma(x^{-1}, u)$ for $x \in \mathcal{G}(\mathfrak{A})$, then:*

(a) $\mathrm{Aut}(\mathcal{G}(\mathfrak{A}))$ *acts transitively on* $\mathcal{G}(\mathfrak{A})$.

(b) $\mathfrak{A} = \mathfrak{A}(\eta, e) \in (\mathcal{G}(\mathfrak{A}), \eta, e)$.

PROOF: (a) Let Π_0 again denote the identity component of the subgroup of $\mathrm{GL}(V)$ generated by $P(x)$ for $x \in \mathrm{Inv}(\mathfrak{A})$. By standard arguments, $\Pi_0 \subset \mathrm{Aut}(\mathcal{G}(\mathfrak{A}), \eta)$. By [1, XI, Satz 2.4], Π_0 acts transitively on $\mathcal{G}(\mathfrak{A})$.

(b) Follows from $\eta'_x(u) = \sigma(x^{-1}, u)$ by differentiation. $\diamond$

Remark 1 In case of a semisimple Jordan algebra $\mathfrak{A}$, the assumptions for Theorem 4.3 are satisfied by $\sigma(u, v) = \mathrm{tr}(L(uv))$.

§5 Examples

We will present some examples in the following. They will illustrate that the notions introduced before are independent of each other and are necessary requirements for most results. We omit the proofs.

(a) Halfspaces

Let $\lambda \neq 0$ be a linear form on V and $\mathscr{G}$ a connected component of those $x \in V$ with $\lambda(x) \neq 0$. For $e \in \mathscr{G}$ and $V_0 = \ker \lambda$ we then have $V = \mathbb{R}e \oplus V_0$ and $\mathscr{G} = \mathbb{R}^+ e \oplus V_0$. Elements $x \in V$ are written uniquely as $x = x_1 e \oplus x_0$, and elements $W \in \mathrm{End}(V)$ as $Wx = (x_1 w_1 + \omega x_0)e \oplus (x_1 w_1 + W_0 x_0)$ and obtains

(a.1) $\mathrm{Aut}(\mathscr{G}) = \{W \in \mathrm{GL}(V) \mid w_1 > 0, \omega = 0\}$.

1 Now consider the subgroup $\Gamma \subset \mathrm{Aut}(\mathscr{G})$ of those $W \in \mathrm{Aut}(\mathscr{G})$ with $W_0 = w_1 \mathrm{id}$. Then:

(a.2) Γ is a closed connected subgroup of $\mathrm{Aut}(\Gamma)$ that acts transitively on $\mathscr{G}$.

(a.3) $R(\mathscr{G}, \Gamma) = \{\alpha x_1^\kappa \exp(-\frac{1}{x_1}\varrho(x_0)) \mid \alpha > 0, \kappa \in \mathbb{R}, \varrho \in V_0^*\}$ (cf. §2.1).

For $\eta \in R(\mathscr{G}, \Gamma)$ we obtain consecutively (cf. §1.1):

(a.4) $\eta'_x(u) = \frac{1}{x}(\varrho(u_0) - \kappa u_1) - \frac{\varrho(x_0)}{x_1^2} u_1$ for $x \in \mathscr{G}$.

(a.5) $\eta''_x(u, v) = \frac{1}{x_1^2}(v_1 \varrho(u_0) + u_1 \varrho(v_0)) - \frac{1}{x_1^2}(\kappa + 2\frac{\varrho(x_0)}{x_1}) u_1 v_1$ for $x \in \mathscr{G}$.

(a.6) η' is rational.

(a.7) η' is injective if and only if either $\dim V = 1$, $\kappa \neq 0$, or if $\dim V = 2$, $\kappa \neq 0$, $\varrho \neq 0$ (see Theorem 3.4).

(a.8) η is exploding if and only if $\varrho = 0$, $\kappa < 0$ (cf. §1.4).

(a.9) η is non-degenerate if and only if either $\dim V = 1, \kappa \neq 0$, or if $\dim V = 2$, $\kappa \neq 0$, $\varrho \neq 0$ (cf. §2.4).

(a.10) $\eta(x) = x_1^{-s}$, $s = \frac{2}{\dim V}$, is a Γ-invariant (cf. §3.4).

(a.11) $\Gamma(\eta') = \{W \in \mathrm{GL}(V) \mid \varrho \circ W_0 = w_1\varrho, \omega = 0\}$ (cf. §1.5).

(a.12) $\mathfrak{L}(\mathcal{G}, \eta) = \mathfrak{Lie}(\Gamma(\eta')) = \{W \in \mathrm{End}(V) \mid \varrho \circ W_0 = w_1\varrho, \omega = 0\}$ (cf. II, Lemma 1.2).

(a.13) $\mathrm{Aut}(\mathcal{G}, \eta) = \{W \in \mathrm{GL}(V) \mid w_1 > 0, \varrho \circ W_0 = w_1\varrho, \omega = 0\}$.

2 If instead of Γ we consider the subgroup Γ' of $\mathrm{Aut}(\mathcal{G})$ of those W with $W_0 =$ id, then

(a.2′) Γ' is a closed connected subgroup of $\mathrm{Aut}(\mathcal{G})$ that acts transitively on $\mathcal{G}$.

(a.3′) $R(\mathcal{G}, \Gamma') = \{\alpha x_1^{\kappa} \mid \alpha > 0, \kappa \in \mathbb{R}\}$.

Properties (a.4) to (a.10) hold accordingly with the additional condition $\varrho = 0$. Moreover, for non-constant $\eta \in R(\mathcal{G}, \Gamma')$:

(a.11′) $\Gamma'(\eta) = \{W \in \mathrm{GL}(V) \mid W_0 = \mathrm{id}, \omega = 0\}$.

3 If $\dim V \leq 2$ and $\eta \in R(\mathcal{G}, \Gamma)$ is non-degenerate, then (cf. §4.1):

(a.14) The algebra $\mathfrak{A}(\eta, e)$ is given by the product

$$(u, v) \mapsto (u_1 v_1)e \oplus (u_1 v_0 + v_1 u_0).$$

(a.15) We see that $\mathfrak{A}(\eta, e)$ is commutative and associative and (cf. §4.2) Γ is the connected components of invertible elements in $\mathfrak{A}(\eta, e)$ that contains e.

We see that $\mathfrak{A}(\eta, e)$ is not semisimple for $\dim V = 2$, but η is non-degenerate.

(b) Quadratic forms

Let $\mu \neq 0$ be a symmetric bilinear form on V and $\mathcal{G}$ a connected component of those $x \in V$ with $\mu(x, x) \neq 0$. On $\mathcal{G}$ define maps $\eta \in \mathrm{C}_+^\infty(\mathcal{G})$ by $\eta(x) = \frac{1}{\mu(x,x)^2}$, $x \in \mathcal{G}$.

From the definition we obtain consecutively:

(b.1) $\eta'_x(u) = 4\frac{\mu(x,u)}{\mu(x,x)}$ for $x \in \mathcal{G}$.

(b.2) $\eta''_x(u,v) = \frac{4}{\mu(x,x)^2}(2\mu(x,v)\mu(x,u) - \mu(x,x)\mu(u,v))$ for $x \in \mathcal{G}$.

(b.3) η is exploding and η' is rational (cf. §1.4, 5).

(b.4) η is non-degenerate if and only if μ is non-degenerate (cf. §2.4).

Let Bk_μ denote the bilinear kernel of μ.

(b.5) $\Gamma(\eta') = \{W \in \mathrm{GL}(V) \mid W\mathrm{Bk}_\mu \subset \mathrm{Bk}_\mu,$ there is $\varrho(W) \in \mathbb{R}$
with $\mu(Wx, Wx) = \varrho(W)\mu(x,x), x \in V\}$.

(b.6) If η is non-degenerate, then $\varrho(W)^{\dim V} = \det(W)^2$.

(b.7) $\mathrm{Aut}(\mathcal{G}, \eta) = \{W \in \mathrm{Aut}(\mathcal{G}) \mid$ there is $\varrho(W) \in \mathbb{R}$
with $\mu(Wx, Wx) = \varrho(W)\mu(x,x), x \in V\}$.

With Theorem §1.7 (c) one obtains

(b.8)

$$\begin{aligned}
\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G},\eta)) &= \mathfrak{Lie}(\Gamma(\eta')) = \mathfrak{L}(\mathcal{G},\eta) \\
&= \{T \in \mathrm{End}(V) \mid \mu(x,Tx)\mu(y,y) = \mu(y,Ty)\mu(x,x), x, y \in V\} \\
&= \mathbb{R}\mathrm{id} + \{T \in \mathrm{End}(V) \mid \mu(v,Tu) + \mu(Tv,u) = 0, u, v \in V\}.
\end{aligned}$$

Now let Γ denote the identity component of $\mathrm{Aut}(\mathcal{G},\eta)$ and $\mathrm{O}(\mu)_0$ the identity component of the orthogonal group $\mathrm{O}(\mu)$ of μ. With (b.8) it follows that

(b.9) $\Gamma = \mathbb{R}^+\mathrm{O}(\mu)_0$.

Moreover (cf. [9]),

(b.10) Γ acts transitively on $\mathcal{G}$.

To compute $R(\mathcal{G},\Gamma)$, write $V = \mathrm{Bk}_\mu \oplus V_1$ such that $\mu_1 = \mu|_{V_1}$ is non-degenerate. With (b.8) and §2.2, we verify

(b.11) $R(\mathcal{G},\Gamma) = \{\alpha\eta^\kappa \mid \alpha > 0, \kappa \in \mathbb{R}\}$ if $\dim V_1 \neq 2$, or if $\dim V_1 = 2$ and $\mu|_{V_1}$ is definite.

If $\dim V_1 = 2$ and $\mu_1 = \mu|_{V_1}$ indefinite, then choose a basis a_1, a_2 of V with $\mu(a_i, a_i) = 0$, $\mu(a_1, a_2) \neq 0$ and verify easily that elements of $\mathfrak{Lie}(\mathrm{O}(\mu_1))$ are diagonal with respect to this basis. Thus the maps $\eta_i : \mathcal{G} \to \mathbb{R}^+$, $x = x_0 + x_1 \mapsto \mu_1(a_i, x_1)^2$ are elements of $R(\mathcal{G}, \Gamma)$, and

(b.12) $R(\mathcal{G}, \Gamma) = \{\alpha \eta_1^\kappa \eta_2^\varrho \mid \alpha > 0, \kappa \in \mathbb{R}\}$ if $\dim V_1 = 2$ and $\mu|_{V_1}$ is indefinite.

(b.13) If $\eta_0 \in R(\mathcal{G}, \Gamma)$ is non-degenerate, then there exists $\alpha > 0$ with $\hat{\eta}_0 = \alpha \eta^\kappa$, where $\kappa = \frac{1}{2} \dim V$ (cf. §2.4).

(b.14) If $\eta_0 \in R(\mathcal{G}, \Gamma)$ is non-degenerate, then η_0^κ is a Γ-invariant for $\kappa = \frac{1}{2} \dim V$ (cf. §3.4).

Now assume $\eta_0 \in R(\mathcal{G}, \Gamma)$ is non-degenerate, $e \in \mathcal{G}$ and $\mu(e, e) = 1$. Then η is non-degenerate by (b.4) and we obtain (cf. §4.1):

(b.15) The product in $\mathfrak{A}(\eta, e)$ is given by

$$(u, v) \mapsto \mu(e, u)v + \mu(e, v)u - \mu(u, v)e.$$

Thus $\mathfrak{A}(\eta, e)$ is a Jordan algebra. Moreover, $A(u) \in \mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta))$ and the results in §4 apply.

If μ is definite and $\dim V = 3$, then there is no right-symmetric algebra $\mathfrak{R}$ with identity e that is contained in $(\mathcal{G}, \eta, e)$ (cf. II, §1.1).

(c) Symmetric matrices

Let V be the vector space of real symmetric $n \times n$-matrices and $\mathcal{G}$ a connected component of those $x \in V$ with $\det(x) \neq 0$.

It is well-known that $\mathcal{G}$ is uniquely determined by the signature of any element $x \in \mathcal{G}$. On $\mathcal{G}$, define a map $\eta \in \mathrm{C}_+^\infty(\mathcal{G})$ by $\eta(x) = \det(x)^{-2}$ for $x \in \mathcal{G}$. From the definition, we obtain:

(c.1) $\eta'_x(u) = 2\,\mathrm{tr}(x^{-1}u)$ for $x \in \mathcal{G}$.

(c.2) $\eta''_x(u, v) = 2\,\mathrm{tr}(x^{-1}ux^{-1}v)$ for $x \in \mathcal{G}$.

(c.3) η is exploding and η' is rational.

(c.4) η is non-degenerate.

By standard arguments from Jordan theory and Theorem 1.8 if then follows:

(c.5) $\Gamma(\eta') = \{\pm W_a \mid a \in \mathrm{GL}_n(\mathbb{R})\}$, where $W_a(x) = axa^\top$.

(c.6) $\mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta)) = \mathfrak{Lie}(\Gamma(\eta')) = \mathfrak{L}(\mathscr{G}, \eta) = \{T_a \mid a \in \mathrm{End}(\mathbb{R}^n)\}$, where $T_a(x) = ax + xa^\top$.

Now let Γ denote the identity component of $\Gamma(\eta')$.

(b.7) Γ acts transitively on $\mathscr{G}$.

