OCTAVES, EXCEPTIONAL GROUPS AND OCTAVE GEOMETRY

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INTRODUCTION

Besides the large classes of simple Lie groups[1] it is well-known that there exist the exceptional groups

$$\mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8,$$

which, in the way they were determined algebraically by E. Cartan [2], are not very impressive or descriptive. E. Cartan himself remarked later and without proof [4] that $\mathbf{G}_2$ is the group of automorphisms of the octaves (the Graves-Cayley numbers). The octaves in their relation to the peculiarities of $\mathbf{D}_4$ were also studied by Cartan [5]. From a more algebraic perspective the study of octaves was taken up again by N. Jacobson [10, 11] with a view towards Lie rings, where he relied on investigations of alternating fields by M. Zorn [15].

A recent major advancement was achieved by C. Chevalley and R.D. Schafer [6]. They discovered that $\mathbf{F}_4$ can be identified with the automorphism group of the so-called Jordan algebra $\mathfrak{J}$ consisting of the Hermitian 3×3-matrices X with octave coefficients and the ring product $X \circ Y = \frac{1}{2}(XY + YX)$. They further showed that these automorphisms (and thus $\mathbf{F}_4$ as their totality) can be charaterised by the property that they leave invariant the quadratic and cubic form

$$\chi(X \circ X) \qquad \text{and} \qquad \chi(X \circ X \circ X).$$

In a certain way everything necessary for this discovery is implicitely contained in E. Cartan [2], who mysteriously missed the cubic form, even though he is aware of an analogous form related to $\mathbf{E}_6$; but the relationship between $\mathbf{F}_4$ and $\mathbf{E}_6$ is also missing in Cartan's work. In Chevalley's and Schafer's work one also finds an interpretation of $\mathbf{E}_6$ in relation to the algebra $\mathfrak{J}$, but in which the relation to a cubic invariant is missing.

Here, we take up all these problems once more and give a more holistic impression of them. Except for some basics on Lie's theory of continuous groups and the Cartan-Weyl theory of semisimple groups [2, 14] we do not assume any previous knowledge and rely on the literature as little as possible. Also, we did not employ the theory of alternative fields (unknown to us while writing this article) or Albert's Jordan algebras.

We develop the theory of octaves from the very basics and give a simple proof of Hurwitz' Theorem which uncovers a remarkable relation between the division algebras and the projective geometries of dimensions -1, 0, 1, 2 in characteristic 2.

Afterwards we investigate $\mathbf{D}_4$ and $\mathbf{B}_3$ in their relationship to the octaves $\mathfrak{C}$, and confirm a wealth of properties, in particular the triality; some methods of proof are perhaps interesing in their own right.

The next subject is $\mathbf{G}_2$, arising as the automorphism group of $\mathfrak{C}$. (We confirm this assertion – as done analogously later – directly by giving the root system.)

[1] Often we denote a group and one of its representations by the same letter if it is clear from the context which representation is meant.

During the investigation of the algebra $\mathfrak{J}$ (see above) we noticed that the infinitesimal automorphisms of $\mathfrak{J}$ are generated by the operations

$$X \mapsto \tilde{A}X = [A, X]$$

where A is skew-symmetric, $\chi(A) = 0$. These operations, together with the automorphisms of $\mathfrak{C}$, form a basis of the automorphism group of $\mathfrak{J}$. The reader should look up further results in this work, like the coincidence of this group with $\mathbf{F}_4$, by himself.

We want to mention the Principal Axis Theorem for $\mathfrak{J}$: The elements of $\mathfrak{J}$ can be diagonalised by the transformations of the group $\mathbf{F}_4$.

We were able the characterise $\mathbf{E}_6$ as the group of linear maps from $\mathfrak{J}$ to itself leaving $\det(X)$ invariant; the Chevalley-Schafer characterisation follows from this.

$\mathfrak{J}$, $\mathbf{F}_4$ and $\mathbf{E}_6$ are related to the projective octave plane. That the trivial incidence axioms of projective geometry must be satisfied by an octave geometry is plausible since the work of R. Moufang [12, 13]; she specified an affine octave geometry. G. Hirsch [9] constructed a plane projective geometry with 8-spheres for straight lines by topological means. We have now completed the same task algebraically.

Points and lines in our geometry are given by irreducible idempotents in the algebra $\mathfrak{J}$, and the incidence relation is $X \circ Y = 0$. The automorphism group of this geometry is $\mathbf{E}_6$. The subgroup $\mathbf{F}_4$ is the one of octave geometry. The lines are (as point sets) 8-spheres: when restricted to a line, the automorphism group of this geometry reduces to $\mathbf{D}_5$, the projective group of the 8-sphere; when restricted to elliptic geometries to $\mathbf{B}_4$, the rotation group of the 8-sphere. The validity of the theorem on the quadriliteral is obtained easily in octave geometry; the Desargue Theorem is known not to hold here.

As we are mainly concerned with the relations to Lie groups we did not strive for the greatest algebraic generality and thus only considered octaves with the real numbers as a base field. For some of our proofs and results this is indispensable.

At the end we list some open problems.

CONTENTS

Remark 1960. Before me, A. Borel used the term 'projective octave plane', but only in a topological sense (Comptes Rendus, Paris 230 (1950), 1378-1380) and P. Jordan has given a representation of the octave plane by idempotents of the ring $\mathfrak{J}$ without proof (Abh. Math. Sem. Hamburg 16 (1949), 74-76). Both works were unknown to me.

Notation

$\mathbf{B}_n = (2n+1)$-dimensional rotation group	
$\mathbf{D}_n = 2n$-dimensional rotation group	
$\mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ exceptional groups	
$\mathfrak{C} =$ algebra of octaves (real base field);	
(x, y) inner product; $e_0, \ldots, e_7$ basis elements	
$\mathfrak{U}$	1.6
G_{ij}, F_{ij}	2.1
$\kappa, K, \pi, \lambda, \pi_1$	2.2
triality	(2.3.6), (2.4.5), 7.10
$\mathfrak{M}_n, \mathfrak{M}_n^+, \mathfrak{M}_n^-$	4.1-4.4
$X \circ Y, (X, Y), (X, Y, Z)$	(4.5.9)-(4.5.11)
$\mathfrak{J}$ (Albert algebra)	(4.5.9)
$\tilde{A}, \mathfrak{R}$	(4.5.12)-(4.5.13)
δ	4.6
$E_{1,2,3}, F_i^a, \delta_i, A_p$	4.7
$\delta_{ij}, \mathfrak{M}_3$	4.9
principal matrix	5.1
characteristic equation	(5.2.4)
$\det(X)$	(5.2.5)
Π (system of irreducible idempotents of $\mathfrak{J}$)	5.3
$\mathcal{P}$ (plane projective geometry)	6.1
P, L	6.2
$\mathcal{P}(\mathfrak{C})$ (octave geometry)	7.1
$X \vee Y$	7.5
$\det(X, Y, Z)$	7.11
real line	7.12
$\Pi^\sharp$	7.12
prospective group	7.13
$\mathfrak{T}$	8.1
$\mathbf{A}$	9.3

1. $\mathfrak{C}$

1.1. The hypercomplex system $\mathfrak{C}$ has the following properties:

1.1.1. $\mathfrak{C}$ is a finite-dimensional linear space (elements: lower case Latin letters) with the real number field (elements: lower case Greek letters) as field of coefficients and endowed with a (positive definite) inner product (x, y), hence also with a vector length $|x| = \sqrt{(x, x)}$.

1.1.2. In $\mathfrak{C}$ a distributive, not neccessarily associative multiplication is defined,

$$x, y \in \mathfrak{C} \;\Rightarrow\; xy \in \mathfrak{C};$$
$$(a-b)(x-y) = ax - bx - ay + by;$$
$$a(\alpha x) = \alpha(ax);$$

1.1.3. with a unit element,

1.1.4. satisfying $|xy| = |x||y|$.

1.2. From 1.1.2 it follows: The left-multiplication

$$L_a x = ax$$

and the right-multiplication

$$R_a x = xa$$

are linear maps. From (1.1.4) it follows that they are isometries for $|a| = 1$, that is, orthogonal maps. Thus they leave invariant the inner products ($|a| = 1$):

$$\big(L_a(\lambda x + \mu y), L_a(\lambda x + \mu y)\big) = (\lambda x + \mu y, \lambda x + \mu y),$$

that is

$$\lambda^2(L_a x, L_a x) + 2\lambda\mu(L_a x, L_a y) + \mu^2(L_a y, L_a y) = \lambda^2(x, x) + 2\lambda\mu(x, y) + \mu^2(y, y),$$

and from this it follows by comparing coefficients that

$$(L_a x, L_a y) = (x, y) \quad \text{for } |a| = 1.$$

Similar for R_a.

The trick employed here when replacing x by $\lambda x + \mu y$ is called ‘polarisation’ in allusion to a practice from invariant theory. In the following we shall omit the explicit computation when using it.

We have shown:

$$(ax, ay) = (x, y) \quad \text{for } |a| = 1.$$

Hence

$$\left(\frac{a}{|a|}x, \frac{a}{|a|}y\right) = (x, y) \quad \text{for } a \neq 0$$

1.2.1. and moreover

$$(ax, ay) = (a, a)(x, y),$$

which also holds for $a = 0$. Also,

1.2.2.

$$(xa, ya) = (a, a)(x, y).$$

Polarisation of (1.2.1) with respect to a yields

1.2.3.

$$(ax, by) + (bx, ay) = 2(a, b)(x, y).$$

1.3. Let e_0 denote the unit element of $\mathfrak{C}$. The adjoint of a linear map A of $\mathfrak{C}$ is denoted by A', that is

$$(Ax, y) = (x, A'y).$$

It follows from (1.2.3) with $b = e_0$ and $(a, e_0) = 0$:

$$(ax, y) + (x, ay) = 0,$$

that is

$$L_a + L'_a = 0 \quad \text{for } (a, e_0) = 0.$$

Moreover,

$$L_{e_0} = L'_{e_0}.$$

If we define $\overline{a}$ to be a function of a such that

$$\overline{a} \text{ is linear in } a$$

and

$$\overline{a} = -a \text{ for } a = e_0,$$

then we have

$$L'_a = L_{\overline{a}},$$

that is

1.3.1.

$$(ax, y) = (x, \overline{a}y).$$

We also can take this equation as the definition of $\overline{a}$. Similarly we have

1.3.2.

$$(xa, y) = (x, y\overline{a}).$$

Applying (1.3.1) and (1.3.2) repeatedly yields:

$$(ax, y) = (x, \overline{a}y) = (x\overline{y}, \overline{a}) = (\overline{y}, \overline{x}\,\overline{a}),$$

from which it follows firstly that

1.3.3.

$$(x, y) = (\overline{x}, \overline{y})$$

and secondly

$$(\overline{ax}, \overline{y}) = (\overline{x}\,\overline{a}, \overline{y}),$$

that is

1.3.4.

$$\overline{xy} = \overline{y}\,\overline{x}.$$

We define

$$\mathrm{Re}(x) = \frac{1}{2}(x + \overline{x}), \quad \mathrm{Ve}(x) = \frac{1}{2}(x - \overline{x}),$$

that is,

$$\begin{aligned} \mathrm{Re}(x) &= (x, e_0)e_0 \\ x &= \mathrm{Re}(x) + \mathrm{Ve}(x) \\ \overline{x} &= \mathrm{Re}(x) - \mathrm{Ve}(x). \end{aligned}$$

Because of (1.3.1) with $x = e_0$ one has

$$(a, y)e_0 = (e_0, \overline{a}y)e_0 = \mathrm{Re}(\overline{a}y),$$

that is

$$(x, y)e_0 = \mathrm{Re}(\overline{x}y) = \mathrm{Re}(\overline{y}x)$$

and because of (1.3.1) to (1.3.4)

$$(x, y)e_0 = \mathrm{Re}(x\overline{y}) = \mathrm{Re}(y\overline{x}).$$

Hence

1.3.5.

$$(x, y)e_0 = \mathrm{Re}(\overline{x}y) = \mathrm{Re}(x\overline{y}) = \mathrm{Re}(\overline{y}x) = \mathrm{Re}(y\overline{x}).$$

From (1.3.1) follows $\mathrm{Re}((ax)\overline{y}) = \mathrm{Re}(x(\overline{y}a)) = \mathrm{Re}((\overline{y}a)x)$, that is

1.3.6.

$$\mathrm{Re}((ab)c) = \mathrm{Re}((bc)a) = \mathrm{Re}((c.a)b)$$

1.4. From (1.2.3), using (1.3.1), one deduces:

$$(\overline{b}(ax), y) + (\overline{a}(bx), y) = 2(a, b)(x, y).$$

This holds for all y. Hence

1.4.1.

$$\overline{b}(ax) + \overline{a}(bx) = 2(a, b)x = (\overline{b}a)x + (\overline{a}b)x.$$

For $a = b$ this is

1.4.2.

$$\overline{a}(ax) = (\overline{a}a)x.$$

As $\mathrm{Re}(a)(ax) = (\mathrm{Re}(a)a)x$ (because $\mathrm{Re}(a) = ae_0$), we also have

1.4.3.

$$a(ax) = a^2x.$$

By polarisation:

1.4.4.

$$a(bx) + b(ax) = (ab)x + (ba)x.$$

Analogously,

1.4.5.

$$(xa)\overline{b} + (xb)\overline{a} = 2(a, b)x,$$

1.4.6.

$$(xa)\overline{a} = x(a\overline{a}),$$

1.4.7.

$$(xa)a = xa^2,$$

1.4.8.

$$(xa)b + (xb)a = x(ab) + x(ba).$$

If we replace b by x and x by y in (1.4.4), and a by y and b by a in (1.4.8), we find:

1.4.9.

$$(ax)y + x(ya) = a(xy) + (xy)a.$$

Now it follows for $y = a$, considering (1.4.7), that:

1.4.10.

$$(ax)a = a(xa).$$

Polarisation:

1.4.11.

$$(ax)b + (bx)a = a(xb) + b(xa).$$

From (1.4.10) it also follows that

1.4.12.

$$(ax)\overline{a} = a(x\overline{a})$$

and by polarisation

1.4.13.

$$(ax)\overline{b} + (bx)\overline{a} = a(x\overline{b}) + b(x\overline{a}).$$

For (1.4.9) we can also write

1.4.14.
$$(L_a x)y = x(R_a y) = (L_a + R_a)(xy);$$
if we replace either x by $L_a x$ or y by $R_a y$ and add up the two equations, we obtain
$$(L_a^2 x)y + 2(L_a x)(R_a y) + x(R_a^2 y) = (L_a + R_a)((L_a x)y + x(R_a y))$$
and because of (1.4.10) this equals
$$(L_a + R_a)^2(xy).$$
With a^2 instead of a in (1.4.14) and considering
$$L_a^2 = L_{a^2}, \quad R_a^2 = R_{a^2}, \quad L_a R_a = R_a L_a$$
(because of (1.4.3), (1.4.7), (1.4.10)) we obtain
$$(L_a x)(R_a y) = L_a R_a(xy)$$
or

1.4.15.
$$(ax)(ya) = a(xy)a.$$
After introduction the ‘associator’
$$\{a, b, c\} = (ab)c - a(bc),$$
we can rewrite (1.4.4), (1.4.9), (1.4.11) as

1.4.16.
$$\{a, b, c\} = \{b, c, a\} = -\{b, a, c\}.$$
This formula is called the ‘alternative law’.

1.5. Let $\{e_0, e_1, \dots, e_{r-1}\}$ be an orthonormal system in $\mathfrak{C}$,
$$\begin{aligned}(e_i, e_j) &= 0 \quad \text{for } i \neq j,\\ &e_0 \quad \text{for } i = j.\end{aligned}$$
By (1.4.1) we have

1.5.1.
$$e_i(e_j x) + e_j(e_i x) = 0 \quad \text{for } i \neq j;\ i, j > 0$$

1.5.2.
$$e_i(e_i x) = -x \quad \text{for } i > 0.$$
In particular, this implies:

1.5.3.
$$e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j;\ i, j > 0,$$

1.5.4.
$$e_i^2 = -e_0 \quad \text{for } i > 0,$$
$$e_i(e_j e_k) = -e_j(e_i e_k) = e_j(e_k e_i) \quad \text{for } i \neq j, i \neq k;\ i, j, k > 0,$$
so

1.5.5.
$$e_i(e_j e_k) = e_j(e_k e_i) = e_k(e_i e_j) \quad \text{for } i \neq j \neq k;\ i, j, k > 0.$$
Moreover:

1.5.6. From $e_i e_j = e_k e_l$ $(i \neq j \neq k \neq l \neq i,\ j \neq l;\ i, j, k, l > 0)$ it follows that

$$e_i e_l = e_j e_k,$$

because the assumption implies $e_j = -e_i(e_k e_l) = e_k(e_i e_l)$.

1.5.7. We can change the basis in such a way that for all i, j it further holds that

$$e_i e_j = \pm e_{k_{i,j}}.$$

Assume this has already been done for $i, j < 2^p$, that is, $e_i e_j = \pm e_k$ for all $i, j < 2^p$ and a suited $k < 2^p$. Then choose f such that

$$\begin{aligned} (f, e_i) &= 0 \quad \text{for } i < 2^p, \\ (f, f) &= e_0 \end{aligned}$$

and let $e_{2^p+i} = fe_i$. Then, because of (1.3.2),

$$(fe_i, e_j) = -(f, e_j e_i) = 0 \quad \text{for } i, j < 2^p, i \neq j,$$

and because of (1.2.1)

$$\begin{aligned} (fe_i, fe_j) &= (e_i, e_j) \text{ for } i, j < 2^p \\ &= \begin{cases} 0 & \text{for } i \neq j, \\ -e_0 & \text{for } i, j < 2^p. \end{cases} \end{aligned}$$

Hence the e_i and fe_i form an orthonormal system and we can apply (1.5.3) to (1.5.5).

$$\begin{aligned} (fe_i)e_j &= (e_i e_j)f = \pm f(e_j e_i) \\ (fe_i)(fe_j) &= (e_i(fe_j))f = -f(f(e_j e_i)) = e_j e_i. \end{aligned}$$

Hence the e_i $(i < 2^{p+1})$ also satisfy this condition.

1.5.8. We now denote the set of the e_i $(i > 0)$ by $\mathfrak{E}$, the elements of $\mathfrak{E}$ are called 'points' and the cyclically ordered triplets $\{e_i, e_j, e_k\}$ are called 'oriented lines', more precisely 'positive' if

$$e_i(e_j e_k) = -e_0 \quad (\text{or } e_i = e_j e_k),$$

and 'negative' if

$$e_i(e_j e_k) = e_0 \quad (\text{or } e_i = -e_j e_k).$$

Two positive lines are said to have the same orientation, and so are two negative lines. A positive and a negative line are said to have opposite orientations. If we neglect the orientation, we simply speak of 'lines'. We use the terminology of projective geometry.

Now, every line contains precisely three points, and every two points are connected by a unique line. The 'plane axiom' holds as well, that is, if we call a line C a 'transversal' of the lines A, B if $C \cap A \neq C \cap B$, $C \cap A$ and $C \cap B$ not empty, then the following holds: If two lines intersect, then their transversals intersect.

Because if $A = \{e_i, e_j, e_k\}$, $B = \{e_i, e_j, e_m\}$, then $e_j e_k = \pm e_l e_m$, hence by 1.5.6, $e_j e_m = \pm e_k e_l$ etc.

The 'plane' AB of two intersecting lines A, B is understood to be the union of A, B and their transversals. We show:

1.5.9. For any two points e_i, e_j the third point of their connecting line is contained in AB.

We may assume that, say e_j, is contained neither in A nor in B. There exists a transversal D of A and B containing e_j; $D = \{e_j, e_k, e_l\}$, $e_k \in A$, $e_l \in B$, $e_i \notin D$. If $e_i \in A \cap B$ and $A = \{e_i, e_k, e_p\}$ then $\{e_i, e_j, \cdot\}$ and $\{e_p, e_l, \cdot\}$ are transversals of A and D, and hence the third point of $\{e_i, e_j, \cdot\}$ is in $\{e_p, e_l, \cdot\}$ and therefore in AB. If on the other hand $e_i \notin B$, $e_i \in A$, then $\{e_i, e_j, \cdot\}$ and B are transversals of A and D and one can proceed in an analogous manner. Finally, if $e_i \notin A \cup B$, and if C is a transversal of A and B containing e_i, then $\{e_i, e_j, \cdot\}$ and A are transversals of C and D, and again the result follows in by an analogous argument.

1.5.10. If the points e_i, e_j, e_k, e_l form a plane quadriliteral and if the sides $\{e_i, e_j, \cdot\}$ and $\{e_k, e_l, \cdot\}$ have the same orientation (that is $e_i e_j = e_k e_l$), then the sides $\{e_i, e_k, \cdot\}$ and $\{e_j, e_l, \cdot\}$ have opposite orientations (follows from 1.5.6).

1.5.11. For any triangle, say with sides $\{e_1, e_3, e_2\}$, $\{e_2, e_6, e_4\}$, $\{e_6, e_7, e_1\}$, there exists exactly one line, $\{e_4, e_3, e_7\}$, being a transversal to every pair of sides. By the positive orientation of the sides, the orientation of the transversal's orientation is determined. With the orientations given above taken to be the positive ones, the sides $\{e_1, e_3, \cdot\}$ and $\{e_6, e_4, \cdot\}$ of the quadriliteral e_1, e_3, e_6, e_4 have the same orientation. Hence $\{e_4, e_3, \cdot\}$ must be the positive transversal.

1.5.12. We now show that $\mathfrak{E}$ is a projective geometry of dimension at most 2.

Under the assumption that this is not the case, through one point we draw three lines $\{e_1, e_2, e_3\}, \{e_1, e_4, e_5\}, \{e_1, e_8, e_9\}$ with these orientations, not lying in one plane, and consider the perspective triangles e_2, e_4, e_8 and e_3, e_5, e_9 (see Figure 1). Corresponding

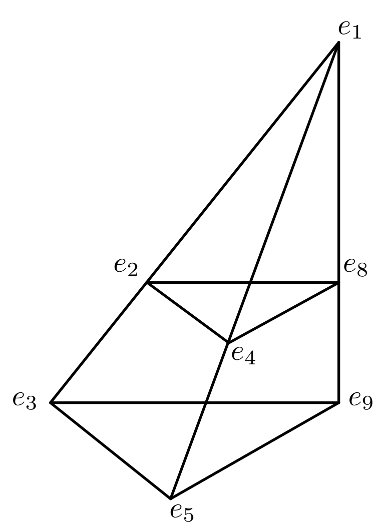

FIGURE 1.

sides have opposite orientation by 1.5.10. They intersect each other in a line which is a transversal for both triangles and obtains opposite orientations from the two triangles by 1.5.11. This is the contradiction we were looking for.