(c.8) $R(\mathscr{G}, \Gamma) = \{\alpha\eta^\kappa \mid \alpha > 0, \kappa \in \mathbb{R}\}$.

(c.9) $\eta^{\frac{n+1}{2}}$ is a Γ-invariant (cf. §3.4).

(c.10) After a choice of $e \in \mathscr{G}$, the product of $\mathfrak{A}(\eta, e)$ is given by

$$(u, v) \mapsto \frac{1}{2}(ue^{-1}w + we^{-1}u).$$

Thus $\mathfrak{A}(\eta, e)$ is a Jordan algebra. Moreover, $A(u) \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}, \eta))$, so that the results of §4 apply.

(d) Indefinite 2 × 2-matrices

Let V denote the vector space of real symmetric 2×2-matrices. The elements of V are written $x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$. Now let $\mathscr{G} = \{x \in V \mid x_3 > 0, \det(x) < 0\}$. Then $\mathscr{G}$ is a simply connected domain that is a proper subset of the indefinite 2×2-matrices discussed in part (c). Contrary to (c), define a map $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ by $\eta(x) = |x_3 \det(x)|^{-1}$. From the definition, we obtain:

(d.1) $\eta'_x(u) = \frac{u_2}{x_3} + \mathrm{tr}(x^{-1})$ for $x \in \mathscr{G}$.

(d.2) $\eta''_x(u, v) = \frac{u_3 v_3}{x_3^2} + \mathrm{tr}(x^{-1}ux^{-1})$ for $x \in \mathscr{G}$.

(d.3) η is non-degenerate (cf. §2.4).

(d.4) η is exploding and η' is rational (cf. §1.4, 5).

Now set $\Gamma = \{W_a \mid a \in \mathrm{GL}_2(\mathbb{R})\}$, where $W_a(x) = axa^\top$, and define $e = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathcal{G}$. Then verify:

(d.5) $\Gamma(\eta') = \{\pm W_a \mid W_a \in \Gamma\}$.

(d.6) $\Gamma = \mathrm{Aut}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G}, \eta)$ operates transitively on $\mathcal{G}$.

(d.7) $\mathfrak{Lie}(\mathrm{Aut}(\mathcal{G}, \eta)) = \mathfrak{Lie}(\Gamma(\eta')) = \mathfrak{L}(\mathcal{G}, \eta) = \{T_a \mid a \text{ upper triangular}\}$, where $T_a(x) = ax + xa^\top$.

For the identity component Γ_0 of Γ it follows:

(d.8) $R(\mathcal{G}, \Gamma_0) = \{\alpha\eta^\kappa\eta_1^\varrho \mid \kappa, \varrho \in \mathbb{R}, \alpha > 0\}$, where $\eta_1(x) = \det(x)$.

(d.9) If $\eta \in R(\mathcal{G}, \Gamma_0)$ and $\eta(Wx) = \det(W)^{-2}\eta(x)$ for $x \in \mathcal{G}$ and $W \in \Gamma_0$, then there exists $\alpha > 0$ with $\eta = \alpha\eta_1^{-3}$.

As the boundary of $\mathcal{G}$ contains points x with $x_3 = 0$ but $\det(x) \neq 0$, it follows:

(d.10) $\mathcal{G}$ has no Γ-invariants.

Chooses again $e = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathcal{G}$. Then (cf. §4.1):

(d.12) The product on $\mathfrak{A}(\eta, e)$ is given by

$$(u, v) \mapsto \frac{1}{2}(ue^{-1}w + we^{-1}u) + \frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & u_2v_2\end{pmatrix}.$$

$\mathfrak{A}(\eta, e)$ is not a Jordan algebra.

(e) Vinberg's cones

Let V be the vector space of real symmetric 3×3-matrices with coefficient 0 in position $(1, 3)$. The elements of V are represented in the form

$$x = \begin{pmatrix}x_1 & x_2 & 0\\ x_2 & x_3 & x_4\\ 0 & x_4 & x_5\end{pmatrix}.$$

Now let p denote the polynomial $p(x) = x_1x_5\det(x)$, and let $\mathcal{G}$ denote a connected component of the set $\mathcal{D}(p) = \{x \in V \mid p(x) \neq 0\}$. On $\mathcal{G}$ define the map $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ by $\eta(x) = \frac{(x_1x_5)^2}{\det(x)^8}$. It is known that the connected of component $\mathscr{K}$ of $\mathscr{D}(p)$ that contains the identity matrix is a regular homogeneous cone. One can show that $\eta^{\frac{1}{4}}$ is the invariant of $\mathscr{K}$. Now let $\mathscr{G}$ be an arbitrary connected component of $\mathscr{D}(p)$. Then verify:

(e.1) $\eta'_x(u) = 8\,\mathrm{tr}(x^{-1}u) - 2\frac{u_1x_5+x_1u_5}{x_1x_5}$ for $x \in \mathscr{G}$.

(e.2) $\eta''_x(u,v) = 8\,\mathrm{tr}(x^{-1}ux^{-1}v) + 2\frac{u_1v_5+v_1u_5}{x_1x_5} - 2\frac{(u_1x_5+x_1u+5)(v_1x_5+x_1v_5)}{(x_1x_5)^2}$ for $x \in \mathscr{G}$.

(e.3) η' is rational and if $\mathscr{G} = \mathscr{K}$, then η is exploding (cf. §1.4, 5).

(e.4) η is non-degenerate and for $x \in \mathscr{K}$, η''_x is positive definite (cf. §2.4).

Now let P denote the group of those matrices $u \in \mathrm{GL}_3(\mathbb{R})$ for which the coefficients at positions $(1,2)$, $(1,3)$, $(3,1)$ and $(3,2)$ are 0, and the coefficients at positions $(1,1)$, $(2,2)$ and $(3,3)$ are positive. Then define $\Gamma = \{W_a \mid a \in P\}$, where $W_a(x) = axa^\top$. We obtain:[1)]

(e.5) Γ is a connected closed subgroup of $\mathrm{Aut}(\mathscr{G})$ that operates transitively on $\mathscr{G}$.

(e.6) $\Gamma(\eta') = \{\pm W \mid W \in \mathrm{Aut}(\mathscr{G})\} = \{\pm W \mid W \in \mathrm{Aut}(\mathscr{G},\eta)\}$.

(e.7) $\mathfrak{Lie}(\Gamma(\eta')) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G})) = \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G},\eta)) = \mathfrak{Lie}(\Gamma) = \{T_a \mid a \in \mathfrak{Lie}(P)\}$, where $T_a(x) = ax + xa^\top$ (cf. Theorem 1.8).

(e.8) $R(\mathscr{G},\Gamma) = \{\alpha x_1^{\kappa_1} x_5^{\kappa_5} \eta(x)^\varrho \mid \alpha > 0, \kappa_1, \kappa_5, \varrho \in \mathbb{R}\}$.

(e.9) For $\mathscr{G} = \mathscr{K}$, $\eta^{\frac{1}{2}}$ is a Γ-invariant.

Now let e be the identity matrix. Then compute (cf. §4.1):

(e.10) The product in $\mathfrak{A}(\eta, e)$ is given by

$$(u,v) \mapsto \begin{pmatrix} u_1v_1+\frac{4}{3}u_2v_2 & \frac{1}{2}(u_1v_2+v_1u_2+u_3v_2+v_3u_2) & 0 \\ \frac{1}{2}(u_1v_2+v_1u_2+u_3v_2+v_3u_2) & v_2u_2+v_3u_3+v_4u_4 & \frac{1}{2}(v_3u_4+v_4u_3+v_4u_5+u_4v_5) \\ 0 & \frac{1}{2}(v_3u_4+v_4u_3+v_4u_5+u_4v_5) & \frac{4}{3}u_4v_4+u_5v_5 \end{pmatrix}.$$

(e.11) $\mathfrak{A}(\eta, e)$ is not a Jordan algebra.

[1)] *Translator's note:* In the original text, the label (e.4) appears twice. So (e.5) here was also called (e.4) in the original text, (e.6) was (e.5), and so on.

(e.12) There is no positive definite bilinear form on V with respect to which $\mathcal{K}$ is a domain of positivity. The tube domain $V + \mathrm{i}\mathcal{K}$ is not symmetric (cf. [16]).

Part II

Algebras and domains

§1 Closed and exact linear forms

1 Let $\mathfrak{R}$ be an algebra defined on V with product $uv = L(u) = R(v)u$ and identity element e. Define the *associator* of $\mathfrak{R}$ by

$$(u, v, w) = (uv)w - u(vw)$$

and call $\mathfrak{R}$ *right-symmetric*, if $(u, v, w) = (u, w, v)$ holds for all $u, v, w \in V$. First, recall some definitions.

The subgroup of $\mathrm{GL}(V)$ generated by the endomorphisms $\exp(R(u))$, $u \in V$, is denoted by $\exp(\mathfrak{R})$. Clearly, $\exp(\mathfrak{R})$ is connected.

The powers of an element $u \in V$ are defined recursively by $u^0 = e$, $u^1 = u$, and $u^{m+1} = uu^m$. In a suitable neighborhood U of e, the logarithm $\log = \log_{\mathfrak{R}} : U \to V$ can be defined by

$$\log(e + x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m.$$

Since $L(e) = \mathrm{id}$, $\det(L(x))$ is not the zero-polynomial. Therefore,

$$\mathcal{J}(\mathfrak{R}) = \{x \in V \mid \det(L(x)) \neq 0\}$$

is open and dense in V. For $q \in \mathcal{J}(\mathfrak{R})$ let $\mathcal{J}_q(\mathfrak{R})$ denote the connected component of $\mathcal{J}(\mathfrak{R})$ that contains q.

2 To every linear form σ on V we assign a rational map $\chi = \chi^{\sigma} : \mathcal{J}(\mathfrak{R}) \to V^*$ by

$$\chi_x(u) = \sigma(L(x)^{-1}u).$$

As in I, §1.5, we write

$$\chi_x(u) = \sigma(L(x)^{-1}u) = \frac{1}{\nu(x)}\pi_x(u) \tag{1.1}$$

for an exact denominator $\nu = \nu_\sigma$ uniquely determined by requiring $\nu(e) = 1$. Set

$$\mathcal{D}(\mathfrak{R}, \sigma) = \mathcal{D}(\nu_\sigma) = \{x \in V \mid \nu_0(x) \neq 0\}$$

and let $\mathcal{G}(\mathfrak{R}, \sigma)$ denote the connected component of $\mathcal{D}(\mathfrak{R}, \sigma)$ that contains e. As $\nu_\sigma(x)$ divides the determinant of $L(x)$, we have

$$\mathcal{J}(\mathfrak{R}) \subset \mathcal{D}(\mathfrak{R}, \sigma), \quad \mathcal{J}_e(\mathfrak{R}) \subset \mathcal{G}(\mathfrak{R}, \sigma) \tag{1.2}$$

for every linear form σ on V. Clearly χ rational and real analytic not only on $\mathcal{J}(\mathfrak{R})$ but even on $\mathcal{D}(\mathfrak{R}, \sigma)$.

In I, §1.5 we assigned a linear algebraic group $\Gamma(\chi)$ to the rational map χ, which is denoted $\Gamma(\mathfrak{R}, \sigma)$ in the present situation. By I, Lemma 1.5, $W \in \mathrm{GL}(V)$ belongs to $\Gamma(\mathfrak{R}, \sigma)$ if and only if $W\mathcal{D}(\mathfrak{R}, \sigma) = \mathcal{D}(\mathfrak{R}, \sigma)$ and

$$\sigma(L(Wx)^{-1}Wu) = \sigma(L(x)^{-1}u) \tag{1.3}$$

for all $x \in \mathcal{D}(\mathfrak{R}, \sigma)$ and $u \in V$. From I, (1.9), it follows for $W \in \Gamma(\mathfrak{R}, \sigma) = \Gamma(\chi)$ that

$$\nu(y)\nu(Wx) = \nu(x)\nu(Wy) \tag{1.4}$$

for all $x, y \in V$. Thus the group $\Gamma(\mathfrak{R}, \sigma)$ acts on $\mathcal{D}(\mathfrak{R}, \sigma)$ and the identity component $\Gamma_0(\mathfrak{R}, \sigma)$ of $\Gamma(\mathfrak{R}, \sigma)$ acts on $\mathcal{G}(\mathfrak{R}, \sigma)$. From (1.4) it follows that

$$\Gamma_0(\mathfrak{R}, \sigma) \subset \mathrm{Aut}(\mathcal{G}(\mathfrak{R}, \sigma), \nu). \tag{1.5}$$

Let $\mathfrak{L}(\mathfrak{R}, \sigma)$ denote the set of $T \in \mathrm{End}(V)$ such that

$$\sigma(L^{-1}(x)Tx) = \sigma(Te), \quad x \in \mathcal{D}(\mathfrak{R}, \sigma). \tag{1.6}$$

Clearly, $R(u)$ belongs to $\mathfrak{L}(\mathfrak{R}, \sigma)$ for every $u \in V$. Since

$$\Delta_x^u \sigma(L(x)^{-1}Tx) = \sigma(L(x)^{-1}Tu) - \sigma(L(x)^{-1}L(u)L(x)^{-1}Tx),$$

$T \in \mathfrak{L}(\mathfrak{R}, \sigma)$ is equivalent to

$$\sigma(L(x)^{-1}L(u)L(x)^{-1}Tx) = \sigma(L(x)^{-1}Tu), \quad x \in \mathcal{D}(\mathfrak{R}, \sigma), u \in V. \tag{1.7}$$

All these definitions are of a purely formal nature. Except for the given trivial relations we cannot expect any additional non-trivial relations to hold for an arbitrary linear form σ. Next, we will show that for certain distinguished linear forms, such relations exist.