1.5.13. If $\dim \mathfrak{E} = -1$, that is $\mathfrak{E}$ is empty, then $\mathfrak{C}$ is essentially the system of real numbers. If $\dim \mathfrak{E} = 0$, that is, there exist no lines, then $\mathfrak{E}$ consists of one element e_1 with $e_1^2 = -e_0$, so $\mathfrak{C}$ is essentially the system of complex numbers. If $\dim \mathfrak{E} = 1$, that is, there exists exactly one line with points e_1, e_2, e_3, then one can assume $e_1 e_2 = e_3$, possibly after renormalising. So one obtains for $\mathfrak{C}$ the system of quaternions.

If $\dim \mathfrak{E} = 2$, then $\mathfrak{E}$ is the plane projective geometry over the prime field of characteristic 2, and $\mathfrak{E}$ has the cardinality 7. We introduce projective coordinates; every element of $\mathfrak{E}$ is given by a triple $(\alpha_0, \alpha_1, \alpha_2)$ with $\alpha_\nu = 0, 1$ (not all 0). We also call this element e_i with $i = \alpha_0 + 2\alpha_1 + 4\alpha_2$. If $e_j = (\beta_0, \beta_1, \beta_2)$ is another element, then $e_i e_j = \pm e_k$ with $e_k = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \alpha_2 + \beta_2) (\bmod 2)$. After a possible renormalising of the points one can assume that the sides of the triangle $e_1 e_2 e_4$ have the positive orientations $\{e_1, e_3, e_2\}, \{e_2, e_6, e_4\}, \{e_4, e_5, e_1\}$. Then the orientation of the transversal is necessarily $\{e_3, e_6, e_5\}$. By renormalising e_7 one can obtain the orientation $\{e_1, e_7, e_6\}$, and this implies uniquely and free of contradiction $\{e_2, e_7, e_5\}, \{e_4, e_7, e_3\}$.

So for $\dim \mathfrak{E} = 2$ there exists essentually one system $\mathfrak{C}$, it is 8-dimensional and is called the (*Graves-Cayley*) *octaves*.

The multiplication table chosen here consists of (1.5.3)-(1.5.5) and Figure 2:

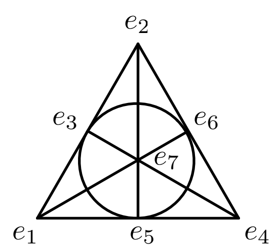

FIGURE 2.

$$\begin{array}{lll} e_1 e_3 = e_2, & e_2 e_6 = e_4, & e_4 e_5 = e_1, \\ & e_3 e_6 = e_5, & \\ e_1 e_7 = e_6, & e_2 e_7 = e_5, & e_4 e_7 = e_3. \end{array}$$

Thus we have proven (see the literature in [8]):

1.5.14. **Hurwitz Theorem.** *The only systems $\mathfrak{C}$ are those of the real numbers, the complex numbers, the quaternions and the octaves.*

In the following, the system $\mathfrak{C}$ will be understood to be the system of octaves.

1.6. If one picks the orthonormal basis $e_0, e_1, \dots, e_7$ arbitrarily (aside from the requirement, that e_0 is 1), then because of their commutator relations (1.5.1) the L_{e_i} generate a finite group $\mathfrak{U}$ represented by orthogonal transformations. This representation depends on the choice of basis, and as a finite group has only finitely many non-equivalent representations in a fixed dimension, the class of this representation is the same with respect to all bases which arise from one another by orthogonal transformations with positive determinant.

This represenation plays a role in certain proofs of the Hurwitz Theorem.

2. $\mathbf{D}_4$, $\mathbf{B}_3$

2.1. As usual $\mathbf{D}_4$ denotes the group of rotations in 8-dimensional space, and $\mathbf{B}_3$ the group of rotations in 7-dimensional space. The associated infinitesmial rings are denoted by the same letters. As is well-known, they consist of the skew-symmetric linear maps in the respective dimensions. We consider $\mathbf{D}_4$ to be a transformation group in $\mathfrak{C}$ and $\mathbf{B}_3$ as its subgroup fixing e_0.

2.2. As a basis of $\mathbf{D}_4$ we can use the G_{ij} defined by

$$G_{ij}e_j = e_i, \quad G_{ij}e_i = -e_j, \quad G_{ij}e_k = 0\ (i, j, k \text{ distinct}).$$

Also,

$$G_{ij} + G_{ji} = 0.$$

Then the following commutator relations hold:

$$\begin{aligned} &[G_{ij}, G_{jk}] = G_{ik}, \\ &[G_{ij}, G_{kl}] = 0 \quad \text{(all indices distinct).} \end{aligned}$$

Define

$$\begin{aligned} F_{i0}x &= \frac{1}{2}e_i x, \quad F_{0i} = -F_{i0}\ (i \neq 0), \\ F_{ij}x &= \frac{1}{2}e_j(e_i x)\ (i \neq j, i \neq 0, j \neq 0), \end{aligned}$$

then again

$$F_{ij} + F_{ji} = 0$$

holds, and moreover, because of the commutator relations (1.5.1):

$$\begin{aligned} [F_{i0}, F_{0j}]x &= \frac{1}{4}e_j(e_i x) - \frac{1}{4}e_i(e_j x) = \frac{1}{2}e_j(e_i x) \\ &= F_{ij}x, \\ [F_{ij}, F_{kl}]x &= -\frac{1}{4}e_l(e_k(e_j(e_i x))) + \frac{1}{4}e_j(e_i(e_l(e_k x))) \\ &= \ldots = 0, \\ [F_{ij}, F_{jk}]x &= \frac{1}{4}e_j(e_i(e_k(e_j x))) - \frac{1}{4}e_k(e_j(e_j(e_i x))) \\ &= \ldots = \frac{1}{2}e_k(e_i x) = F_{ik}x. \end{aligned}$$

The relation

$$G_{ij} \mapsto F_{ij}$$

generates an automorphism which we call π,

$$\pi G_{ij} = F_{ij}.$$

It must be an outer automorphism, for the G_{ij} with determinant 0 are mapped to the F_{ij} with determinant 1.

If we further set

$$Kx = \overline{x},$$

then for $A \in \mathbf{D}_4$

$$A \mapsto KAK$$

is also an automorphism, which we call κ,

$$\kappa A = KAK.$$

From

$$\kappa G_{i0} = -G_{i0}, \quad \kappa G_{ij} = G_{ij}$$

it follows that κ acts with determinant -1 on the space $\mathbf{D}_4$, that is, κ is an outer automorphism.

A closer inspection of 1.5.14 yields:

2.2.1.

$$\begin{aligned} 2F_{70} &= +G_{70} - G_{61} - G_{52} - G_{34}, \\ 2F_{61} &= -G_{70} + G_{61} - G_{52} - G_{34}, \\ 2F_{52} &= -G_{70} - G_{61} + G_{52} - G_{34}, \\ 2F_{34} &= -G_{70} - G_{61} - G_{52} + G_{34}. \end{aligned}$$

So π maps the commutative subring $\mathbf{H}$ generated by $G_{70}, G_{61}, G_{52}, G_{34}$ to itself, and the matrix of this map is

2.2.2.

$$\pi \text{ in } \mathbf{H}: \quad \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

The matrix of κ is

2.2.3.

$$\kappa \text{ in } \mathbf{H}: \quad \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

π and κ generate a group $\mathfrak{S}$:

2.2.4.

$$\pi\kappa = \lambda = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$\lambda^2 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

so

2.2.5.

$$\kappa^2 = 1, \quad \lambda^3 = 1, \quad \lambda\kappa = \kappa\lambda^2 \quad (\text{in } \mathbf{H})$$

and from this it follows that $\mathfrak{S}$ is isomorphic to the symmetric group in three symbols ($\kappa \mapsto (1\ 2)$, $\lambda \mapsto (1\ 2\ 3)$).

2.2.6. The same conclusions can be drawn for analogous subgroups containing G_{i0} ($i = 1, \ldots, 6$), and thus the constraint ‘in $\mathbf{H}$’ in (2.2.5) can be dropped.

2.2.7. The roots for the subring **H** of the

$$\alpha_0 G_{70} + \alpha_1 G_{61} + \alpha_2 G_{52} + \alpha_3 G_{34}$$

are

$$(\pm\alpha_\mu \pm \alpha_\nu)\mathrm{i} \quad \text{(all sign combinations)};$$

for example, one checks the corresponding roots

$$\begin{array}{rll} (\alpha_0 + \alpha_1)\mathrm{i} & \text{corresponds to} & +G_{76} + G_{10} + \mathrm{i}G_{71} + \mathrm{i}G_{06} \\ (-\alpha_0 + \alpha_1)\mathrm{i} & \text{corresponds to} & -G_{76} + G_{10} - \mathrm{i}G_{71} + \mathrm{i}G_{06} \\ (\alpha_0 - \alpha_1)\mathrm{i} & \text{corresponds to} & +G_{76} - G_{10} - \mathrm{i}G_{71} + \mathrm{i}G_{06} \\ (-\alpha_0 - \alpha_1)\mathrm{i} & \text{corresponds to} & -G_{76} - G_{10} + \mathrm{i}G_{71} + \mathrm{i}G_{06} \end{array}$$

and these are all complex, of course.

The transformations of $\mathfrak{S}$ leave invariant all the elements of **H** with $\alpha_1 + \alpha_2 + \alpha_3 = 0$, in particular the roots $\pm(\alpha_1 - \alpha_2)\mathrm{i}$, $\pm(\alpha_2 - \alpha_3)\mathrm{i}$, $\pm(\alpha_3 - \alpha_1)\mathrm{i}$. The others are permuted by λ as follows:

$$\lambda: \ (\alpha_\nu + \alpha_0)\mathrm{i} \mapsto (\alpha_\nu - \alpha_0)\mathrm{i} \mapsto -(\alpha_\mu + \alpha_\rho)\mathrm{i} \mapsto (\alpha_\nu + \alpha_0)\mathrm{i}.$$

κ permutes $\pm\alpha_0$ and permutes the roots accordingly.

2.3. The same (and more) is obtained independently in the following way:

In addition to

$$L_a x = ax \quad \text{and} \quad R_a x = xa$$

define

$$T_a = L_a + R_a.$$

One can write (1.4.9) as

2.3.1.

$$L_a x \cdot y + x \cdot R_a y = T_a(xy).$$

For $a = e_i$ $(i \neq 0)$

$$L_a x = e_i x - 2F_{i0} x$$

and

$$T_a x = e_i x + x e_i = \begin{cases} 2e_i & \text{for } x = e_0 \\ -2e_0 & \text{for } x = e_i \\ 0 & \text{for } x = e_j,\ j \neq i \end{cases},$$

so $L_a = 2F_{i0}$, $T_a = 2G_{i0}$. But then it also holds generally that

2.3.2.

$$\pi T_a = L_a \quad \text{for } \mathrm{Re}(a) = 0.$$

2.3.3. Moreover,

$$\kappa L_a = -R_a \quad \text{for } \mathrm{Re}(a) = 0.$$

Also,

$$L_a \mapsto R_a \quad \text{for } \mathrm{Re}(a) = 0$$

generates an automorphism of $\mathbf{D}_4$, which we call π_1 here, but will later see to be identical to λ,

2.3.4.

$$\pi_1 L_a = R_a \quad \text{for } \mathrm{Re}(a) = 0.$$

we can now write (2.3.1) as:

(2.3.1.a) $$L_a x \cdot y + x \cdot (\pi_1 L_a) y = \pi^{-1} L_a(xy).$$

Consider for a moment the direct sum $\mathfrak{C} + \mathfrak{C} + \mathfrak{C}$ of three copies of $\mathfrak{C}$ and within it the direct sum Q_a of $L_a, \pi_1 L_a, \pi^{-1} L_a$, that is

$$Q_a(x, y, z) = (L_a x, \pi_1 L_a y, \pi^{-1} L_a z),$$

then (2.3.1.a) says that

2.3.5.

$$xy - z = 0$$

is invariant with respect to the infinitesimal transformation Q_a. As the L_a ($\mathrm{Re}(a) = 0$) generate all of $\mathbf{D}_4$, a representation of $\mathbf{D}_4$ is given by $L_a \mapsto Q_a$, which also leaves invariant (2.3.5).

2.3.6. So for all $A \in \mathbf{D}_4$:

$$Ax \cdot y + x \cdot (\pi_1 A) y = (\pi^{-1} A)(xy).$$

Thus we have the
Infinitesimal principle of triality of $\mathbf{D}_4$. *For every $A \in \mathbf{D}_4$ there exist precisely one B and precisely on C in $\mathbf{D}_4$ such that*

$$Ax \cdot y + x \cdot BY = C(xy)$$

holds.

The ‘precisely’ remains to be proven: If

$$x \cdot B_1 y = C_1(xy), \quad B_1, C_1 \in \mathbf{D}_4,$$

then for x respectively $y = e_0$ with $Be_0 = a$ one has

$$B_1 y = C_1 y \quad \text{and} \quad C_1 x = xa,$$

hence

$$x(ya) = (xy)a,$$

which is only possible for $\mathrm{Ve}(a) = 0$ in x, y, whereas the skew-symmetry of C_1 requires that $\mathrm{Re}(a) = 0$. Thus $B_1 = C_1 = 0$, from which the claim follows.

Remark: When formulating the principle of triality, one can just as well start with B or C instead of A.

We can apply (1.4.4) with x, a, y instead of a, b, x and write

$$x \cdot L_a y + L_a(xy) = T_a x \cdot y,$$

hence

$$T_a x \cdot y - x \cdot L_a y = L_a(xy).$$

From (2.3.6) it follows now

$$-L_a = \pi_1 T_a, \quad L_a = \pi^{-1} T_a.$$

Together with (2.3.2), (2.3.3) and (2.3.4) and because of $T_a = L_a + R_a$, this yields for $\mathrm{Re}(a) = 0$:

$$\begin{aligned} \pi: &\quad L_a \mapsto T_a \mapsto L_a, \quad R_a \mapsto -R_a \mapsto R_a; \\ \pi_1: &\quad L_a \mapsto R_a \mapsto -T_a - L_a; \\ \kappa: &\quad L_a \mapsto -R_a \mapsto L_a, \quad T_a \mapsto -T_a \mapsto T_a. \end{aligned}$$

From this it follows that $\pi_1 = \pi\kappa$, that is, $\pi_1 = \lambda$.

The isomorphism of $\mathfrak{S}$ with $\mathfrak{S}_3$ has been established once more. In the plane determined by L_a and R_a in $\mathbf{D}_4$, the group $\mathfrak{S}$ is represented by

$$\lambda = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \kappa = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \pi = \lambda\kappa = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$\lambda^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \lambda^2\kappa = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

2.3.7. Instead of (2.3.6) we can now write:

$$Ax \cdot y + x \cdot \lambda A y = \pi A(xy).$$

Here, $\pi A = \kappa\lambda^2 A = K(\lambda^2 A)K$. If one forms the inner product of (2.3.7) with $\overline{z}$, then, because of (1.3.5)-(1.3.6) and the skew-symmetry of $\kappa\lambda^2 A$, one obtains:

$$\begin{aligned} \mathrm{Re}(Ax \cdot y \cdot z) + \mathrm{Re}(x \cdot \lambda A y \cdot z) &= (\pi A(xy), \overline{z}) \\ &= -(K(xy), \lambda^2 A K \overline{z}) = -(\overline{xy}, \lambda^2 A z), \end{aligned}$$

that is,

2.3.8.
$$\mathrm{Re}((Ax \cdot y)z + (x \cdot (\lambda A)y)z + (xy \cdot (\lambda^2 A)z)) = 0.$$

2.4. We can achieve the same thing by studying the group $\mathbf{D}_4$ itself rather than its infinitesimal ring. We continue from 1.6.

Let $\Phi \in \mathbf{B}_3$ be an orthogonal transformation of $\mathfrak{C}$ such that $\Phi e_0 = e_0$, $\det(\Phi) = 1$. Assuming the orthonormal basis $e_0, \ldots, e_7$, we find a new one: $\Phi e_0, \ldots, \Phi e_7$. The representations of $\mathfrak{U}$ generated by

$$e_i \mapsto L_{e_i} \quad \text{and} \quad e_i \mapsto L_{\Phi e_i}$$

are both orthogonal and equivalent to one another. So there exists a $\hat{\Phi}$ such that

2.4.1.
$$L_{\Phi e_i} = \hat{\Phi} L_{e_i} \hat{\Phi}^{-1}.$$

As the ring generated by the L_{e_i} is all of $\mathbf{D}_4$, these representations are even irrducible, and hence $\hat{\Phi}$ is determined up to a scalar factor, and as $\hat{\Phi}$ can be assumed to be real and orthogonal, this factor is ± 1.

From (2.4.1) it follows by taking linear combinations

2.4.2.
$$L_{\Phi x} = \hat{\Phi} L_x \hat{\Phi}^{-1}$$

and this implies that

2.4.3.
$$\Phi \mapsto \pm\hat{\Phi}$$
is a two-valued representation of $\mathbf{B}_3$ by an other subgroup of $\mathbf{D}_4$.

By differentiating $\Phi e_0 = e_0$ in $\mathbf{B}_3$ at $\Phi = 1$, one obtains $Ce_0 = 0$; by differentiating (2.4.2)-(2.4.3) at $\Phi = 1$, one obtains a homomorphism of the associated infinitesimal rings
$$C \mapsto \hat{C}$$
$$L_{Cx} \mapsto [\hat{C}, L_x].$$
In detail, the last equations reads

2.4.4.
$$Cx \cdot y + x \cdot \hat{C}y = \hat{C}(xy),$$
so for $Ce_0 = 0$:
$$\pi_1 C = \lambda C = \hat{C}.$$
Writing (2.4.2) in the form

2.4.5.
$$\Phi x \cdot \hat{\Phi} y = \hat{\Phi}(xy)$$
one obtains the 'finite' analogue of the infinitesimal formula (2.4.4).

This is a special case of the 'finite' version of the

Principle of triality in $\mathbf{D}_4$. *For very $\Theta \in \mathbf{D}_4$ there exists up to sign precisely one Θ_1 and precisely one Θ_2, such that*

2.4.6.
$$\Theta x \cdot \Theta_1 y = \Theta_2(xy)$$
holds. (Again, one can also start with Θ_1 or Θ_2.)

For a Θ with $\Theta e_0 = e_0$ we have already proved the existence; if Θ is of the form $\Theta = L_a$ ($|a| = 1$), then it follows from (1.4.15) that
$$L_a x \cdot R_a y = L_a R_a(xy),$$
that is, $\Theta_1 = R_a$, $\Theta_2 = L_a R_a$.

An arbitray $\Theta \in \mathbf{D}_4$ can be written as $\Theta = L_a \Phi$, where $a = \Theta e_0$, $\Phi \in \mathbf{B}_3$ and then
$$\Theta_1 = R_a \hat{\Phi}, \quad \Theta_2 = L_a R_a \hat{\Phi}.$$
As $\Theta \mapsto \Theta_1$ and $\Theta_2 \mapsto \Theta_2$ are (multi-valued) automorphisms, we only need to prove uniqueness (up to a factor ± 1) for the case $\Theta = 1$. One can rewrite (2.4.6) as $L_x \Theta_1 = \Theta_2 L_x$; as the L_x generate an irreducible group, $\Theta_1 = \Theta_2 = \gamma 1$, and as $\Theta_1, \Theta_2 \in \mathbf{D}_4$, we have $\gamma = \pm 1$.

2.4.7. It can be seen by infinitesimal methods that the automorphisms π, λ, etc. are globally 2-2-valued: If
$$H = \alpha_0 G_{70} + \alpha_1 G_{61} + \alpha_2 G_{52} + \alpha_3 G_{34},$$
and if we define
$$E_{ij} e_i = e_i, \quad E_{ij} e_j = e_j, \quad E_{ij} e_k = 0 \; (k \neq i, k \neq j),$$
then
$$H^2 = -\alpha_0^2 E_{70} - \alpha_1^2 E_{61} - \alpha_2^2 E_{52} - \alpha_3^2 E_{34},$$

that is,

$$\begin{aligned}\exp(H) = {} & \cos(\alpha_0)E_{70} + \cos(\alpha_1)E_{61} + \cos(\alpha_2)E_{52} + \cos(\alpha_3)E_{34} \\ & + \sin(\alpha_0)G_{70} + \sin(\alpha_1)G_{61} + \sin(\alpha_2)G_{52} + \sin(\alpha_3)G_{34}.\end{aligned}$$

So the lattice points $\alpha_\nu = m_\nu \cdot 2\pi$ are mapped to group's identity element by the exponential map. The automorphisms π, λ, etc. do not map this lattice to itself, but only the sublattice on which $\sum m_\nu \equiv 0 \bmod 2$ holds. A lattice point with $\sum m_\nu \equiv 1 \bmod 2$ is mapped by λ, π, etc. to a half-integer point satisfying $\exp(H) = -1$.

3. $\mathbf{G}_2$

Theorem 3.1. *The group of (continuous) automorphisms of $\mathfrak{C}$ is $\mathbf{G}_2$ (in Cartan's classification).*

This can be deduced from the above by counting parameters, but it is not hard to set up the root system:

3.2. An automorphism Φ of $\mathfrak{C}$ is neccessarily linear and satisfies

3.2.1.

$$\Phi(x) \cdot \Phi(y) = \Phi(xy).$$

$|\Phi(x)|$ is thus a new absolute value for the elements of $\mathfrak{C}$, which is also invariant under left-multiplication by a (with $|a| = 1$): as these generate an irreducible group, we have $|\Theta(x)| = \alpha|x|$ for some fixed α, and as

3.2.2.

$$\Theta(e_0) = e_0,$$

we have $\alpha = 1$. So $\Theta \in \mathbf{D}_4$.