3 In the following, let u, v, w denote arbitrary elements of V. Relations in such elements hold in general. A linear form σ on V is called

- $\mathfrak{R}$-*commutative*, if $\sigma(uv) = \sigma(vu)$ holds,
- $\mathfrak{R}$-*associative*, if $\sigma(uv \cdot w) = \sigma(u \cdot vw)$ holds,
- $\mathfrak{R}$-*closed*, if $\Delta_x^u \sigma(L(x)^{-1}v) = -\sigma(L(x)^{-1}L(u)L(x)^{-1}v)$, $x \in \mathcal{J}(\mathfrak{R})$, is symmetric in u, v,
- $\mathfrak{R}$-*exact*, if there is a differentiable map $\varphi : \mathcal{G}(\mathfrak{R}, \sigma) \to \mathbb{R}$ with $\Delta_x^u \varphi(x) = \sigma(L(x)^{-1}u)$, $x \in \mathcal{G}(\mathfrak{R}, \sigma)$.

Clearly an $\mathfrak{R}$-exact form is always $\mathfrak{R}$-closed and a $\mathfrak{R}$-closed linear form is always $\mathfrak{R}$-commutative. Let $K(\mathfrak{R})$ denote the vector space of $\mathfrak{R}$-closed linear forms on V, and $K_1(\mathfrak{R})$ the subspace of $\mathfrak{R}$-exact linear forms. Note that σ is already $\mathfrak{R}$-closed if the defining identity holds for all x in a non-empty open subset of $\mathcal{J}(\mathfrak{R})$, and that in this case the identity holds for all $x \in \mathcal{D}(\mathfrak{R}, \sigma)$. Finally, we will call an algebra $\mathfrak{R}$ *exact* if $K_1(\mathfrak{R}) = K(\mathfrak{R})$.

4 A characterisation of $\mathfrak{R}$-closed forms is given by:

Lemma 1.1 *For an algebra $\mathfrak{R}$ on V with identity e and linear form σ on V, the following are equivalent:*

(a) *σ is $\mathfrak{R}$-closed.*

(b) *$\sum_{n=0}^{m} \sigma(L(w)^n L(u) L(w)^{m-n} v)$ is symmetric in u and v for all $m \geq 0$.*

(c) *For $x \in \mathcal{D}(\mathfrak{R}, \sigma)$, $\sigma(L(x)^{-1}(xu \cdot v))$ is symmetric in u and v.*

(d) *σ is $\mathfrak{R}$-commutative and $\sigma(L(w)^n(w, u, v))$ is symmetric in u and v for all $n \geq 0$.*

(e) *For $m \geq 0$ we have $\Delta_x^u \sigma(x^{m+1}) = (m+1)\sigma(L(x)^m u)$.*

(f) *$\Delta_y^u \sigma(\log_{\mathfrak{R}}(y)) = \sigma(L(y)^{-1}u)$ for all y in a neighborhood of e.*

From part (d) we also infer the validity of

$$\sigma(u \cdot wv) + \sigma(w \cdot uv) = \sigma(v \cdot wu) + \sigma(w \cdot vu), \quad \sigma \in K(\mathfrak{R}). \tag{1.8}$$

PROOF: (a) $\Rightarrow$ (b): For τ in a neighborhood of 0, replace x by $e + \tau w$ and consider the geometric series for $\frac{1}{\mathrm{id}+\tau L(w)}$.

(a) $\Rightarrow$ (c): Replace u and v by xu and xv, respectively.

(c) $\Rightarrow$ (d): For τ in a neighborhood of 0, set $x = e + \tau w$ and consider the geometric series for $\frac{1}{\mathrm{id}+\tau L(w)}$.

(b) $\Rightarrow$ (e): Since for $m \geq 1$,

$$\Delta_x^u \sigma(L(x)^m v) = \sum_{n=0}^{m-1} \sigma(L(x)^n L(u) L(x)^{m-n} v)$$

the left-hand side is symmetric in u and v, thus

$$\sigma((\Delta_x^u L(x)^m)v) = \sigma((\Delta_x^v L(x)^m)u).$$

For $v = x$, it follows with Euler's differential equation

$$\sigma((\Delta_x^u L(x)^m)x) = \sigma((\Delta_x^x L(x)^m)u) = m\sigma(L(x)^m u).$$

On the other hand,

$$\Delta_x^u \sigma(x^{m+1}) = \Delta_x^u \sigma(L(x)^m x) = \sigma((\Delta_x^u L(x)^m)x) + \sigma(L(x)^m u),$$

so that the claim follows.

(e) $\Rightarrow$ (f): For $y = e + x$ substitute (e) in the series for $\sigma(\log(y))$.

(f) $\Rightarrow$ (a): Apply Δ_y^v to part (f) and obtain the claim for all y in a neighborhood of e. But then the claim holds for all $y \in \mathcal{J}(\mathfrak{R})$.

(d) $\Rightarrow$ (e): By assumption,

$$\sigma(L(x)^n(xu \cdot v - xv \cdot u)) = \sigma(L(x)^{n+1}(uv - vu))$$

for $n \geq 0$. For $0 \leq n \leq m-1$, substitute $v = x^{m-n-1}$ and obtain

$$\sigma(L(x)^n(xu \cdot x^{m-n-1} - x^{m-n}u)) = \sigma(L(x)^{n+1}(ux^{m-n-1} - x^{m-n-1}u)).$$

Summation over n yields

$$\sum_{n=0}^{m-1} \sigma(L(x)^n(xu \cdot x^{m-n-1})) = \sigma(x^m u) - \sigma(L(x)^m u) + \sigma_{n=0}^{m-1} \sigma(L(x)^{n+1}(ux^{m-n-1})),$$

hence

$$\Delta_x^{xu}\sigma(x^m) = \sigma(x^m u) - \sigma(L(x)^m u) + \Delta_x^u\sigma(x^{m+1}) - \sigma(ux^m).$$

As σ is $\mathfrak{R}$-commutative by assumption,

$$\Delta_x^u\sigma(x^{m+1}) = \Delta_x^{xu}\sigma(x^{m+1}) + \sigma(L(x)^m u). \tag{$*$}$$

Prove part (e) by induction over m. The case $m = 0$ is trivial. Suppose the claim has been proven for m. Substitute u by xu and obtain $(*)$ with

$$\begin{aligned}(m+1)\sigma(L(x)^{m+1}u) = (m+1)\sigma(L(x)^m(xu)) &= \Delta_x^{xu}\sigma(x^{m+1}) \\ &= \Delta_x^u\sigma(x^{m+2}) - \sigma(L(x)^{m+1}u),\end{aligned}$$

which is the claim for $m + 1$ instead of m. ◇

From part (d) we obtain:

Corollary 1 *If the algebra $\mathfrak{R}$ is right-symmetric, then $K(\mathfrak{R})$ consists of the $\mathfrak{R}$-commutative linear forms on V.*

5 Let $\Gamma(\mathfrak{R}, \sigma)$ and $\mathfrak{L}(\mathfrak{R}, \sigma)$ be defined as in 2. For $\sigma \in K(\mathfrak{R})$, a comparison of (1.3) and (1.7) shows that $\mathfrak{Lie}(\Gamma(\mathfrak{R}, \sigma))$ is a subset of $\mathfrak{L}(\mathfrak{R}, \sigma)$.

Lemma 1.2 *For $\sigma \in K(\mathfrak{R})$ we have:*

(a) $\exp(\mathfrak{R}) \subset \Gamma(\mathfrak{R}, \sigma)$.

(b) $\mathfrak{L}(\mathfrak{R}, \sigma) = \mathfrak{Lie}(\Gamma(\mathfrak{R}, \sigma))$.

(c) $\sigma([T, S]e) = 0$ *for* $T, S \in \mathfrak{L}(\mathfrak{R}, e)$.

(d) $\sigma(\log(\exp(T)x)) = \sigma(\log(x)) + \sigma(Te)$ *for T in a neighborhood of 0 in $\mathfrak{L}(\mathfrak{R}, \sigma)$ and x in a neighborhood of e.*

PROOF: For $T, S \in \mathfrak{L}(\mathfrak{R}, \sigma)$ substitute Sx for u in (1.7) and obtain

$$\sigma(L(x)^{-1}TSx) = \sigma(L(x)^{-1}L(Sx)L(x)^{-1}Tx)$$

Since $\sigma \in K(\mathfrak{R})$, the right-hand side is symmetric in T and S, so that $\sigma(L(x)^{-1}[T, S]x) = 0$ follows. For $x = e$, we obtain (c) and then $[T, S] \in \mathfrak{L}(\mathfrak{R}, \sigma)$ from (1.6).

We use the following well-know auxiliary result:

If $\mathfrak{T}$ is a Lie algebra of endomorphisms of V and $\Lambda \subset \mathfrak{T}^*$ is given such that $\lambda([T, S]) = 0$ for all $T, S \in \mathfrak{T}$ and $\lambda \in \Lambda$, then $\lambda(WSW^{-1}) = \lambda(S)$ for all $S \in \mathfrak{T}$, $\lambda \in \Lambda$ and $W = \exp(T)$, $T \in \mathfrak{T}$.

Apply this result to $\mathfrak{T} = \mathfrak{L}(\mathfrak{R}, \sigma)$, $\Lambda = \{(T \mapsto \sigma(L(x)^{-1}Tx)) \mid x \in \mathcal{D}(\mathfrak{R}, \sigma)\}$ and obtain

$$\sigma(L(x)^{-1}WSW^{-1}x) = \sigma(L(x)^{-1}Sx)$$

where $W = \exp(T)$, $T, S \in \mathfrak{L}(\mathfrak{R}, \sigma)$. Here, the right-hand side equals $\sigma(Se)$. Now substitute $S = R(u)$, so $\sigma(L(x)^{-1}W(W^{-1}x \cdot u)) = \sigma(u)$, and substitute x by Wx, and u by $L(x)^{-1}u$. By (1.3), it follows that $W \in \Gamma(\mathfrak{R}, \sigma)$, that is, $\mathfrak{L}(\mathfrak{R}, \sigma) \subset \mathfrak{Lie}(\Gamma(\mathfrak{R}, \sigma))$. Now (b) follows from $\mathfrak{Lie}(\Gamma(\mathfrak{R}, \sigma)) \subset \mathfrak{L}(\mathfrak{R}, \sigma)$.

As $R(u) \in \mathfrak{L}(\mathfrak{R}, \sigma)$ for $u \in V$, part (a) follows.

For $T \in \mathfrak{L}(\mathfrak{R}, \sigma)$ and $\tau \in \mathbb{R}$ let $W = \exp(\tau T)$. For sufficiently small $|\tau|$, $\varphi(\tau) = \sigma(\log(Wx))$ is differentiable and $\dot{W} = WT$. By (a), $W \in \Gamma(\mathfrak{R}, \sigma)$, so that from Lemma 1.1 (f), (1.3) and (1.4),

$$\begin{aligned}\dot{\varphi}(\tau) &= \Delta_{Wx}^{\dot{W}x}\sigma(\log(Wx)) = \sigma(L(Wx)^{-1}\dot{W}x)\\ &= \sigma(L(Wx)^{-1}WTx)\\ &= \sigma(L(x)^{-1}Tx) = \sigma(Te)\end{aligned}$$

follows. Hence $\varphi(\tau) = \tau\sigma(Te) + \varphi(0)$, that is, (d). ◇

6 The results so far are tested on a special type of algebras. The algebra $\mathfrak{R}$ is called *almost nilpotent* if $\mathfrak{R}$ has can be represented as a direct sum of vector spaces $\mathfrak{R} = \mathbb{R}e \oplus \mathfrak{R}_0$, such that $\mathfrak{R}_0$ is a subalgebra and $L(x_0)$ is nilpotent for every $x_0 \in \mathfrak{R}_0$. If $\mathfrak{R}$ is almost nilpotent, then for every $x_0 \in \mathfrak{R}_0$ there exists m with $x_0^m = 0$. The write the elements of $\mathfrak{R}$ as $x = \xi e + x_0$ and obtain $\mathcal{J}(\mathfrak{R}) = \{x \in V \mid \xi \neq 0\}$. Set $\mathcal{G}(\mathfrak{R}) = \{x \in V \mid \xi > 0\}$ and define $\log : \mathcal{G}(\mathfrak{R}) \to V$ by

$$\log(x) = \log(\xi)e + \log_{\mathfrak{R}}\left(e + \frac{1}{\xi}x_0\right), \quad x \in \mathcal{G}(\mathfrak{R}).$$

Since the elements of $\mathfrak{R}_0$ are nilpotent, the sequence for $\log_{\mathfrak{R}}(e + \frac{1}{\xi}x_0)$ is finite. Thus log is real-analytic on $\mathcal{G}(\mathfrak{R})$. One verifies that $\log(x)$ and $\log_{\mathfrak{R}}(x)$ coincide on a neighborhood of e. For $\sigma \in K(\mathfrak{R})$ define

$$\eta_{\mathfrak{R},\sigma}(x) = \exp(-\sigma(\log(x))), \quad x \in \mathcal{G}(\mathfrak{R}).$$

Lemma 1.3 *If the algebra $\mathfrak{R}$ is almost nilpotent, we have:*

(a) $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathscr{G}(\mathfrak{R}), \eta_{\mathfrak{R},\sigma})$ *for all* $\sigma \in K(\mathfrak{R})$.

(b) *If* $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathscr{G}(\mathfrak{R}), \eta)$ *for some* $\eta \in \mathrm{C}^\infty_+(\mathscr{G}(\mathfrak{R}))$, *then there is* $\sigma \in K(\mathfrak{R})$ *and* $\alpha \in \mathbb{R}^+$ *with* $\eta = \alpha\eta_{\mathfrak{R},\sigma}$.

PROOF: (a) For $W = \exp(R(u))$, $u \in V$, Lemma 1.2 (d) immediately yields $\sigma(\log(Wx)) = \sigma(\log(x)) + \sigma(u)$ for all x in a beighborhood of e. As $\mathfrak{R}_0$ is an ideal in $\mathfrak{R}$, it follows that $W \in \mathrm{Aut}(\mathscr{G}(\mathfrak{R}))$ and thus $\sigma(\log(Wx)) = \sigma(\log(x)) + \sigma(u)$ for all $x \in \mathscr{G}(\mathfrak{R})$.

(b) By assumption, $\mathfrak{R} \in (\mathscr{G}(\mathfrak{R}), \eta, e)$, so that $\eta'_x(u) = \sigma(L(x)^{-1}u) = (\eta_{\mathfrak{R},\sigma})'_x(u)$ by Lemma 1.1 (f). $\diamond$

Corollary 1 *Every almost nilpotent algebra is exact.*

For if $\eta_{\mathfrak{R},\sigma}$ is defined on $\mathscr{G}(\mathfrak{R}, \sigma)$, then $\nu_\sigma(\xi e + x_0)$ is a power of ξ.

§2 Applications

1 It shall now be explicated how the formal constructions of §1 can be applied to those maps $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ for which the orbit of $\mathrm{Aut}(\mathscr{G}, \eta)$ through a point e of $\mathscr{G}$ is open. By Corollary 1 to Lemma 3.2 in part I, such an orbit is open if and only if $(\mathscr{G}, \eta, e)$ is not empty. With the notation of §1.1 and §1.2 and I, §3.2, set

$$\mathbf{G}(V) = \{(\mathscr{G}, \eta, e\mathfrak{R}) \mid \eta \in \mathrm{C}^\infty_+(\mathscr{G}), \eta(e) = 1, \mathfrak{R} \in (\mathscr{G}, \eta, e)\}$$
$$\mathbf{H}(V) = \{(\mathfrak{R}, e, \sigma) \mid \mathfrak{R} \text{ is an algebra on } V \text{ with identity element } e, \sigma \in K(\mathfrak{R})\}.$$

From the examples from part I, §4.2 we infer that $\mathbf{G}(V)$ is always non-empty. We have:

Theorem 2.1 *For* $(\mathscr{G}, \eta, e, \mathfrak{R}) \in \mathbf{G}(V)$ *we have that:*

(a) *The linear form* $\sigma = \eta'_e$ *is* $\mathfrak{R}$*-closed and* $\mathscr{G} \subset \mathscr{G}(\mathfrak{R}, \sigma)$.

(b) *For all* y *in a neighborhood of* e *we have*

$$\eta(y) = \exp(-\sigma(\log_{\mathfrak{R}}(y))).$$

(c) $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathcal{G}, \eta)$.

(d) *If η is exploding, then $\sigma = \eta'_e$ is an $\mathfrak{R}$-exact linear form and $\mathcal{G} = \mathcal{G}(\mathfrak{R}, \sigma)$.*

PROOF: By (3.1) in part I, we have

$$\eta'_x(u) = -\Delta_x^u \log(\eta(x)) = \sigma(L(x)^{-1}u), \quad x \in \mathcal{G}. \qquad (*)$$

Thus σ is an $\mathfrak{R}$-closed linear form. The remainder of (a) follows from I, Lemma 1.6.

Comparing $(*)$ with part (f) in Lemma 1.1, we find $\Delta_y^u(\log(\eta(y))+\sigma(\log_{\mathfrak{R}}(y))) = 0$ for all y in a neighborhood of e. This is part (b).

Part (c) follows directly from the definition of the algebras in $(\mathcal{G}, \eta, e)$.

For part (d) we can apply I, Theorem 1.8 because of $(*)$. Thus $\mathcal{G}$ is a connected component of $\mathcal{D}(\mathfrak{R}, \sigma)$, that is, $\mathcal{G} = \mathcal{G}(\mathfrak{R}, \sigma)$ and σ is $\mathfrak{R}$-exact. ◇

Corollary 1 *For $(\mathcal{G}, \eta, e, \mathfrak{R}) \in \mathbf{G}(V)$ and $\sigma = \eta'_e$, the expressions $\sigma(x^m)$, with $m \geq 2$, do not depend on the choice of algebra $\mathfrak{R}$ from $(\mathcal{G}, \eta, e)$. For, by part (b), $\sigma(\log_{\mathfrak{R}}(y))$ depends only on η and e.*

Corollary 2 *If η is exploding and $(\mathcal{G}, \eta, e)$ not empty, then there exists a homogeneous polynomial ν with*

(a) $\deg \nu \leq \dim V$.

(b) *$\mathcal{G}$ is a connected component of $\mathcal{D}(\nu)$.*

(c) *For all $\mathfrak{R} \in (\mathcal{G}, \eta, e)$, ν divides the polynomial $\det(L(x))$ and we have $\mathcal{G} = \mathcal{G}(\mathfrak{R}, \eta'_e)$.*

Choose ν as an exact divisor of η' and deduce from part I, (3.1) that $\nu(x)$ is a divisor of $\det(L(x))$ for all $\mathfrak{R} \in (\mathcal{G}, \eta, e)$. The claim follows from I, Theorem 1.8 (a) and Theorem 2.1 (d).

2 Because of Theorem 2.1 (a) we define a map

$$F : \mathbf{G}(V) \to \mathbf{H}(V), \quad (\mathcal{G}, \eta, e, \mathfrak{R}) \mapsto (\mathfrak{R}, e, \eta'_e).$$

Examples demonstrate that in general F is neither injective nor surjective. However, a suitable restriction is bijective: It was already remarked in I, §1.4 that for fixed η an arbitrary shrinking of the has to be excluded if we wish to prove more than formal results. For short, we say that η *defines the domain* $\mathscr{G}$, if:

$\eta \in C^\infty_+(\mathscr{G})$, $\quad \eta'$ is rational. (2.1)

If ν is an exact denominator of η', then $\mathscr{G}$ is a connected component of $\mathscr{D}(\nu)$. (2.2)

Theorem 1.8 in part I says that $\eta \in C^\infty_+(\mathscr{G})$ defines the domain $\mathscr{G}$ if η is exploding. Let

$$\mathbf{G}_1(V) = \{(\mathscr{G}, \eta, e, \mathfrak{R}) \in \mathbf{G}(V) \mid \eta \text{ defines } \mathscr{G}\} \subset \mathbf{G}(V).$$
$$\mathbf{H}_1(V) = \{(\mathfrak{R}, e, \sigma) \in \mathbf{H}(V) \mid \sigma \in K_1(\mathfrak{R})\} \subset \mathbf{H}(V).$$

For $(\mathscr{G}, \eta, e, \mathfrak{R}) \in \mathbf{G}_1(V)$ and every $\mathfrak{R} \in (\mathscr{G}, \eta, e)$ we write as in I, (3.1):

$$-\Delta^u_x \log(\eta(x)) = \eta'_x(u) = \sigma(L(x)^{-1}u), \quad \sigma = \eta'_e, \tag{2.3}$$

and denote an exact denominator of η' by ν. With the notations of §1.2 we then have $\nu = \nu_\sigma$ and $\mathscr{G}$ is a connected component of $\mathscr{D}(\nu_\sigma) = \mathscr{D}(\mathfrak{R}, \sigma)$. Thus $\mathscr{G} = \mathscr{G}(\mathfrak{R}, \eta'_e)$ for every $\mathfrak{R}$ in $(\mathscr{G}, \eta, e)$. The relation (2.3) further implies that $\sigma = \eta'_e$ is an $\mathfrak{R}$-exact linear form. Thus F maps the set $\mathbf{G}_1(V)$ into $\mathbf{H}_1(V)$.

Theorem 2.2 *The map* $(\mathscr{G}, \eta, e, \mathfrak{R}) \mapsto (\mathfrak{R}, e, \eta'_e)$ *defines a bijection from* $\mathbf{G}_1(V)$ *to* $\mathbf{H}_1(V)$.

PROOF: First, note that the map is injective, for $(\mathfrak{R}, e, \eta'_e) = (\tilde{\mathfrak{R}}, \tilde{e}, \tilde{\eta}'_e)$ implies $\mathfrak{R} = \tilde{\mathfrak{R}}$, $e = \tilde{e}$ and $\eta'_e = \tilde{\eta}'_e$. Since $\mathscr{G}(\mathfrak{R}, \eta'_e) = \mathscr{G}(\tilde{\mathfrak{R}}, \tilde{\eta}'_e)$, η and $\tilde{\eta}$ are both defined on the domain $\mathscr{G} = \mathscr{G}(\mathfrak{R}, \eta'_e)$, and (2.3) shows that $\log(\eta)$ and $\log(\tilde{\eta})$ differ only by an additive constant. From $\eta(e) = 1 = \tilde{\eta}(e)$ it follows that $\eta = \tilde{\eta}$.

To prove surjectivity, consider $(\mathfrak{R}, e, \sigma) \in \mathbf{H}_1(V)$. Note that by assumption there exists a differentiable map $\varphi : \mathscr{G}(\mathfrak{R}, \sigma) \to \mathbb{R}$ with $\Delta^u_x \varphi(x) = \sigma(L(x)^{-1}u)$ and $\varphi(e) = 0$. Thus φ is real-analytic, and Lemma 1.1 (f) shows the validity of $\varphi(x) = \sigma(\log_{\mathfrak{R}}(x))$ on a neighborhood of e. Now Lemma 1.2 (d) implies that

$$\varphi(\exp(R(u))x) = \varphi(x) + \sigma(u) \tag{$*$}$$

for u in a neighborhood 0 and x in a neighborhood of e. By Lemma 1.2 (a), $\exp(\mathfrak{R})$ is contained in the identity component of $\Gamma(\mathfrak{R}, \sigma)$, so that $\exp(\mathfrak{R})$ acts on $\mathscr{G}(\mathfrak{R}, \sigma)$ because of §1.2. Thus $(*)$ holds for all u and all $x \in \mathscr{G}(\mathfrak{R}, \sigma)$. Now set $\mathscr{G} = \mathscr{G}(\mathfrak{R}, \sigma)$, $\eta(x) = e^{-\varphi(x)}$, and obtain $\sigma = \eta'_e$ and $\mathfrak{R} \in (\mathscr{G}, \eta, e)$. Moreover, $\mathscr{G}$ is defined by η, so $(\mathscr{G}, \eta, e, \mathfrak{R})$ belongs to $\mathbf{G}_1(V)$ and has $(\mathfrak{R}, e, \sigma)$ as image. $\diamond$

§3 Construction of exact linear forms

1 Let $\mathfrak{R}$ be an algebra on V with identity e. In Lemma 1.2 we saw that the subgroup $\exp(\mathfrak{R})$ of $\mathrm{GL}(V)$ generated by all endomorphisms $\exp(R(u))$, $u \in V$, is contained in all groups $\Gamma(\mathfrak{R}, \sigma)$, $\sigma \in K(\mathfrak{R})$. The simplest examples of relative invariants for $\exp(\mathfrak{R})$ (cf. I, §2.1) are polynomials: To every polynomial $\omega : V \to \mathbb{R}$ assign as in I, §1.5 the subset

$$\mathcal{D}(\omega) = \{x \in V \mid \omega(x) \neq 0\}$$

of V and denote by $\mathcal{D}_e(\omega)$ the connected component of $\mathcal{D}(\omega)$ containing e.

Let $P(\mathfrak{R}, e)$ denote the set of homogeneous polynomials $\omega : V \to \mathbb{R}$ with $\omega(e) = 1$ for which there is a map $\alpha = \alpha_\omega : \exp(\mathfrak{R}) \to \mathbb{R}$ such that $\omega(Wx) = \alpha(W)\omega(x)$ holds for all $W \in \exp(\mathfrak{R})$. By construction, ω is positive on $\mathcal{D}_e(\omega)$ and the algebra $\mathfrak{R}$ is contained in $(\mathcal{D}_e(\omega), \omega, e)$. So by Theorem 2.1 (a), $\omega \mapsto \omega'_e$ defines an injection of $P(\mathfrak{R}, e)$ into $K(\mathfrak{R})$. With the notation of I, §2.1, $\omega \in R(\mathcal{D}_e(\omega), \exp(\mathfrak{R}))$ holds.

Conversely, if we consider $\sigma \in K(\mathfrak{R})$, then as in (1.1), we can write

$$\sigma(L(x)^{-1}u) = \frac{1}{\nu(x)}\pi_x(u), \quad x \in \mathcal{G}(\mathfrak{R}, \sigma)$$

with a normalized exact denominator $\nu = \nu_\sigma$. In the notation of 1. and §1.2,

$$\mathcal{D}(\nu_\sigma) = \mathcal{D}(\mathfrak{R}, \sigma), \quad \mathcal{D}_e(\nu_\sigma) = \mathcal{D}(\mathfrak{R}, \sigma).$$

Since $\exp(\mathfrak{R})$ is contained in $\Gamma(\mathfrak{R}, \sigma)$, it follows from (1.4) that $\nu_\sigma \in P(\mathfrak{R}, e)$. So $\sigma \mapsto \nu_\sigma$ defines a map of $K(\mathfrak{R})$ into $P(\mathfrak{R}, e)$ and ν_σ is positive and real-analytic on $\mathcal{G}(\mathfrak{R}, \sigma)$.