With the principle of triality (2.4.6) it follows from (3.2.1) that $\Phi = \pm\Phi_1 = \pm\Phi_2$, and more precisely because of (3.2.2):

3.2.3.

$$\hat{\Phi} = \lambda\Phi = \pm\Phi.$$

Considering the infinitesimal ring, instead of (3.2.1)-(3.2.3) one has for an infinitesimal automorphism A:

3.2.4.

$$Ax \cdot y + x \cdot Ay = A(xy),$$

3.2.5.

$$Ae_0 = 0,$$

3.2.6.

$$\lambda A = A.$$

Writing an automorphism in the form

$$\sum \alpha_{ij} G_{ij} \quad (\alpha_{ij} + \alpha_{ji} = 0)$$

one obtains $\alpha_{i0} = 0$ from (3.2.5) and $\sum \alpha_{ij} F_{ij} e_0 = 0$ because of (3.2.6), hence

$$\sum \alpha_{ij} e_j e_i = 0.$$

This is satisfied by the expressions

$$\begin{aligned}
&\alpha G_{32} + \beta G_{45} + \gamma G_{76}\\
&\alpha G_{13} + \beta G_{64} + \gamma G_{75}\\
&\alpha G_{21} + \beta G_{65} + \gamma G_{47}\\
&\alpha G_{26} + \beta G_{51} + \gamma G_{73}\\
&\alpha G_{14} + \beta G_{36} + \gamma G_{27}\\
&\alpha G_{42} + \beta G_{53} + \gamma G_{17}\\
&\alpha G_{61} + \beta G_{52} + \gamma G_{34},
\end{aligned}$$

all with $\alpha + \beta + \gamma = 0$, and their linear combinations, and comprises the whole automorphism group. It was shown before (2.2.7) that these elements are invariant under all of $\mathfrak{S}$.

To compute the roots we choose a maximal abelian subalgebra, say

$$\alpha G_{32} + \beta G_{45} + \gamma G_{67} \quad (\text{with } \alpha + \beta + \gamma = 0).$$

To the roots

$$\begin{aligned}
\pm \mathrm{i}(\alpha + \beta) &\text{ belongs } (G_{43} + G_{53} - 2G_{17}) \pm \mathrm{i}(-2G_{16} + G_{52} + G_{34}),\\
\pm \mathrm{i}(\alpha - \beta) &\text{ belongs } (G_{42} - G_{53}) \mp \mathrm{i}(G_{52} - G_{34}),\\
&\text{etc.}
\end{aligned}$$

so that (with $\alpha + \beta + \gamma = 0$) the root system is formed by

$$\begin{array}{ccc}
\pm\, \mathrm{i}(\alpha + \beta), & \pm\, \mathrm{i}(\alpha - \beta), & \pm \mathrm{i}(\beta + \gamma),\\
\pm\, \mathrm{i}(\gamma - \alpha), & \pm\, \mathrm{i}(\gamma + \alpha), & \pm(\beta - \gamma),
\end{array}$$

and this is precisely the root system of $\mathbf{G}_2$.

4. $\mathfrak{J}$, $\mathbf{F}_4$

4.1. Let $\mathfrak{M}_n$ denote the ring of n-by-n matrices with coefficients in $\mathfrak{C}$.
$A^* = \overline{A}'$ is the conjugate-tranpose of A,
$\mathfrak{M}_n^+$ is the set of $A \in \mathfrak{M}_n$ such that $A = A^*$,
$\mathfrak{M}_n^-$ is the set of $A \in \mathfrak{M}_n$ such that $A + A^* = 0$.
The trace of A is denoted by $\chi(A)$.

4.2. Let $X = (x_{ij}) \in \mathfrak{M}_n$, $Y = (y_{ij}) \in \mathfrak{M}_n$. Then

$$\chi(XY) = \sum x_{ij} y_{ji} \quad \text{and} \quad \chi(X'Y') = \sum x_{ji} y_{ij},$$

that is,

4.2.1.
$$\chi(XY) = \chi(X'Y').$$
Moreover, because of (1.3.4) we have
$$\overline{XY} = \sum_j \overline{x_{ij} y_{jk}} = \sum_j \overline{y}_{jk}\overline{x}_{ij} = (\overline{Y}'\overline{X}')'$$
that is,

4.2.2.
$$(XY)^* = Y^* X^*.$$

4.3. We define an inner product in $\mathfrak{M}_n$:
$$(X, Y)e_0 = \mathrm{Re}(\chi(XY));$$
it is linear in X and Y and also symmetric, because by (1.3.5):
$$\mathrm{Re}(\chi(XY)) = \mathrm{Re}\big(\sum x_{ij} y_{ji}\big) = \sum \mathrm{Re}(x_{ij} y_{ji}) = \sum \mathrm{Re}(y_{ji} x_{ij}),$$

4.3.1. so
$$(X, Y) = (Y, X).$$
Moreover,

4.3.2.
$$(X, X^*) = \sum |x_{ij}|^2 > 0 \quad \text{for } X \neq 0,$$
so the inner product is not degenerate.

Because of (4.2.2), $\mathrm{Re}(\chi(XY)) = \frac{1}{2}(\chi(XY) + \overline{\chi(XY)}) = \chi(\frac{1}{2}(XY + Y^*X^*))$, so

4.3.3.
$$(X, Y)e_0 = \chi\left(\frac{1}{2}(XY + Y^*X^*)\right).$$

4.3.4. Thus
$$\begin{aligned}(X, Y)e_0 &= \chi\left(\frac{1}{2}(XY + YX)\right) && \text{for } X, Y \in \mathfrak{M}_n^+ \text{ and } X, Y \in \mathfrak{M}_n^-,\\ &= \chi\left(\frac{1}{2}(XY - YX)\right) && \text{for } X \in \mathfrak{M}_n^+, Y \in \mathfrak{M}_n^-,\end{aligned}$$
and hence, because of (4.3.1):

4.3.5.
$$(X, Y) = 0 \quad \text{for } X \in \mathfrak{M}_n^+, Y \in \mathfrak{M}_n^-.$$

4.3.6. This implies that (X, Y) is not degenerate on $\mathfrak{M}_n^+$ and $\mathfrak{M}_n^-$.

4.3.7. Moreover:
(X, X) is positive definite on $\mathfrak{M}_n^+$,
(X, X) is negative definite on $\mathfrak{M}_n^-$.

4.4. From (1.4.9) it follows that

$$\chi((AX)Y) + \chi(X(YA)) = \chi(A(XY)) + \chi((XY)A).$$

Going to the real parts yields because of (4.3.1):

$$(AX, Y) + (X, YA) = 2(A, XY).$$

Cyclic permutation yields

$$(XY, A) + (Y, AX) = 2(X, YA).$$

From these two equations and (4.3.1) it follows: $(A, XY) = (X, YA)$ or in other words

4.4.1.

$$(XY, Z) = (X, YZ) \quad \text{cyclically symmetric in } X, Y, Z.$$

One also has

$$(ZX, Y) = (X, YZ) \quad \text{and} \quad (XZ, Y) = (X, ZY),$$

thus if

$$[Z, X] = ZX - XZ$$

denotes the Lie commutator:

4.4.2.

$$([Z, X], Y) + (X, [Z, Y]) = 0.$$

As infinitesimal transformations, the $\tilde{Z}$ defined by

$$\tilde{Z}X = [Z, X] \quad \text{with } Z \in \mathfrak{M}_n$$

leave invariant the inner product.

4.4.3. Moreover,

$$[\mathfrak{M}_n^+, \mathfrak{M}_n^+] \subset \mathfrak{M}_n^-,$$
$$[\mathfrak{M}_n^-, \mathfrak{M}_n^-] \subset \mathfrak{M}_n^-,$$
$$[\mathfrak{M}_n^+, \mathfrak{M}_n^-] \subset \mathfrak{M}_n^+.$$

4.5. From now on we assume $n = 3$. Consider for $X \in \mathfrak{M}_3^+$

$$X(XX) - (XX)X.$$

The (i, l)-entry is

$$\sum_j \sum_k \big(x_{ij}(x_{ijk}x_{kl}) - (x_{ij}x_{jk})x_{kl}\big).$$

By 1.4 one can compute a product $a(bc)$ associatively if one element is real or two elements are identical or conjugate. As $x_{\mu\mu}$ is real and $x_{\mu\nu} = \overline{x}_{\nu\mu}$, if the expression is not to vanish, i, j, k have to be mutually distinct, and thus $l = i$ holds. So all elements not on the main diagonal vanish. As by (1.4.16) $(ab)c - a(bc)$ is cyclically symmetric, all diagonal entries are identical.[2]

4.5.6. So

$$X(XX) - (XX)X = a \cdot 1 \quad \text{for } X \in \mathfrak{M}_3.$$

$XX \in \mathfrak{M}_3^+$ and by (4.4.3), $a \cdot 1 \in \mathfrak{M}_3^-$, so

[2]*Translator's note:* In the original German article, the equation labelled 4.5.1 to 4.5.5 are missing, so that the first equation in section 4.5 has the number 4.5.6. For consistency, we keep the mislabelling of the German article.

4.5.7.

$$\mathrm{Re}(a) = 0.$$

If $\chi(A) = 0$ and $Z = a \cdot 1$, then $(A, Z) = \mathrm{Re}(\chi(AZ)) = 0$. So if $\chi(A) = 0$, then (by (4.4.1) and (4.5.6)), for $X \in \mathfrak{M}_3^+$:

$$(AX, XX) = (A, X(XX)) = (A, (XX)X) = (XA, XX),$$

hence

$$([A, X], XX) = 0.$$

Polarisation yields in $\mathfrak{M}_3^+$:

4.5.8.

$$([A, X], YZ + ZY) + ([A, Y], ZX + XZ) + ([A, Z], XY + YX) = 0.$$

We now define a commutative product in $\mathfrak{M}_3^+$:

4.5.9.

$$X \circ Y = \frac{1}{2}(XY + YX)$$

and denote $\mathfrak{M}_3^+$ endowed with this product by $\mathfrak{J}$. Then, by (4.3.4):

4.5.10.

$$(X, Y)e_0 = \chi(X \circ Y),$$

by (4.4.1):

$$(X \circ Y, Z) = (X, Y \circ Z),$$

so

4.5.11.

$$(X, Y, Z) = \chi(X \circ Y \circ Z) \quad \text{symmetric in } X, Y, Z.$$

By (4.4.3), $A \in \mathfrak{M}_3^-$, considerer as an infinitesimal transformation, leaves invariant $\mathfrak{M}_3^+$:

4.5.12.

$$[A, \mathfrak{M}_3^+] \subset \mathfrak{M}_3^- \quad \text{for } A \in \mathfrak{M}_3^-.$$

If we take $\mathfrak{N}$ to be the set of those $A \in \mathfrak{M}_3^-$ with $\chi(A) = 0$, then we further obtain from (4.4.2) and (4.5.8) that $\tilde{A}$, defined by $\tilde{A} = [A, X]$ ($X \in \mathfrak{J}$, $A \in \mathfrak{N}$), leaves invariant the bilinear and trilinear forms (X, Y) and (X, Y, Z) in $\mathfrak{J}$:

4.5.13.

$$([A, X], Y) + (X, [A, Y]) = 0,$$

4.5.14.

$$([A, X], Y, Z) + (X, [A, Y], Z) + (X, Y, [A, Z]) = 0.$$

4.6. Every linear mal δ from $\mathfrak{J}$ into itself, which, as an infinitesimal transformation, leaves invariant (X, Y) and (X, Y, Z), is an infinitesimal automorphism of $\mathfrak{J}$:

$$\begin{aligned}(\delta X \circ Y + X \circ \delta Y - \delta(X \circ Y), Z) &= (\delta X, Y, Z) + (X, \delta Y, Z) - (\delta(X \circ Y), Z) \\ &= (\delta X, Y, Z) + (X, \delta Y, Z) + (X, Y, \delta Z) \\ &= 0\end{aligned}$$

for every $Z \in \mathfrak{J}$, that is,

$$\delta X \circ Y + X \circ \delta Y = \delta(X \circ Y).$$

In particular, the

$$\tilde{A} X = [A, X], \quad A \in \mathfrak{R},$$

generate an infinitesimal ring of automorphisms $\tilde{A}$ of $\mathfrak{J}$, which we shall call $\tilde{\mathfrak{R}}$. We will see that $\tilde{\mathfrak{R}}$ is in fact the whole automorphism ring of $\mathfrak{J}$ and identical to $\mathbf{F}_4$.