2 By combining the maps $\omega \mapsto \omega'_e$ and $\sigma \mapsto \nu_\sigma$ we obtain self-maps $\sigma \mapsto \sigma^\#$ of $K(\mathfrak{R})$ and $\omega \mapsto \omega^\#$ of $P(\mathfrak{R}, e)$ via

$$\sigma^\# = (\nu_\sigma)'_e, \quad \omega^\# = \nu_\sigma, \quad \text{if } \sigma = \omega'_e.$$

By Theorem 2.1 (b),

$$\nu_\sigma(x) = \exp(-\sigma^\#(\log_{\mathfrak{R}}(x)))$$

for all x in a neighborhood of e. By Lemma 1.1 (f),

$$\sigma^\#(L(x)^{-1}u) = \Delta_x^u \sigma^\#(\log_{\mathfrak{R}}(x)) = -\Delta_x^u \log(\nu_\sigma(x)) = -\frac{1}{\nu_\sigma(x)}\Delta_x^u \nu_\sigma(x). \tag{3.1}$$

By construction of $\omega^{\#}(x)$ is the exact normalized denominator of

$$\omega_e'(L(x)^{-1}u) = \omega_x'(u) = -\frac{1}{\omega(x)}\Delta_x^u\omega(x)m \tag{3.2}$$

cf. I, (3.1).

Lemma 3.1 *For $\sigma, \varrho \in K(\mathfrak{R})$ and $\omega \in P(\mathfrak{R}, e)$:*

(a) *$\sigma^{\#} = \varrho^{\#}$ if and only if $\nu_\sigma = \nu_\varrho$.*

(b) *$\omega^{\#}$ equals the product of the distinct normalized irreducible factors of ω.*

(c) *$\omega^{\#\#} = \omega^{\#}$, $\nu_{\sigma^{\#}} = (\nu_\sigma)^{\#}$, $\sigma^{\#\#\#} = \sigma^{\#\#}$.*

(d) *$\mathcal{D}(\mathfrak{R}, \sigma^{\#}) = \mathcal{D}(\mathfrak{R}, \sigma)$, $\mathcal{G}(\mathfrak{R}, \sigma^{\#}) = \mathcal{G}(\mathfrak{R}, \sigma)$ and $\sigma^{\#} \in K_1(\mathfrak{R})$.*

(e) *$\Gamma(\mathfrak{R}, \sigma^{\#})$*
$= \{W \in \mathrm{GL}(V) \mid \nu_\sigma(y)\nu_\sigma(Wx) = \nu_\sigma(x)\nu_\sigma(Wy)$ for $x, y \in V\}$
$= \Gamma(\nu_\sigma')$.

PROOF: (a) As ν_σ and ν_ϱ are normalized to 1 at the point e, the claim follows directly from (3.1).

(b) To determine the normalized exact denominator $\nu(x)$ of $\Delta_x^u \log(\omega(x))$, we write ω as a product of powers of normalized irreducible factors $\omega_1, \ldots, \omega_s$ and verify that $\nu = \omega_1 \cdots \omega_s$. The claim now follows from (3.2).

(c) The first claim follows directly from part (b). By definition, $\nu_{\sigma^{\#}}(x)$ is the normalized exact denominator of $\sigma^{\#}(L(x)^{-1}u)$. On the other hand, $(\nu_\sigma)^{\#}$ is the normalized exact denominator of $-\frac{1}{\nu_\sigma(x)}\Delta_x^u\nu_\sigma(x)$. The second claim now follows from (3.1). Now we have

$$\nu_{\sigma^{\#\#}} = (\nu_{\sigma^{\#}})^{\#} = (\nu_\sigma)^{\#\#} = (\nu_\sigma)^{\#} = \nu_{\sigma^{\#}}.$$

With part (a), the last claim follows.

(d) By $\nu_{\sigma^{\#}} = (\nu_\sigma)^{\#}$ and part (b),

$$\mathcal{D}(\mathfrak{R}, \sigma^{\#}) = \{x \in V \mid (\nu_\sigma)^{\#}(x) \neq 0\} = \mathcal{D}(\mathfrak{R}, \sigma)$$

so that $\mathcal{G}(\mathfrak{R}, \sigma^{\#}) = \mathcal{G}(\mathfrak{R}, \sigma)$ follows. By (3.1), $\sigma^{\#}$ is thus $\mathfrak{R}$-exact.

(e) Let $W \in \mathrm{GL}(V)$. By (3.1), $W \in \Gamma(\mathfrak{R}, \sigma^{\#})$ is equivalent to $\Delta_x^u \log(\nu_\sigma(Wx)) = \Delta_{Wx}^{Wu} \log(\nu_\sigma(Wx)) = \Delta_x^u \log(\nu_\sigma(x))$, that is, with $\nu_\sigma(y)\nu_\sigma(Wx) = \nu_\sigma(x)\nu_\sigma(Wy)$ for $x, y \in V$. $\diamond$

The following claim follows analogously from (3.1):

Corollary 1 *$K(\mathfrak{R}) \neq \{0\}$ implies $K_1(\mathfrak{R}) \neq \{0\}$.*

For $\sigma^{\#} \neq 0$ if $\sigma \neq 0$.

Corollary 2 *$\sigma^{\#\#} = \sigma^{\#}$ if and only if ν_σ is a product of distinct irreducible factors.*

This follows from (3.1) because of (c).

Corollary 3 *If $\det(L(x))$ is a product of distinct irreducible factors, then $\sigma^{\#\#} = \sigma^{\#}$ for all $\sigma \in K(\mathfrak{R})$.*

Corollary 4 *For $\sigma \in K(\mathfrak{R})$, $\sigma \neq 0$, we have that $\frac{1}{\nu_\sigma} : \mathcal{G}(\mathfrak{R}, \sigma) \to \mathbb{R}$ is exploding, and that $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathcal{G}(\mathfrak{R}, \sigma), \frac{1}{\nu_\sigma})$.*

From (e) and Lemma 1.2 (a) we immediately obtain

$$\exp(\mathfrak{R}) \subset \Gamma(\mathfrak{R}, \sigma^{\#}) \subset \mathrm{Aut}(\mathcal{G}(\mathfrak{R}, \sigma), \tfrac{1}{\nu_\sigma}).$$

In addition to Theorem 2.1 (d), we have:

Theorem 3.2 *If $\mathfrak{R}$ is an algebra on V with identity, then the following are equivalent:*

(a) $K(\mathfrak{R}) \neq \{0\}$.

(b) $K_1(\mathfrak{R}) \neq \{0\}$.

(c) *There is a domain $\mathcal{G}$ in V and an exploding map $\eta : \mathcal{G} \to \mathbb{R}^+$ with $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathcal{G}, \eta)$.*

PROOF: (a) $\Leftrightarrow$ (b) follows from Corollary 1.

(a) $\Rightarrow$ (c) follows from Corollary 4.

(c) $\Rightarrow$ (a) follows from Theorem 2.1. $\diamond$

§4 Dual domains

1 For an algebra $\mathfrak{R}$ with identity element e, a $\mathfrak{R}$-commutative linear form σ is called *non-degenerate* if the symmetric bilinear form $(u, v) \mapsto \sigma(u, v) = \sigma(uv)$ is non-degenerate.

Suppose the $\mathfrak{R}$-commutative linear form σ is non-degenerate. Then we can define a new algebra $\mathfrak{R}^\sigma$ on V whose product is $(u, v) \mapsto u \triangle_\sigma v = L_\sigma(u)v = R_\sigma(v)u$, defined by

$$\sigma(u, v \triangle_\sigma w) = \sigma(uw, v). \tag{4.1}$$

Clearly, the product is determined by $R_\sigma(w) = R(W)^\sigma$. Moreover, e is the identity element of $\mathfrak{R}^\sigma$ and σ is $\mathfrak{R}^\sigma$-commutative. Finally, $(\mathfrak{R}^\sigma)^\sigma = \mathfrak{R}$.

Let ν, $\tilde{\nu}$ denote the normalized exact denominators of $\sigma(L(x)^{-1}u)$, $\sigma(L_\sigma(x)^{-1}u)$, respectively. Since σ is non-degenerate, there exist rational functions $h = h_{\mathfrak{R},\sigma}$, $\tilde{h} = \tilde{h}_{\mathfrak{R},\sigma}$ with

$$\sigma(L(x)^{-1}u) = \sigma(h(x), u), \qquad \text{that is, } h(x) = L^\sigma(x)^{-1}e \tag{4.2}$$

and

$$\sigma(L(x)^{-1}u) = \sigma(\tilde{h}(x), u), \qquad \text{that is, } \tilde{h}(x) = L^\sigma_\sigma(x)^{-1}e \tag{4.3}$$

respectively. With the notation of §1.2, the maps $h : \mathcal{D}(\mathfrak{R}, \sigma) \to V$ and $\tilde{h} : \mathcal{D}(\mathfrak{R}^\sigma, \sigma) \to V$ are real analytic and $h(e) = e = \tilde{h}(e)$. From (4.2) and (4.3) it follows that

$$\Delta^u_x h(x)|_{x=e} = \Delta^u_x \tilde{h}(x)|_{x=e} = -u. \tag{4.4}$$

Lemma 4.1 *For non-degenerate $\sigma \in K(\mathfrak{R})$ we have:*

(a) *h and $\tilde{h}$ are birational and inverse to one another.*

(b) *$\sigma \in K(\mathfrak{R}^\sigma)$.*

(c) *If U is a neighborhood of e and $\varphi : U \to \mathbb{R}$ differentiable with $\Delta^u_x\varphi(x) = \sigma(L(x)^{-1}u)$, then $\tilde{\varphi}(y) = -\varphi(\tilde{h}(y))$ is differentiable in a neighborhood of e and $\Delta^u_y\tilde{\varphi}(y) = \sigma(L_\sigma(x)^{-1}u)$ holds.*

PROOF: (a) Clearly, h and $\tilde{h}$ are real-analytic in a neighborhood of e. By (4.4) h and $\tilde{h}$ are invertible in a neighborhood of e. Substitute $x = h^{-1}(y)$ in $\sigma(h(x), xu) = \sigma(u)$ to obtain $\sigma(u) = \sigma(y, h^{-1}(y)u) = \sigma(y \triangle_\sigma u, h^{-1}(y))$, that is, $h^{-1}(y) = \tilde{h}(y)$ in a neighborhood of e. As h and $\tilde{h}$ are rational, it follows formally that $h \circ \tilde{h} = \mathrm{id}$, and analogously $\tilde{h} \circ h = \mathrm{id}$.

(b) By (4.1),

$$\begin{aligned}\varphi(x; u, v) &= \sigma(L_\sigma(x)^{-1}((x \triangle_\sigma u) \triangle_\sigma v)) = \sigma(\tilde{h}(x), (x \triangle_\sigma y) \triangle_\sigma v) \\ &= \sigma((\tilde{h}(x) v) u, x),\end{aligned}$$

that is, $\varphi(h(y); u, v) = \sigma((yv)u, h(y)) = \sigma(L(y)^{-1}((yv)u))$. By Lemma 1.1 (c), $\varphi(h(y); u, v)$ is symmetric in u and v. But this means $\sigma \in K(\mathfrak{R}^\sigma)$.

(c) Direct verification. ◇

2 Again, let $\sigma \in K(\mathfrak{R})$ be non-degenerate. Let $\mathcal{N}(\mathfrak{R}, \sigma)$ denote the set of $x \in \mathcal{D}(\mathfrak{R}, \sigma)$ for which the symmetric bilinear form $(u, v) \mapsto \sigma(L(x)^{-1} L(u) L(x)^{-1} v)$ is non-degenerate. Obviously $\mathcal{N}(\mathfrak{R}, \sigma)$ is open and contains e. Examples show that $\mathcal{N}(\mathfrak{R}, \sigma)$ can be a proper subset of $\mathcal{D}(\mathfrak{R}, \sigma)$.

Theorem 4.2 *If $\sigma \in K(\mathfrak{R})$ is non-degenerate, then the maps $h : \mathcal{N}(\mathfrak{R}, \sigma) \to \mathcal{N}(\mathfrak{R}^\sigma, \sigma)$ and $\tilde{h} : \mathcal{N}(\mathfrak{R}^\sigma, \sigma) \to \mathcal{N}(\mathfrak{R}, \sigma)$ are birational, real-analytic and inverses of one another.*

PROOF: By $\sigma(L(x)^{-1} L(u) L(x)^{-1} v) = -\Delta_x^u \sigma(L(x)^{-1} v) = -\sigma(\Delta_x^u h(x), v)$, the map h is locally invertible at every point in $\mathcal{N}(\mathfrak{R}, \sigma)$. For $x \in \mathcal{N}(\mathfrak{R}, \sigma)$ there exists thus a neighborhood $U \subset \mathcal{N}(\mathfrak{R}, \sigma)$ such that the local inverse $f : h(U) \to U$ of h is real-analytic. Now h is birational by Lemma 4.1 (b) with inverse $\tilde{h}$, so that f coincides with $\tilde{h}$ on $\mathcal{D}(\mathfrak{R}^\sigma, \sigma) \cap h(U)$. We thus obtain $h(U) \subset \mathcal{D}(\mathfrak{R}^\sigma, \sigma)$. As the product of the functional determinant of $\tilde{h}$ at the point $h(x)$ with the functional determinant of h at the point x equals 1, it follows that $h(x) \in \mathcal{D}(\mathfrak{R}^\sigma, \sigma)$. ◇

Lemma 4.3 *If $\sigma \in K(\mathfrak{R})$ is non-degenerate, then for all $W \in \mathrm{GL}(V)$:*

(a) *$W \in \Gamma(\mathfrak{R}, \sigma)$ is equivalent to $h(Wx) = (W^\sigma)^{-1} h(x)$ for all $x \in \mathcal{D}(\mathfrak{R}, \sigma)$.*

(b) *$W \in \Gamma(\mathfrak{R}^\sigma, \sigma)$ is equivalent to $\tilde{h}(Wx) = (W^\sigma)^{-1} \tilde{h}(x)$ for all $x \in \mathcal{D}(\mathfrak{R}^\sigma, \sigma)$.*

(c) *$\Gamma(\mathfrak{R}, \sigma)^\sigma = \Gamma(\mathfrak{R}^\sigma, \sigma)$.*

PROOF: (a) Compare the defining property (1.3) for $\Gamma(\mathfrak{R}, \sigma)$ with the definition of h according to (4.2).

(b) Analogusly.

(c) For $W \in \Gamma(\mathfrak{R}, \sigma)$ apply $\tilde{h}$ to part (a) and set $x = \tilde{h}(y)$. Then $W\tilde{h}(y) = \tilde{h}((W^\sigma)^{-1} y)$ follows, so that $(W^\sigma)^{-1} \in \Gamma(\mathfrak{R}^\sigma, \sigma)$. This means $\Gamma(\mathfrak{R}, \sigma)^\sigma \subset \Gamma(\mathfrak{R}^\sigma, \sigma)$, and $(\mathfrak{R}^\sigma)^\sigma = \mathfrak{R}$ establishes the claim. ◇

3 Now let $\mathscr{G}$ be a domain in V and $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$. In I, §2.4 we called η non-degenerate if the symmetric bilinear form $\eta''_x : V \times V \to \mathbb{R}$ is non-degenerate for all $x \in V$. Define $\mathscr{G}^\eta = \{\eta'_x \mid x \in \mathscr{G}\}$ and obtain an open map $\eta' : \mathscr{G} \to \mathscr{G}^\eta$. By I, Theorem 3.4, for non-degenerate η the map $\eta' : \mathscr{G} \to \mathscr{G}^\eta$ is a bijection if $\mathrm{Aut}(\mathscr{G}, \eta)$ acts transitively on $\mathscr{G}$. In the following we will study this map under the weaker assumption $(\mathscr{G}, \eta, e) \neq \emptyset$.

Let $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ be non-degenerate, $e \in \mathscr{G}$ and $(\mathscr{G}, \eta, e) \neq \emptyset$. After a choice of $\mathfrak{R} \in (\mathscr{G}, \eta, e)$, set $\sigma = \eta'_e$ and obtain $\sigma \in K(\mathfrak{R})$ by Theorem 2.1 (a). By I, §3.1 and I, §3.2, in the notation of §1.1, we have

$$\begin{aligned} \eta'_x(u) &= \sigma(L(x)^{-1}u) = \sigma(h(x), u) \\ \eta''_x(u, v) &= \sigma(L(x)^{-1}L(u)L(x)^{-1}v), \quad x \in \mathscr{G}. \end{aligned} \tag{4.5}$$

Here, h depends only on η and e, but not on the choice of $\mathfrak{R}$ in $(\mathscr{G}, \eta, e)$. As $\tilde{h}$ is the inverse of h, also $\tilde{h}$ depends only on η and e. In particular, $\mathscr{G} \subset \mathcal{N}(\mathfrak{R}, \sigma)$ and the linear form σ is non-degenerate. Thus we can apply the results of 1. and 2.

Define an isomorphism $i : V \to V^*$ by $i_a(u) = \sigma(a, u)$ and obtain $\eta'_x = i_{h(x)}$, with $x \in \mathscr{G}$, from (4.5). If we set

$$\mathscr{G}^{\eta,e} = \{h(x) \mid x \in \mathscr{G}\}$$

then we obtain the following commutative diagram

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\eta'} & \mathscr{G}^{\eta} \\ & {\scriptstyle h}\searrow & \uparrow{\scriptstyle i} \\ & & \mathscr{G}^{\eta,e} \end{array} \tag{4.6}$$

Theorem 4.4 *If $\eta \in \mathrm{C}^\infty_+(\mathscr{G})$ is non-degenerate, $e \in \mathscr{G}$ and $(\mathscr{G}, \eta, e) \neq \emptyset$, then all maps in* (4.6) *are real analytic and birational.*

PROOF: For $(\mathscr{G}, \eta, e)$ construct $h = h_{\mathfrak{R},\sigma}$ and $\sigma = \eta'_x$ according to (4.2) and thus obtain $\mathscr{G} \subset \mathcal{N}(\mathfrak{R}, \sigma)$. Now the claim follows from Theorem 4.2. ◇

4 Assume once more $e \in \mathcal{G}$, $\eta \in \mathrm{C}^{\infty}_{+}(\mathcal{G})$ non-degenerate and that $(\mathcal{G}, \eta, e) \neq \emptyset$. Then set $\sigma = \eta'_e$,

$$\tilde{\eta} : \mathcal{G}^{\eta,e} \to \mathbb{R}, \quad \tilde{\eta}(y) = \eta(\tilde{h}(y))^{-1} \tag{4.7}$$

and obtain $\tilde{\eta} \in \mathrm{C}^{\infty}_{+}(\mathcal{G}^{\eta,e})$.

Theorem 4.5 *If $\eta \in \mathrm{C}^{\infty}_{+}(\mathcal{G})$ is non-degenerate, $e \in \mathcal{G}$ and $(\mathcal{G}, \eta, e) \neq \emptyset$, then:*

(a) *$\tilde{\eta}$ is non-degenerate and $e \in \mathcal{G}^{\eta,e}$.*

(b) *$\tilde{\eta}'_y(u) = \sigma(\tilde{h}(y), u)$ for $y \in \mathcal{G}^{\eta,e}$, $\tilde{\eta}'_e = \eta'_e = \sigma$.*

(c) *$(\mathcal{G}^{\eta,e})^{\tilde{\eta},e} = \mathcal{G}$.*

(d) *$\mathrm{Aut}(\mathcal{G}^{\eta,e}, \tilde{\eta}) = \mathrm{Aut}(\mathcal{G}, \eta)^{\sigma}$.*

(e) *For every $\mathfrak{R} \in (\mathcal{G}, \eta, e)$ we have $\mathfrak{R}^{\sigma} \in (\mathcal{G}^{\eta,e}, \tilde{\eta}, e)$.*

PROOF: (b) By the chain rule, (4.5) and Lemma 4.1 (a) it first follows for $y \in \mathcal{G}^{\eta,e}$ that

$$\tilde{\eta}'_y(u) = -\eta'_{\tilde{h}(y)}(\Delta^u_y \tilde{h}(y)) = -\sigma(y, \Delta^u_y \tilde{h}(y)).$$

From $\sigma(\tilde{h}(y), y) = \sigma(e)$ it follows that $\sigma(\Delta^u_y \tilde{h}(y), y) = -\sigma(\tilde{h}(y), u)$.

(a) Define $H : \mathcal{G} \to \mathrm{End}(V)$ and $\tilde{H} : \mathcal{G}^{\eta,e} \to \mathrm{End}(V)$ by $H(x)v = -\Delta^v_x h(x)$ and $\tilde{H}(y)v = -\Delta^v_y \tilde{h}(y)$, respectively, and obtain from (4.5) and (b)

$$\eta''_x(u, v) = \sigma(u, H(x)b), \quad \tilde{\eta}''_y(u, v) = \sigma(u, \tilde{H}(y)v).$$

Since $h \circ \tilde{h} = \mathrm{id}$, $H(\tilde{h}(y))\tilde{H}(y) = \mathrm{id}$, so that $H(x)$ invertible, for $x \in \mathcal{G}$, implies $\tilde{H}(y)$ invertible, for $y \in \mathcal{G}^{\eta,e}$.

(c) h and $\tilde{h}$ are inverses of each other.

(d) Follows from Lemma 4.3.

(e) $\exp(\mathfrak{R}) \subset \mathrm{Aut}(\mathcal{G}, \eta)$ implies $\exp(\mathfrak{R})^{\sigma} \subset \mathrm{Aut}(\mathcal{G}^{\eta,e}, \tilde{\eta})$ by (d). But by definition of $\mathfrak{R}^{\sigma}$, $\exp(\mathfrak{R})^{\sigma} = \exp(\mathfrak{R}^{\sigma})$. ◇

Corollary 1 *Let $\eta \in \mathrm{C}^{\infty}_{+}(\mathcal{G})$ be non-degenerate and $(\mathcal{G}, \eta, e) \neq \emptyset$. Then $\tilde{\eta}$ is non-degenerate and $(\mathcal{G}^{\eta,e}, \tilde{\eta}, e) \neq \emptyset$.*

Corollary 2 *For non-degenerate $\eta \in \mathrm{C}^{\infty}_{+}(\mathcal{G})$, the following are equivalent:*

(a) $\mathrm{Aut}(\mathscr{G}, \eta)$ *acts transitively on* $\mathscr{G}$.

(b) $\mathrm{Aut}(\mathscr{G}^{\eta,e}, \tilde{\eta})$ *acts transitively on* $\mathscr{G}^{\eta,e}$.

Corollary 3 *For non-degenerate* $\eta \in \mathrm{C}^{\infty}_{+}(\mathscr{G})$*, we have* $\mathscr{G} \cap \mathscr{G}^{\eta,e} \neq \emptyset$.

This result was proved by Ochiai [12] for regular convex cones.

These results justify calling $\mathscr{G}^{\eta,e}$ the (η, e)*-dual domain of* $\mathscr{G}$.

5 For non-degenerate and homogeneous $\eta \in \mathrm{C}^{\infty}_{+}(\mathscr{G})$ we defined in I, §4.1 after a choice of $e \in \mathscr{G}$ a commutative algebra $\mathfrak{A}(\eta, e)$ on V with product $(u, v) \mapsto A(u)v = u \cdot v$. If $\mathfrak{R} \in (\mathscr{G}, \eta, e)$, then we can obtain $\mathfrak{A}(\eta, e)$ from $\mathfrak{R}$ and $\mathfrak{R}^{\sigma}$: For by I, §4.6

$$uv + u\triangle_{\sigma}v = 2u \cdot v. \tag{4.8}$$

By $(\mathfrak{R}^{\sigma})^{\sigma} = \mathfrak{R}$ and Theorem 4.5 (e) it follows that

$$\mathfrak{A}(\tilde{\eta}, e) = \mathfrak{A}(\eta, e). \tag{4.9}$$

In addition to I, Theorem 4.3, we have:

Theorem 4.6 *Let* $\eta \in \mathrm{C}^{\infty}_{+}(\mathscr{G})$ *be non-degenerate and homogeneous. If* $(\mathscr{G}, \eta, e) \neq \emptyset$ *for some* $e \in \mathscr{G}$*, then the following are equivalent:*

(a) $h(x) = \tilde{h}(x)$ *for all* x *in a neighborhood of* e.

(b) $\mathfrak{A}(\eta, e) \in (\mathscr{G}, \eta, e)$.

If this is the case, then $\mathfrak{A}(\eta, e)$ *is a Jordan algebra and* $\sigma = \eta'_e$ *is* $\mathfrak{A}(\eta, e)$*-exact.*

PROOF: (a) $\Rightarrow$ (b): By Theorem 4.5 (b), η and $\tilde{\eta}$ coincide on a neighborhood of e. For $\mathfrak{R} \in (\mathscr{G}, \eta, e)$, $\exp(\mathfrak{R})$ belongs to $\Gamma(\mathfrak{R}, \sigma)$ by Lemma 1.2 (a). With Lemma 4.3 it then follows that $\exp(\mathfrak{R}^{\sigma}) \subset \Gamma(\mathfrak{R}, \sigma)$ and thus $\eta'_{Wx}(Wu) = \eta'_x(u)$ for $W \in \exp(\mathfrak{R}^{\sigma})$. We obtain $\mathfrak{R}^{\sigma} \in (\mathscr{G}, \eta, e)$, and (4.8) implies $\mathfrak{A}(\eta, e) \in (\mathscr{G}, \eta, e)$.

(b) $\Rightarrow$ (a): Choose $\mathfrak{R} = \mathfrak{A}(\eta, e)$ and obtain $h = \tilde{h}$.

By I, Theorem 4.2, $\mathscr{G}$ is a connected component of $\mathrm{Inv}(\mathfrak{A}(\eta, e))$, and the exact denominators of x^{-1} and $\eta'_x(u) = \sigma(x^{-1}, u)$ are proportional. Thus $\mathscr{G} = \mathscr{G}(\mathfrak{A}(\eta, e), \sigma)$ and η is exact. ◇

§5 Associative algebras

1 Our investigations so far left open which $\mathfrak{R}$-closed linear forms are also $\mathfrak{R}$-exact, or which algebras are exact (cf. §1.3). Now we demonstrate how to treat these questions for associative algebras. We will see that the radical of the algebra does not cause any problems.