4.7. For the idempotents

$$E_1 = \begin{pmatrix} e_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_0 \end{pmatrix}$$

of $\tilde{\mathfrak{J}}$ we have

$$E_i E_j = \begin{cases} E_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}.$$

Thus for every infinitesimal automorphism δ of $\mathfrak{J}$ it holds that

$$\begin{aligned}&\delta E_i \circ (e \cdot 1 - 2E_i) = 0, \\ &\delta E_i \circ E_j + E_i \circ \delta E_j = 0 \quad \text{for } i \neq j.\end{aligned}$$

An easy calculation shows

$$\delta E_1 = \begin{pmatrix} 0 & -a_3 & -\overline{a}_2 \\ -\overline{a}_3 & 0 & 0 \\ -a_2 & 0 & 0 \end{pmatrix}, \quad \delta E_2 = \begin{pmatrix} 0 & a_3 & 0 \\ \overline{a}_3 & 0 & a_1 \\ 0 & \overline{a}_1 & 0 \end{pmatrix}, \quad \delta E_3 = \begin{pmatrix} 0 & 0 & \overline{a}_2 \\ 0 & 0 & -a_1 \\ a_2 & -\overline{a}_1 & 0 \end{pmatrix}.$$

If one sets

$$A = \begin{pmatrix} 0 & a_3 & \overline{a}_2 \\ -\overline{a}_3 & 0 & a_1 \\ -a_2 & -\overline{a}_1 & 0 \end{pmatrix}$$

then $A \in \mathfrak{R}$,

$$[A, E_i] = \delta E_i,$$

that is

$$(\delta - \tilde{A}) X = \delta X - [A, X] = 0 \quad \text{for } X = E_i.$$

If we want to show that $\tilde{\mathfrak{R}}$ comprises the infinitesimal automorphisms of $\mathfrak{J}$, then we can restrict to the δs satisfying

$$\delta X = 0 \quad \text{for } X = E_i.$$

Set

$$F_1^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \overline{a} & 0 \end{pmatrix}, \quad F_2^a = \begin{pmatrix} 0 & 0 & \overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad F_3^a = \begin{pmatrix} 0 & a & 0 \\ \overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$E_i \circ F_i^a = 0, \qquad \text{hence } E_i \circ \delta F_i^a = 0$$

$$E_j \circ F_i^a = \frac{1}{2} F_i^a \quad (i \neq j), \qquad \text{hence } (2E_j - 1) \circ \delta F_i^a = 0,$$

and therefore it holds for the set $\mathfrak{F}_i$ of the F_i^a:

$$\delta \mathfrak{F}_i \subset \mathfrak{F}_i.$$

We now define $\delta_i a$ by

$$\delta F_i^a = F_i^{\delta_i a},$$

such that $\delta_i a$ is linear in a. As

$$F_i^a \circ F_i^b = 0 \bmod E_1, E_2, E_3,$$

we obtain

$$F_i^{\delta_i a} \circ F_i^b + F_i^a \circ F_i^{\delta_i b} = 0,$$

that is,

$$\delta_i a \cdot \overline{b} + b \cdot \overline{\delta_i a} + a \cdot \overline{\delta_i b} + \delta_i b \cdot \overline{a} = 0,$$

hence

$$(\delta_i a, b) + (a, \delta_i b) = 0,$$

which means $\delta_i \in \mathbf{D}_4$. From

$$F_i^{2a} \circ F_{i+1}^{2b} = F_{i+2}^{2(\overline{ab})} \quad (\text{with } i \text{ cyclic})$$

one deduces:

$$F_i^{2\overline{\delta}_i a} \circ F_{i+1}^{2b} + F_i^{2a} \circ F_{i+1}^{2\delta_{i+1} b} = F_{i+2}^{2\delta_{i+2} \overline{ab}},$$

so

$$\overline{\delta_i a \cdot b} + \overline{a \cdot \delta_{i+1} b} = \delta_{i+2}(\overline{ab}),$$

and by the Triality Theorem this shows

$$\delta_{i+1} = \lambda \delta_i.$$

As $\delta_1 \in \mathbf{D}_4$, we can assume

$$\delta_1 a = pa \quad \text{or} \quad \delta_1 a = q(pa),\ \mathrm{Re}(p) = \mathrm{Re}(q) = (p, q) = 0.$$

In the first case, set

$$A_p = \begin{pmatrix} -p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

$$[A_p, F_1^a] = F_1^{pa}, \quad [A_p, F_2^a] = F_2^{ap}, \quad [A_p, F_3^a] = F_3^{-pa-ap}$$

and hence

$$\tilde{A}_p X = [A_p, X] = \delta X$$

holds for $X = F_i^a$, so it holds in general. In the second case, with the same notation, set

$$2\Delta X = [A_q, [A_p, X]] - [A_p, [A_q, X]],$$

and then we obtain

$$2\Delta F_1^a = F_1^{q(pa)} - F_1^{p(qa)} = 2F_1^{q(pa)} \quad \text{etc.}$$

so that $\Delta = \delta$.

We have thus proven:

4.8. *The infinitesimal ring of the automorphisms of $\mathfrak{J}$ is generated by the elements $\tilde{A}$ defined by*

$$\tilde{A}X = [A, X], \quad \text{where } A \in \mathfrak{R} \text{ (that is } A \in \mathfrak{M}_3^+,\ \chi(A) = 0).$$

The elements of the automorphism ring can be written uniquely in the form

4.8.1.

$$\tilde{A} = \Delta + \tilde{A}_0,$$

where $A_0 \in \mathfrak{R}$ has all 0 on the main diagonal and Δ is an infinitesimal automorphism leaving invariant the E_ν ($\nu = 1, 2, 3$). Conversely, all transformations (4.8.1) are infinitesimal automorphisms of $\mathfrak{J}$. They leave invariant the bilinear and trilinear forms (X, Y) and (X, Y, Z), and are determined by these properties. Those of them leaving invariant the E_ν ($\nu = 1, 2, 3$) transform every $\mathfrak{F}_i$ into itself and give rise to a representation of $\mathbf{D}_4$ in the $\mathfrak{F}_i$ and $\mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3$:

$$\delta F_i^a = F_i^{\delta_i a}, \quad \delta_{i+1} = \lambda \delta_i.$$

4.9. If X is a generic element of $\mathfrak{J}$,

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad (x_{ij} = \overline{x}_{ji})$$

and δ an infinitesimal automorphism leaving invariant the E_ν, then δX may be written as

4.9.1.

$$\delta X = \begin{pmatrix} \delta_{11}x_{11} & \delta_{12}x_{12} & \delta_{12}x_{13} \\ \delta_{21}x_{21} & \delta_{22}x_{22} & \delta_{23}x_{23} \\ \delta_{31}x_{31} & \delta_{32}x_{32} & \delta_{33}x_{33} \end{pmatrix}$$

if one defines

$$\delta_{12} = \delta_3, \quad \delta_{23} = \delta_1, \quad \delta_{31} = \delta_2$$

and $\delta_{ji}x = \overline{\delta_{ij}\overline{x}}$, that is,

$$\delta_{ij} = \kappa \delta_{ji} \quad \text{and} \quad \delta_{ii} = 0.$$

In this notation the principle of triality reads as follows:

4.9.2.

$$\delta_{ij}x \cdot y + x \cdot \delta_{jk}y = \delta_{ik}(xy) \quad (i, j, k \text{ distinct}).$$

δX is now reasonably defined by (4.9.1) for all $X \in \mathfrak{M}_3$. Set $X \in \mathfrak{M}_3^{\mathrm{r}}$ if all entries in the main diagonal of X are real numbers (that is, real multiples of e_0). With

$$A_p = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\mathrm{Re}(p) = 0)$$

we have

$$\tilde{A}_p X = [A_p, X] = \begin{pmatrix} 0 & -px_{12} - x_{12}p & -px_{13} \\ px_{12} + x_{21}p & 0 & px_{23} \\ x_{31}p & -x_{32}p & 0 \end{pmatrix},$$

so $\tilde{A}_p$ is a δ as in (4.9.1). Similarly one sees that all δ can be generated by the $\tilde{A}_p$.

For $X, Y \in \mathfrak{M}_3^{\mathrm{r}}$ define

$$X \circ Y = \frac{1}{2}(XY + Y^* X^*) \quad \left(= \frac{1}{2}(XY + YX) \text{ in } \mathfrak{J}\right),$$

then we have

4.9.3.
$$\delta X \circ Y + X \circ \delta Y = \delta(X \circ Y).$$

This is because
$$\delta_{ij} x_{ij} \cdot y_{jk} + x_{ij} \cdot \delta_{jk} y_{jk} = \delta_{ik}(x_{ij} y_{jk})$$
by (4.9.2) for distinct i, j, k, and trivially for $i = j \neq k$ and $i \neq j = k$, as x_{ii} is real and $\delta_{ii} = 0$, respectively y_{kk} is real and $\delta_{kk} = 0$. As a consequence,
$$Z = (\delta X)Y + X(\delta Y) - \delta(XY)$$
is a diagonal matrix with ith entry:
$$\sum_j (\delta_{ij} x_{ij} \cdot y_{ji} + x_{ij} \cdot \delta_{ji} y_{ji}).$$
Therefore, $\delta X \circ Y + X \circ \delta Y - \delta(X \circ Y) = \frac{1}{2}(Z + Z^*)$ is also a diagonal matrix with ith entry
$$\sum_j (\delta_{ij} x_{ij} \cdot y_{ji} + x_{ij} \cdot \delta_{ji} y_{ji}) = \sum_j (\delta_{ij} x_{ij}, \overline{y}_{ji}) + (x_{ij}, \overline{\delta_{ji} y_{ji}}) = 0,$$
because $\overline{\delta_{ji} y_{ji}} = \delta_{ij} \overline{y}_{ji}$ and the skew-symmetry of $\delta_{ij} \in \mathbf{D}_4$. So $\frac{1}{2}(Z + Z^*) = 0$, which means (4.9.3) holds.