Let $\mathfrak{R}$ be an associative algebra on V with identity element e and product $uv = L(u)v = R(v)u$. The set $\mathcal{J}(\mathfrak{R})$ then consists precisely of the invertible elements of $\mathfrak{R}$ (cf. §1.1). For $x \in \mathcal{J}(\mathfrak{R})$ let x^{-1} denote the inverse of x.

If σ is a linear form on V, then define as in §1.2

$$\chi_x(u) = \sigma(x^{-1}u) = \frac{1}{\nu(x)}\pi_x(u), \quad \nu = \nu_\sigma, \quad \nu(e) = 1,$$

and

$$\mathcal{D}(\mathfrak{R}, \sigma) = \{x \in V \mid \nu_\sigma(x) \neq 0\},$$

and denote by $\mathcal{G}(\mathfrak{R}, \sigma)$ the connected component of $\mathcal{D}(\mathfrak{R}, \sigma)$ that contains e. Moreover, $\Gamma(\mathfrak{R}, \sigma)$ be the group of those $W \in \mathrm{GL}(V)$ for which $W\mathcal{D}(\mathfrak{R}, \sigma) = \mathcal{D}(\mathfrak{R}, \sigma)$ and

$$\sigma((Wx)^{-1}Wu) = \sigma(x^{-1}u), \quad x \in \mathcal{D}(\mathfrak{R}, \sigma),$$

hold.

By the Corollary to Lemma 1.1, $K(\mathfrak{R})$ consists of the $\mathfrak{R}$-commutative linear forms.

Let $\mathcal{G}(\mathfrak{R})$ denote the connected component of $\mathcal{J}(\mathfrak{R})$ that contains e. For all $\sigma \in K(\mathfrak{R})$ it follows that

$$\mathcal{J}(\mathfrak{R}) \subset \mathcal{D}(\mathfrak{R}, \sigma) \quad \text{and} \quad \mathcal{G}(\mathfrak{R}) \subset \mathcal{G}(\mathfrak{R}, \sigma).$$

Further define

$$\Gamma(\mathfrak{R}) = \{R(y) \mid y \in \mathcal{G}(\mathfrak{R})\}$$

and observe that $\Gamma(\mathfrak{R})$ is contained in $\mathrm{Aut}(\mathcal{G}(\mathfrak{R}))$, acts transitively on $\mathcal{G}(\mathfrak{R})$ and is closed in $\mathrm{GL}(V)$.

Since $\exp(R(u)) = R(\exp(u))$ it follows that $\mathfrak{Lie}(\Gamma(\mathfrak{R})) = \{R(u) \mid u \in V\}$.

Lemma 5.1 *For all* $\sigma \in K(\mathfrak{R})$*:*

(a) $\nu_\sigma(xy) = \nu_\sigma(x)\nu_\sigma(y)$ *for* $x, y \in \mathfrak{R}$.

(b) $\Gamma(\mathfrak{R}) \subset \mathrm{Aut}(\mathscr{G}(\mathfrak{R}, \sigma), \nu_\sigma)$.

(c) $\mathfrak{R} \in (\mathscr{G}(\mathfrak{R}, \sigma), \eta, e)$ *if* $\eta \in \mathrm{C}^\infty_+(\mathscr{G}(\mathfrak{R}, \sigma))$ *and* $\eta'_x(u) = \sigma(x^{-1}u)$ *for all* $x \in \mathscr{G}(\mathfrak{R}, \sigma)$.

PROOF: (a) For $\nu = \nu_\sigma$, $x, y \in \mathscr{J}(\mathfrak{R})$ we have

$$\frac{1}{\nu(xy)}\pi_{xy}(uy) = \sigma((xy)^{-1}(uy)) = \sigma(x^{-1}u) = \frac{1}{\nu(x)}\pi_x(u),$$

since σ is commutative. Thus $\nu(x)$ is a divisor of $\nu(xy)$, so that $\nu(xy) = \varrho(y)\nu(x)$ follows. The claim follows for $x = e$.

(b) Follows from (a).

(c) By definition (cf. I, §3.2), we need to show $R(u) \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}(\mathfrak{R}, \sigma), \eta))$ for all $u \in \mathfrak{R}$. By $\sigma(x^{-1}R(u)x) = \sigma(u)$ and I, Lemma 3.1, we only need to show $R(u) \in \mathfrak{Lie}(\mathrm{Aut}(\mathscr{G}(\mathfrak{R}, \sigma)))$ for all $u \in \mathfrak{R}$. As $\exp(R(u)) = R(\exp(u))$, this follows from (b). ◇

2 Define a linear form tr by $\mathrm{tr}(u) = \mathrm{trace}(L(u))$ and obtain $\mathrm{tr} \in K(\mathfrak{R})$. For $\eta(x) = \det(L(x))^{-1}$, $\eta \in \mathrm{C}^\infty_+(\mathscr{G}(\mathfrak{R}))$ is exploding and we have

$$\eta'_x(u) = \Delta^u_x \log\det(L(x)) = \mathrm{tr}(x^{-1}u), \quad x \in \mathscr{G}(\mathfrak{R}).$$

It follows that $\mathscr{G}(\mathfrak{R}) = \mathscr{G}(\mathfrak{R}, \mathrm{tr})$ and tr is $\mathfrak{R}$-exact. Since $R(x) \in \mathrm{Aut}(\mathscr{G}(\mathfrak{R}, \eta))$ for $x \in \mathscr{G}(\mathfrak{R})$, $\mathrm{Aut}(\mathscr{G}(\mathfrak{R}, \eta))$ acts transitively on $\mathscr{G}(\mathfrak{R})$ and $\mathfrak{R} \in (\mathscr{G}(\mathfrak{R}), \eta, e)$.

Lemma 5.2 *Every centrally-simple associative algebra is exact.*

PROOF: Every $\mathfrak{R}$-commutative linear form is a multiple of tr. ◇

Remark 1 The case $\mathfrak{R} = \mathbb{C}$ shows that we cannot drop the assumption of centrality.

3 A direct sum decomposition $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_0$ is called *Wedderburn decomposition* of $\mathfrak{R}$ if

$$\mathfrak{R} \text{ is a semisimple subalgebra of } \mathfrak{R}, \tag{WZ.1}$$

$$\mathfrak{R}_0 = \mathrm{Rad}\,\mathfrak{R}, \text{ that is, } \mathfrak{R}_0 \text{ is the maximal nilpotent ideal of } \mathfrak{R}. \tag{WZ.2}$$

The elements of $\mathfrak{R}$ can be written uniquely as $u = u_1 + u_0$. It is well-known that such a Wedderburn decomposition of $\mathfrak{R}$ exists and for every such decomposition, $e \in \mathfrak{R}_1$. Moreover,

$$\begin{aligned}\operatorname{Rad}\mathfrak{R} &= \{u \in V \mid \operatorname{tr}(uv) = 0 \text{ for all } v \in V\} \\ &= \{u \in V \mid uv \text{ nilpotent for all } v \in V\}.\end{aligned}$$

For later use note that

$$\mathcal{G}(\mathfrak{R}) = \mathcal{G}(\mathfrak{R}_1) + \mathfrak{R}_0 \tag{5.1}$$

since $\mathcal{J}(\mathfrak{R}) = \mathcal{J}(\mathfrak{R}_1) + \mathfrak{R}_0$ and the projection from $\mathfrak{R}$ to $\mathfrak{R}_1$ is continous.

Lemma 5.3 *Let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_0$ be a Wedderburn decomposition of $\mathfrak{R}$ and let the natural number k be chosen such that $\mathfrak{R}_0^k = \mathbf{0}$ holds. For $\sigma \in K(\mathfrak{R})$ let σ_1 denote the restriction of σ to $\mathfrak{R}_1$. Then*

$$\sigma(\log_{\mathfrak{R}}(x)) - \sigma_1(\log_{\mathfrak{R}_1}(x_1)) = \sum_{m=1}^{k-1} \frac{(-1)^m}{m} \sigma((x_1^{-1}x_0)^m), \tag{5.2}$$

$$\sigma(x^{-1}u) = \sigma(x_1^{-1}u) = \sum_{m=1}^{k-1} \sigma((x_1^{-1}x_0)^m x_1^{-1}u) \tag{5.3}$$

for all x in a neighborhood of e and all $u \in \mathfrak{R}$.

PROOF: For x in a suitable neighborhood of e we have $x = x_1(e + x_1^{-1}x_0)$ and

$$\sigma(x^{-1}u) = \sigma((e + x_1^{-1}x_0)^{-1}x_1^{-1}u) = \sum_{m=0}^{\infty} (-1)^m \sigma_m(x;u) \tag{1}$$

with

$$\sigma_m(x;u) = \sigma((x_1^{-1}x_0)^m x_1^{-1}u). \tag{2}$$

In addition, one easily verifies

$$\sigma_m(x;u) = \frac{(-1)^m}{m!} (\Delta_{x_1}^{x_0})^m \sigma(x_1^{-1}u). \tag{3}$$

After choosing k, $\sigma_m(x;u) = 0$ holds for $m \geq k$, so that with (1) and (3) it follows that

$$\sigma(x^{-1}u) = \sigma(x_1^{-1}u) + \sum_{m=1}^{k-1} \frac{1}{m!} (\Delta_{x_1}^{x_0})^m \sigma(x_1^{-1}u). \tag{4}$$

With (2), this proves (5.3).

To prove (5.2), apply Δ_x^u to the difference of both sides of (5.2) and obtain, using Lemma 1.1 (f) and (2), (3) and (4) above,

$$\sigma(x^{-1}u) - \sigma(x_1^{-1}u_1) - \sum_{m=1}^{k-1}(-1)^m(\sigma_m(x;u_1) - \sigma_{m-1}(x;u_0))$$
$$= \sigma_0(x;u) + \sum_{m=1}^{k-1}(-1)^m\sigma_m(x;u) - \sigma_0(x;u_1) - \sum_{m=1}^{k-1}(-1)^m(\sigma_m(x;u_1) - \sigma_{m-1}(x;u_0))$$
$$= \sigma_0(x;u_0) + \sum_{m=1}^{k-1}(-1)^m\sigma_m(x;u_0) + \sum_{m=1}^{k-1}(-1)^m\sigma_{m-1}(x;u_0) = 0.$$

Thus (5.2) holds up to an additive constant α. But for $x = e$, $\alpha = 0$ follows. ◇

Corollary 1

(a) $\nu_\sigma(x) = \nu_\sigma(x_1)$ *for all* $x \in \mathfrak{R}$.

(b) $\nu_{\sigma_1}(x_1)$ *divides* $\nu_\sigma(x_1)$.

PROOF: Part (b) is clear by definition of ν_σ and ν_{σ_1}. To prove part (a), consider (5.3), (2) and (3), and observe that $\nu_\sigma(x)$ divides a power of $\nu_\sigma(x_1)$. This implies the claim. ◇

Corollary 2

(a) $\mathcal{D}(\mathfrak{R},\sigma) = (\mathcal{D}(\mathfrak{R},\sigma) \cap \mathfrak{R}_1) + \mathfrak{R}_0$.

(b) $\mathcal{G}(\mathfrak{R},\sigma) = (\mathcal{G}(\mathfrak{R},\sigma) \cap \mathfrak{R}_1) + \mathfrak{R}_0$.

4 With the notation of 1. and I, §2, we set

$$R(\mathfrak{R}) = R(\mathcal{G}(\mathfrak{R}), \Gamma(\mathfrak{R})).$$

For $\eta \in R(\mathfrak{R})$ it follows that $\eta(xy) = \alpha(R(y))\eta(x)$, where $x, y \in \mathcal{G}(\mathfrak{R})$, so that $\alpha(R(y)) = \eta(y)(\eta(e)^{-1}$ follows. The elements of $R(\mathfrak{R})/\mathbb{R}^+$ thus correpsond bijectively to the elements of $H(\Gamma(\mathfrak{R}))$, the space of continuous homomorphisms from $\Gamma(\mathfrak{R})$ to $\mathbb{R}^+$. Clearly, $\mathfrak{R} \in (\mathcal{G}(\mathfrak{R}), \eta, e)$ for all $\eta \in R(\mathfrak{R})$.

Lemma 5.4 *Let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_0$ be a Wedderburn decomposition of $\mathfrak{R}$ and $\eta \in R(\mathfrak{R})$. Then there exists $\xi \in R(\mathfrak{R}_1)$ and $\zeta \in R(\mathbb{R}e + \mathfrak{R}_0)$ with*

(a) $\eta(x_1 + x_0) = \xi(x_1)\zeta(e + x_1^{-1}x_0)$, $x_1 \in \mathscr{G}(\mathfrak{R})$, $x_0 \in \mathfrak{R}_0$,

(b) $\zeta(e + x_1x_0) = \zeta(e + x_0x_1)$, $x_1 \in \mathfrak{R}_1$, $x_0 \in \mathfrak{R}_0$.

PROOF: By (5.1), we have $\mathscr{G}(\mathbb{R}e + \mathfrak{R}_0) = \mathbb{R}^+e + \mathfrak{R}_0 \subset \mathscr{G}(\mathfrak{R})$. We can thus define $\zeta(\alpha e + x_0) = \eta(\alpha e + x_0)$ for all $\alpha \in \mathbb{R}^+$, $x_0 \in \mathfrak{R}_0$. From $\eta(x_1 + x_0) = \eta(x_1)\eta(e + x_1^{-1}x_0) = \eta(e + x_0x_1^{-1}\eta(x_1)$ and with $\xi(x_1) = \eta(x_1)$, the claim now follows. ◇

With Lemma 5.4, the analysis of $R(\mathfrak{R})$ can be reduced to the those of $R(\mathfrak{R}_1)$ and $R(\mathbb{R}e + \mathfrak{R}_0)$. The algebras of the form $\mathbb{R}e + \mathfrak{R}_0$ are almost nilpotent, and these were already considered in §1.6. By the corollary to Lemma 1.3, every such algebra is exact.