For $X, Y \in \mathfrak{J}$, (4.9.3) is nothing new. For $X \in \mathfrak{J}$ and $Y = A \in \mathfrak{R}$ with 0 on the main diagonal one can also write (because $A + A^* = 0$)
$$[\delta A, X] + [A, \delta X] = \delta[A, X],$$
or
$$\delta \tilde{A} - \tilde{A}\delta = \widetilde{\delta A},$$
or

4.9.4.
$$[\delta, \tilde{A}] = \widetilde{\delta A}.$$

4.10. In order to determine the structure of the infinitesimal automorphism ring of $\mathfrak{J}$ more precisely, we choose as a maximal abelian subring $\mathbf{H}$ the same one as in 2.2 for $\mathbf{D}_4$; that is, we set
$$\delta_j = \alpha_0^{(j)} G_{70} + \alpha_1^{(j)} G_{61} + \alpha_2^{(j)} G_{52} + \alpha_3^{(j)} G_{34},$$
where the vector $\alpha^{(j+1)}$ is obtained from the vector $\alpha^{(j)}$ by application of the matrix λ (see (2.2.4)). As roots, we firstly find those already known from $\mathbf{D}_4$ (see (2.2.7),
$$(\pm\alpha_\mu \pm \alpha_\nu)\mathrm{i},$$
(where we omit the upper index because it does not matter which one we choose; the root system is invariant under λ) with eigenvectors corresponding to those given there. Because of (4.8.1), we will look for further eigenvectors amongst the A with 0 on the diagonal. In fact, the
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix} \text{ with } a = e_0 \mp \mathrm{i}e_7, e_1 \mp \mathrm{i}e_6, e_2 \mp \mathrm{i}e_5, e_4 \mp \mathrm{i}e_3$$

belong to the roots $\pm\alpha_0^{(1)}\mathrm{i}$, $\pm\alpha_1^{(1)}\mathrm{i}$, $\pm\alpha_2^{(1)}\mathrm{i}$, $\pm\alpha_3^{(1)}\mathrm{i}$. The

$$\begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \text{ with } a = e_0 \mp \mathrm{i}e_7, \text{ etc.}$$

belong to the roots $\pm\alpha_0^{(2)}\mathrm{i}$, etc. and the

$$\begin{pmatrix} 0 & a & 0 \\ -\overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } a = e_0 \mp \mathrm{i}e_7, \text{ etc.}$$

belong to the roots $\pm\alpha_0^{(3)}\mathrm{i}$, etc. So the totality of roots is found to be (see (2.2.4))

$$\pm\mathrm{i}\alpha_\nu, \quad \pm\mathrm{i}\alpha_\mu \pm \mathrm{i}\alpha_\nu, \quad \frac{1}{2}(\pm\mathrm{i}\alpha_0 \pm \mathrm{i}\alpha_1 \pm \mathrm{i}\alpha_2 \pm \mathrm{i}\alpha_3)$$

(where all combinations of signs $\pm$ appear). But this is precisely the root system of the exceptional group $\mathbf{F}_4$.

Now one easily confirms that the automorphism ring of $\mathfrak{J}$ is irreducible on the subspace with $\chi(X) = 0$. The unit component of the automorphism group is thus a direct product of simple groups and because the roots coincide it is just $\mathbf{F}_4$. (This can be easily seen by a direct calculation if one computes the structure further.) If there were an additional component $\omega\mathbf{F}_4$, then ω would generate an outer automorphism of $\mathbf{F}_4$ (but such an automorphism cannot exist; Cartan [5]), or $\omega = -1$ would hold because of the irreducibility on the subspace with $\chi(X) = 0$ (but this is not an automorphism of $\mathfrak{J}$). So the following holds:

Theorem 4.11. *The automorphism group of* $\mathfrak{J}$ *is* $\mathbf{F}_4$.

4.12. The automorphism group of $\mathfrak{J}$ leaves invariant $E_1 + E_2 + E_3 = 1$. The subgroup leaving invariant the E_ν ($\nu = 1, 2, 3$) is isomorphic to $\mathbf{D}_4$ (locally). The subgroup leaving invariant *one* of the E_ν (say E_1) is a 16-dimensional representation of $\mathbf{B}_4$ (the group of rotations in 9-dimensional space). For the invariance of E_1 implies A_0 being of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix},$$

so except for the roots of $\mathbf{D}_4$, only the $\pm\alpha_\nu^{(1)}\mathrm{i}$ are preserved. Together, these are just the roots of $\mathbf{B}_4$.

4.13. The infinitesimal automorphisms δ of $\mathfrak{J}$ leaving invariant all purely real elements of $\mathfrak{J}$ must satisfy $\delta_i e_0 = 0$ (see 4.8), and thus belong to $\mathbf{G}_2$ by 3.2.

With $A, B \in \mathfrak{R}$ and $X \in \mathfrak{J}$ one can compute associatively if X is purely real. Then the Jacobi relation holds,

$$[A, [B, X]] - [B, [A, X]] = [[A, B], X].$$

The map

$$[\tilde{A}, \tilde{B}] - \widetilde{[A, B]} + \frac{1}{3}\chi([A, B])$$

is an infinitesimal automorphism of $\mathfrak{J}$ which vanishes when applied to a real X, and is thus generated by an element of $\mathbf{G}_2$. Hence

$$\widetilde{[A, B]} \equiv [\tilde{A}, \tilde{B}] + \frac{1}{3}\chi([A, B]) \quad \text{mod automorphisms of } \mathfrak{C}.$$

Moreover, one observes that $\dim \mathbf{F}_4 = 52 = \dim \mathfrak{R} + \dim \mathbf{G}_2 = 38 + 14$.

5. Π

5.1. **Principal Axis Transform in $\mathfrak{J}$.** *The elements of $\mathfrak{J}$ can be transformed to diagonal form (principal matrix) by transformations in $\mathbf{F}_4$; the diagonal elements (eigenvalues) are then uniquely determined up to order, and characterise the equivalence class.*

A consequence of this theorem is: *If $X_1 \in \mathfrak{J}$ and inductively $X_{n+1} = X_1 \circ X_n$, then $X_i \circ X_j = X_{i+j}$.*

For $X_i \circ X_j = X_{i+j}$ holds for a principal element X_1 and as this relation is $\mathbf{F}_4$-invariant, it holds in general.

Proof. As a continuous automorphism group of $\mathfrak{J}$, the group $\mathbf{F}_4$ is closed in the general linear group, and as a rotation group it is compact. We call two elements of $\mathfrak{J}$ equivalent if they can be transformed into one another by transformations on $\mathbf{F}_4$. Then the equivalence classes are closed and compact. Among the

$$X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_1 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}$$

of an equivalence class we are looking for an element with $\xi_1^2 + \xi_2^2 + \xi_3^2$ maximal. We claim that this element X_0 is a principal matrix. Assume to the contrary say $x_1^{(0)} \neq 0$. Consider the curve X_τ through X_0 defined by the differential equation

5.1.1.

$$\frac{\mathrm{d}X_\tau}{\mathrm{d}\tau} = [A, X_\tau], \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}$$

which is contained in the equivalence class of X_0 because $A \in \mathbf{F}_4$. More precisely, the differential equation is

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}\xi_2}{\mathrm{d}\tau} = 2(a, x_1), \quad \frac{\mathrm{d}\xi_3}{\mathrm{d}\tau} = -2(a, x_1), \quad \frac{\mathrm{d}x_1}{\mathrm{d}\tau} = a(\xi_3 - \xi_2) \quad \text{etc.}$$

Now $\mathrm{d}(\xi_1^2 + \xi_2^2 + \xi_3^2) = 4(a, x_1)(\xi_2 - \xi_3)\mathrm{d}\tau$, and this expression has to vanish for all a at $\tau = 0$; thus (as $x_1 \neq 0$) $\xi_2 = \xi_3$ at $\tau = 0$. From the first three differential equations it follows along the curve: $\xi_1 = \text{const}$, $\xi_2 + \xi_3 = \text{const}$, and this means at $\tau = 0$ there is a minimum rather than a maximum, contradicting our assumption.

One treats the cases $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$ analogously. This shows that each equivalence class contains a principal matrix.

We further show that two principal matrices identical diagonal elements (up to order) are equivalent. We only need to show how a transformation in $\mathbf{F}_4$ exchanges entries in a principal matrix, say ξ_2 and ξ_3. Again, this is achieved by the curve defined by (5.1.1). The differential equation implies

$$\frac{\mathrm{d}^2(\xi_2 - \xi_3)}{\mathrm{d}\tau^2} = 4\Big(a, \frac{\mathrm{d}x_1}{\mathrm{d}\tau}\Big) = -4(a, a)(\xi_3 - \xi_2)$$

with the initial condition

$$\frac{\mathrm{d}(\xi_2 - \xi_3)}{\mathrm{d}\tau} = 0 \quad \text{for } \tau = 0.$$

The solution

$$\xi_2 - \xi_3 = (\xi_2^{(0)} - \xi_3^{(0)}) \cos(2|a|\tau)$$

becomes for $\tau = \frac{\pi}{2}|a|$:

$$\xi_2 - \xi_3 = -(\xi_2^{(0)} - \xi_3^{(0)}),$$

and on the other hand $\xi_2 + \xi_3$ is constant along the curve. For $\tau = \frac{\pi}{2}|a|$ the original ξ_2, ξ_3 have indeed be exchanged.

The uniqueness of the main diagonal elements (eigenvalues) follows readily.

5.2. Instead of $X \circ X$ and $X \circ X \circ X$ we will also write X^2 and X^3, respectively. Invariants under $\mathbf{F}_4$ are $e_0 \cdot 1$, (X, X) and (X, X, X), and so are $\chi(X) = (X, e_0 \cdot 1)$, $\chi(X^2)$ and $\chi(X^3)$.

5.2.1. Thus

$$\chi(X) = \sum \rho_\nu^2, \quad \chi(X^2) = \sum \rho_\nu^2, \quad \chi(X^3) = \sum \rho_\nu^3,$$

if ρ_1, ρ_2, ρ_3 are diagonal elements of a principal matrix equivalent to X. By (5.2.1) the ρ_ν are determined up to order, and from this follows the rest of Theorem 5.1.

A principal matrix X sataisfies its characteristic equation $\varphi(\rho) = \prod(\rho_\nu - \rho) = 0$. As $\mathbf{F}_4$ is the automorphism group of $\mathfrak{J}$, the relation $\varphi(X) = 0$ is invariant under $\mathbf{F}_4$, that is, every $X \in \mathfrak{J}$ satisfies a characteristic equation. More precisely:

$$\varphi(\rho) = -\rho^3 + (\sum \rho_\nu)\rho^2 - (\sum \rho_\nu \rho_\mu)\rho + (\rho_1 \rho_2 \rho_3);$$

here, $\sum \rho_\nu = \chi(X)$, $\sum \rho_\nu \rho_\mu = \frac{1}{2}(\chi(X)^2 - \chi(X^2))$; the product $\rho_1 \rho_2 \rho_3$ shall be denoted by $\det(X)$ for now.