Lemma 5.5 *If $\mathfrak{R} = \mathfrak{R}_1 \oplus \ldots \oplus \mathfrak{R}_s$ is a direct sum of simple ideal $\mathfrak{R}_i$ and η_i is defined as the absolute value of the determinant of left-multiplication in $\mathfrak{R}_i$, then*

$$R(\mathfrak{R}) = \{\alpha\eta_1^{\kappa_1}\cdots\eta_s^{\kappa_s} \mid \alpha \in \mathbb{R}^+, \kappa_1, \ldots, \kappa_s \in \mathbb{R}\}.$$

PROOF: As $\mathscr{G}(\mathfrak{R}) = \mathscr{G}(\mathfrak{R}_1) + \ldots + \mathscr{G}(\mathfrak{R}_s)$, we see that every $\eta \in R(\mathfrak{R})$ can be written as a product of elements $\varrho_i \in \mathscr{G}(\mathfrak{R}_i)$. We may thus assume $\mathfrak{R}$ to be simple. In this case, $\mathfrak{R} = \mathbb{K}e + [\mathfrak{R}, \mathfrak{R}]$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Now we see that η'_e is uniquely determined by its values on $\mathbb{K}$, and we deduce from Theorem 2.1 that η is uniquely determined by its values on $\mathscr{G}(\mathfrak{R}) \cap \mathbb{K}$. Thus we may assume $\mathfrak{R} = \mathbb{R}$ or $\mathfrak{R} = \mathbb{C}$. Now R($\mathbb{K}$) is one-dimensional, so that the claim follows since $\eta_i \in R(\mathfrak{R})$. ◇

5 Now the exact associative algebras shall be determined in full generality. We need:

Lemma 5.6 *Let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_0$ be a Wedderburn decomposition of $\mathfrak{R}$ and let $\mathfrak{R}_1 = \mathfrak{R}^{(1)} \oplus \ldots \oplus \mathfrak{R}^{(s)}$ be the decomposition of $\mathfrak{R}_1$ into simple ideals. Further, write $x_1 = x^{(1)} + \ldots + x^{(s)}$, denote by $\sigma^{(i)}$ the restriction of some $\sigma \in K(\mathfrak{R})$ to $\mathfrak{R}^{(i)}$, and let η_i denote the absolute value of the determinant of left-multiplications in $\mathfrak{R}^{(i)}$. Then there exist polynomials ν_i on $\mathfrak{R}^{(i)}$ and $\alpha_i, \kappa_i \in \mathbb{R}$ with*

(a) $\nu_\sigma(x) = \nu_\sigma(x_1) = \prod_{i=1}^{s} \nu_i(x^{(i)})$,

(b) $\nu_i(x^{(i)} = \alpha_i \eta_i(x^{(i)})^{\kappa_i}$,

(c) $\mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}_1 = \bigoplus_{i=1}^{s} \mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}^{(i)}$,

(d) $\mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}^{(i)} = \mathscr{G}(\mathfrak{R}^{(i)}, \sigma^{(i)}) = \mathscr{G}(\mathfrak{R}^{(i)})$, *if* $\sigma^{(i)} \neq 0$.

PROOF: (a) The first equality follows from Corollary 1 to Lemma 5.3. To prove the second equality, deduce from Lemma 5.1 (b) that $\nu = \nu_\sigma$ is an element in $R(\mathfrak{R}_1)$. Now (a) and (b) follow from Lemma 5.5. Part (c) follows from (a). For $\sigma^{(i)} \neq 0$, the exact denominator μ_i of $\sigma^{(i)}(x^{(i)-1}u^{(i)})$ is a power of η_i according to parts (a) and (b) above applied to $\mathfrak{R}^{(i)}$. The exponent appearing here and κ_i are non-zero, as otherwise $\sigma^{(i)}$ would be zero. Thus the statements $\nu_i(x^{(i)}) \neq 0$, $\mu_i(x^{(i)}) \neq 0$ and $\eta_i(x^{(i)}) \neq 0$ are equivalent, which proves the claim. ◇

Remark 1 If we equip the vector space $\mathfrak{R} = \mathrm{Mat}(n, \mathbb{R}) \oplus \mathrm{Mat}(n, \mathbb{R})$ with the product $(A \oplus B)(\tilde{A} \oplus \tilde{B}) = A\tilde{A} \oplus (A\tilde{B} + B\tilde{A})$, then $\mathfrak{R}$ is an associative algebra with identity element. We easily see that the given decomposition of $\mathfrak{R}$ is a Wedderburn decomposition. Clearly, $\sigma(A \oplus B) = \mathrm{tr}(B)$ is an $\mathfrak{R}$-commutative linear form. We obtain $\mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}_1 \neq \mathscr{G}(\mathfrak{R}_1, \sigma_1) = \mathfrak{R}_1$.

Lemma 5.7 *Let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_0$ be a Wedderburn decomposition of $\mathfrak{R}$. For $\sigma \in K(\mathfrak{R})$ let σ_1 denote the restriction of σ to $\mathfrak{R}_1$. Then: σ is $\mathfrak{R}$-exact if and only if σ_1 is $\mathfrak{R}_1$-exact.*

PROOF: Suppose the restriction σ_1 is $\mathfrak{R}_1$-exact. Then $\varrho(x_1) = \exp(-\sigma(\log_{\mathfrak{R}_1}(x_1)))$ can be extended analytically to $\mathscr{G}(\mathfrak{R}_1, \sigma_1)$ (cf. Lemma 5.1 and Theorem 2.1). Now deduce $\mathscr{G}(\mathfrak{R}_1, \sigma_1) \supset \mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}_1$ from Corollary 1 of Lemma 5.3. Since the right-hand side of (5.2) is real-analytic on $\mathscr{G}(\mathfrak{R}, \sigma)$, σ is an $\mathfrak{R}$-exact linear form.

Now let σ be an $\mathfrak{R}$-exact linear form and let η be chosen as in Lemma 5.1 (c). Then the restriction of $\tilde{\eta}$ of η to $\mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}_1$ is defined and $\tilde{\eta}'_{x_1}(u_1) = \sigma_1(x_1^{-1}u_1)$ holds. Now it remains to show that $\tilde{\eta}$ can be extended to $\mathscr{G}(\mathfrak{R}_1, \sigma_1)$. For this, note that $\eta(R(y)x) = \frac{\eta(y)}{\eta(e)}\eta(x)$ holds for all $x \in \mathscr{G}(\mathfrak{R}, \sigma)$, $y \in \mathscr{G}(\mathfrak{R})$ by Lemma 5.1. Since $\mathscr{G}(\mathfrak{R}_1) = \bigoplus_{i=1}^{s} \mathscr{G}(\mathfrak{R}^{(i)}) \subset \mathscr{G}(\mathfrak{R})$, $\tilde{\eta}$ is thus the product of maps $\tilde{\eta}_i : \mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}^{(i)} \to \mathbb{R}^+$. By Lemma 5.6 (d) we only need to consider the case $\sigma^{(i)} = 0$. Using Theorem 2.1 we have $\tilde{\eta}_i(x^{(i)}) = \exp(-\sigma^{(i)}(\log_{\mathfrak{R}^{(i)}}(x^{(i)})))$ on a neighborhood of the identity element $e^{(i)}$ of $\mathfrak{R}^{(i)}$, so that $\tilde{\eta}_i = 1$ follows on a neighborhood of $e^{(i)}$. As η and hence $\tilde{\eta}_i$ are real-analytic by I, Theorem 3.3, it follows that $\tilde{\eta}_i = 1$ on $\mathscr{G}(\mathfrak{R}, \sigma) \cap \mathfrak{R}^{(i)}$, and $\tilde{\eta}_i$ can be extended to $\mathscr{G}(\mathfrak{R}^{(i)}, \sigma^{(i)}) = \mathfrak{R}^{(i)}$. ◇

Corollary 1 *$\mathfrak{R}$ is exact if and only if $\mathfrak{R}/\operatorname{Rad}\mathfrak{R}$ is exact.*

PROOF: As $\mathfrak{R}_1$ and $\mathfrak{R}/\operatorname{Rad}\mathfrak{R}$ are isomorphic, we only need to show that $\mathfrak{R}$ is exact if and only if $\mathfrak{R}_1$ is exact. So suppose $\mathfrak{R}$ is exact and $\sigma \in K(\mathfrak{R}_1)$. Then there exists $\tau \in K(\mathfrak{R})$ with $\tau_1 = \sigma$. By assumption, τ is exact and thus by Lemma 5.7 σ is also exact. Conversely, let $\mathfrak{R}_1$ be exact and $\sigma \in K(\mathfrak{R})$. Then σ_1 is exact by assumption and by Lemma 5.7, σ is also exact. ◇

Theorem 5.8 *An associative algebra $\mathfrak{R}$ with identity element is exact if and only if the simple summands of $\mathfrak{R}/\operatorname{Rad}\mathfrak{R}$ are central.*

PROOF: By the above corollary we may assume that $\mathfrak{R}$ is semisimple. If we write $\mathfrak{R}$ as a direct sum of simple ideals $\mathfrak{R}_i$, then we obtain a direct decomposition of $K(\mathfrak{R})$ into the sum of the $K(\mathfrak{R}_i)$. Thus $\mathfrak{R}$ is exact if and only if every summand is exact. The centrally-simple summands are exact by Lemma 5.2. If there is a non-central simple summand, then it is isomorphic to a full matrix algebra $\operatorname{Mat}(n, \mathbb{C})$, taken as an algebra over $\mathbb{R}$. Such an algebra is not exact, as the imaginary part of the trace of an element is commutative, but not exact. ◇

6 Finally, let us consider the algebra $\mathfrak{A}(\eta, e)$ as defined in I, §4.1. Let $\eta \in R(\mathfrak{R})$ be non-degenerate. Then, on the one hand, $\mathfrak{R} \in (\mathcal{G}(\mathfrak{R}), \eta, e)$, and on the other hand, $\sigma = \eta'_e$ is an $\mathfrak{R}$-commutative linear form by I, (3.2), and $\eta'_x(u) = \sigma(x^{-1}u)$ holds for $x \in \mathcal{G}(\mathfrak{R})$. We compute easily that

$$\eta'''_x(u, v, w) = \sigma(uv_vu, w).$$

In summary, we obtain:

Lemma 5.9 *If $\mathfrak{R}$ is an associative algebra with identity element e and $\eta \in R(\mathfrak{R})$ is non-degenerate, then the product of the algebra $\mathfrak{A}(\eta, e)$ is given by $u \cdot v = \frac{1}{2}(uv + vu)$ for $u, v \in \mathfrak{R}$. In particular, $\mathfrak{A}(\eta, e)$ is a special Jordan algebra with a non-degenerate linear form.*

References

[1] H. Braun, M. Koecher: *Jordan-Algebren*, Springer Berlin, Heidelberg, New York, 1966

[2] C. Chevalley: *Théorie des Groupes de Lie*, Hermann Paris, 1968

[3] J. Dorfmeister: *Eine Theorie der homogenen regulären Kegel*, Dissertation, Münster, 1974

[4] J. Dorfmeister: *Homogene Siegel-Gebiete*[2)], Habilitation thesis, Münster, 1978

[5] D.K. Harrison: *Commutative nonassociative algebras and cubic forms*, Journal of Algebra 32, 1974, 518-528

[6] S. Helgason: *Differential geometry and symmetric spaces*, Academic Press New York, 1962

[7] G. Hochschild: *The structure of Lie groups*, Holden-Day San Francisco, 1965

[8] M. Koecher: *Jordan algebras and their applications*, Lecture notes, University of Minnesota, 1962

[9] T.Y. Lam: *Algebraic theory of quadratic forms*, Benjamin New York, 1973

[10] B. Malgrange: *Ideals of differentiable functions*, Oxford University Press London, 1966

[11] G.D. Mostow: *Self-adjoint groups*, Annals of Mathematics 62, 1955, 44-55

[12] T. Ochiai: *A lemma on open convex cones*, Journal of the Faculty of Science, the University of Tokyo 12, 1960, 231-234

[13] O.S. Rothaus: *Domains of positivty*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 24, 1960, 189-236

[14] M. Sato, T. Kimura: *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Mathematical Journal 65, 1977, 1-155

[15] H. Shima: *On locally symmetric homogeneous domains of completely reducible linear Lie groups*, Mathematische Annalen 217, 1975, 93-95

[2)] *Translator's note:* An English version is published as *Homogeneous Siegel domains*, Nagoya Mathematical Journal 86, 1982.

[16] È.B. Vinberg: *The theory of convex homogeneous cones*, Translations of the Moscow Mathematical Society 12, 1963, 340-403

[17] H. Whitney: *Elementary structure of real algebraic varieties*, Annals of Mathematics 66, 1957, 545-556

Index

Original: *Relative Invarianten und nicht-assoziative Algebren*, Mathematische Annalen 228, 1977, 147-186

Translation by Wolfgang Globke, Version of March 27, 2018.