One easily verifies:

5.2.2.

$$\chi(X^2) = \sum \xi_\nu^2 + 2 \sum x_\nu \overline{x}_\nu$$

5.2.3.

$$\begin{aligned}\chi(X^3) = \sum & \xi_\nu^3 \\ & + 3\big(\xi_1(x_2\overline{x}_2 + x_3\overline{x}_3) + \xi_2(x_3\overline{x}_3 + x_1\overline{x}_1) + \xi_3(x_1\overline{x}_1 + x_2\overline{x}_2)\big) \\ & + 6\mathrm{Re}(x_1x_2x_3).\end{aligned}$$

By taking the trace in

5.2.4.

$$\begin{aligned}\varphi(X) &= -X^3 + \chi(X)X^2 - \frac{1}{2}\big(\chi(X)^2 - \chi(X^2)\big)X + \det(X) \cdot 1 \cdot e_0 \\ &= 0\end{aligned}$$

one obtains after a short calculation:

5.2.5.

$$\begin{aligned}\det(X) &= \frac{1}{3}\chi(X^3)\frac{1}{2}\chi(X^2)\chi(X) + \frac{1}{6}\chi(X)^3 \\ &= \xi_1\xi_2\xi_3 - \xi_1x_1\overline{x}_1 - \xi_2x_2\overline{x}_2 - \xi_3x_3\overline{x}_3 + 2\mathrm{Re}(x_1x_2x_3).\end{aligned}$$

This is a sensible generalisation of the usual definition of the determinant.

The determinant is of course also $\mathbf{F}_4$-invariant.

5.3. Let $\Pi \subset \mathfrak{J}$ denote the set of irreducible idempotents; that is, $X \in \Pi$ if and only if:

5.3.1.

$$X = X \circ X \neq 0,$$

5.3.2. and if $X = X_1 + X_2$ and

$$X_\mu \circ X_\nu = \begin{cases} 0 & \text{for } \mu \neq \nu, \\ X_\mu & \text{for } \mu = \nu, \end{cases}$$

then $X_1 = 0$ or $X_2 = 0$.

The relation (5.3.1) is $\mathbf{F}_4$-invariant and only possible for a principal matrix if all eigenvalues are 0 or 1; including (5.3.2) the only possibility is 0, 0, 1.

5.3.3. Hence

$$\chi(X) = \chi(X^2) = \chi(X^3) = 1 \quad \text{for } X \in \Pi.$$

Conversely, if (5.3.3) holds, then X is equivalent to a principal matrix with 0, 0, 1, that is, $X = X^2 \neq 0$; and if $X = X_1 + X_2$ etc. holds, then the eigenvalues of the X_1, X_2 are 0 or 1 and $1 = \chi(X) = \chi(X_1) + \chi(X_2)$, so at least on $\chi(X_\nu) = 0$, so at least one $X_\nu = 0$.

The equation (5.3.3) thus characterises the elements of Π.

In the same way we find the elements X of Π to be characterised by

$$X = X \circ X, \quad \chi(X) = 1.$$

5.4. The equation $X^2 = X$ can be written in more detail as

5.4.1.

$$\begin{aligned} \xi_i &= \xi_i^2 + x_{i+1}\overline{x}_{i+1} + x_{i+2}\overline{x}_{i+2} \quad \text{(cyclic)} \\ x_i &= x_{i+2}\overline{x}_{i+1} + (\xi_{i+1} + \xi_{i+2})x_i \quad \text{(cyclic).} \end{aligned}$$

Because of $\sum \xi_\nu = 1$ it follows for $X \in \Pi$:

$$\xi_i \overline{x}_i = x_{i+1}x_{i+2}.$$

5.4.2.

$$\xi_i x_i \overline{x}_i = x_i(x_{i+1}x_{i+2}) = \mathrm{Re}(x_i x_{i+1} x_{i+2}),$$

and as this is cyclically symmetric:

$$\xi_i x_i \overline{x}_i \text{ independent of } i.$$

Moreover, as $\xi_i x_i \overline{x}_i = \xi_i \overline{x}_i x_i$,

5.4.3.

$$x_i(x_{i+1}x_{i+2}) = (x_{i+1}x_{i+2})x_i = \xi_i x_i \overline{x}_i \quad \text{independent of } i.$$

The $x_i\overline{x}_i$ thus behave like the $\xi_{i+1}\xi_{i+2}$; substituting in (5.4.1) gives the factor of proportionality, which is 1; so

5.4.4.

$$x_i \overline{x}_i = \xi_{i+1}\xi_{i+2}.$$

6. $\mathcal{P}$

6.1. A *plane projective geometry* $\mathcal{P}$ is a system consisting of *points*, *lines* and an *incidence relation* with the following property: For every two points (lines) there exists *one* line (*one* point) incident to both.

Under the assumption that the system of points (the *plane*) is a manifold and the system of lines (taken as sets of their points) satisfies certain regularity conditions, G. Hirsch [9] proved that the plane can be only of the dimensions 2^n (and the lines are thus 2^{n-1}-dimensional). The real, complex and quaternionic projective geometry are examples for the cases $n = 1, 2, 3$; for the fourth case G. Hirsch constructed an example by topological means which (rather indirectly) is related to the octaves.

In [7] we will provide an algebraic example. We do not know whether it is the only one possible (under assumptions of regularity) or not.

6.2. If it is known that a plane projective geometry admits a group G of the type of the real projective group, then the points, lines and incidence can also be characterised in a group theoretic manner. For a point p_0 and a line l_0 incident with it we study the subgroups P and L of transformations f in G leaving invariant p_0 and l_0, respectively. Then an arbitrary point p (or an arbitrary line l) is characterised by the left-coset of transformations in G which move p_0 to p (or l_0 to l). An incidence of p and l then corresponds to a non-empty intersection of the corresponding cosets aP and bL.

These considerations lead to the following *group theoretic* definition of projective geometry:

Given a group G with two subgroups P and L. The left-cosets of P and L are called *points* and *lines*, respectively. The relation $aP \cap bL \neq \emptyset$ is called *incidence*.

(Note: If $aP \cap bL \neq \emptyset$, then it is a left-coset of $P \cap L$. For if $c, d \in aP \cap bL$, then $c^{-1}d \in P$, $c^{-1}d \in L$, so $c^{-1}d \in P \cap L$, and with $c \in aP \cap bL$ and $u \in P \cap L$ we have $cu \in aP \cap bL$.)

6.3. We now interprete the incidence axioms.

6.3.1. *For very two points there exists at least one line incident to both:* The line through P have to cover the whole plane. We can write them as aL for some $a \in P$. The set PL has to meet all cP. In other words:

$$PLP = G.$$

One easily checks that this condition is sufficient as well.

6.3.2. *Every two lines have at least one point of intersection:* Analogously, we obtain

$$LPL = G$$

as a neccessary and sufficient condition.

6.3.3. *For very two points there exists at most one line incident to both:* If $a_\nu \in P$, $b_\nu \in L$ and $a_1 b_1 = b_2 a_2$, then the lines L and $a_1 L$ coincide in the points P and $b_2 P$. The axiom now says: $a_1 \in L$ or $b_2 \in P$. Thus

$$PL \cap LP \subset P \cup L,$$

where the inclusion can be replaced by an equality. One easily checks that this condition is sufficient as well.

6.4. G generates a transformation group in the thus defined $\mathcal{P}$, if $a \in G$ corresponds to the transformation f_a with

$$f_a(cP) = acP, \quad f_a(cL) = acL.$$

Points are mapped to points, lines are mapped to lines, incidence are mapped to incidences. The group is transitive.

The subgroup fixing P is precisely f_P; the one fixing L is f_L. The subgroup fixing aP is $f_{aPa^{-1}}$ for $a \in L$. The quotient group of L by this normal subgroup can be called the projective group of lines in the geometry $\mathcal{P}$.

6.5. We now intend to choose $\mathbf{F}_4$ for the group G and for the subgroup P and L those subgroups of $\mathbf{F}_4$ fixing the elements E_1 and E_2, respectively, and both are isomorphic to $\mathbf{B}_4$. One only need to check whether 6.3.1-6.3.3 are satisfied; this direct approach is a hard one, though. Nevertheless, we wanted to begin with the considerations 6.1-6.4 to make the following approach plausible. Comparing dimensions ($\dim \mathbf{F}_4 = 52$, $\dim \mathbf{B}_4 = 36$) tells us that we have to expect a 16-dimensional $\mathcal{P}$ (the dimension of the coset space).

The coset space G/P can be modelled in $\mathfrak{J}$ by the equivalence classes of E_1; but this is just the set Π of irreducible idempotents in $\mathfrak{J}$. It is also useful to model G/L. As an incidence relation $X \circ Y = 0$ offers itself. This is satisfied for E_1, E_2 and it is $\mathbf{F}_4$-invariant.

The following definitions are now sufficiently motivated by heuristics.

7. $\mathcal{P}(\mathfrak{C})$

7.1. We define two 'genera' of 'entities, the genus of *points* and the genus of *lines*. Each genus is a bijective image of the system Π in a well-defined manner. If $X \in \Pi$, then 'the point X' is short for 'the point corresponding under the given map to the element X in Π'.

Between two entities of different genera an incidence relation exists,

$$X \circ Y = 0,$$

that is, the point X is incident with the line Y if $X \circ Y = 0$.

This is the *octave plane*.

7.2. *Neccessary and sufficient for the incidence of the point X and the line Y is* $(X, Y) = 0$.

We only need to show:

$$(X, Y) = \chi(X \circ Y) = 0 \quad \text{implies} \quad X \circ Y = 0.$$

For this we may assume $X = E_1$, that is $\chi(X, Y) = \eta_1$, where

$$Y = \begin{pmatrix} \eta_1 & y_3 & \overline{y}_2 \\ \overline{y}_3 & \eta_2 & y_1 \\ y_2 & \overline{y}_1 & \eta_3 \end{pmatrix}.$$

So $\eta_1 = 0$ and thus by (5.4.1): $y_2 = y_3 = 0$. So

7.2.1.

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \overline{y}_1 & \eta_3 \end{pmatrix}.$$

So indeed $X \circ Y = 0$.

7.2.2. We further remark that, because of (5.4.1),

$$\eta_2 + \eta_3 = 1 \quad \text{and} \quad \eta_2 = \eta_2^2 + y_1\overline{y}_1.$$

The points on X are thus determined by η_2 and y_1, which have to satisfy the relation

$$\Big(\eta_2 - \frac{1}{2}\Big)^2 + y_1\overline{y}_1 = \frac{1}{4},$$

and one easily checks that this condition is also sufficient.

So the line is, as a set of points, an 8-dimensional sphere.

7.3. We show: For $X \in \Pi$, $Y \in \mathfrak{J}$, we have

$$X \circ (X \circ Y) = \frac{1}{2} X \circ Y + \frac{1}{2}(X, Y)X.$$

We only need to prove it for $X = E_1$. Then $X \circ Y$ is of the form

$$\begin{pmatrix} \eta_1 & \frac{1}{2}y_3 & \frac{1}{2}\overline{y}_2 \\ \frac{1}{2}\overline{y}_3 & 0 & 0 \\ \frac{1}{2}y_2 & 0 & 0 \end{pmatrix}.$$

So $(X, Y) = \chi(X \circ Y) = \eta_1$ and

$$X \circ (X \circ Y) = \begin{pmatrix} \eta_1 & \frac{1}{4}y_3 & \frac{1}{4}\overline{y}_2 \\ \frac{1}{4}\overline{y}_3 & 0 & 0 \\ \frac{1}{4}y_2 & 0 & 0 \end{pmatrix},$$

which implies the assertion.

7.4. Let $X, Y \in \Pi$,

$$Z = X - Y.$$

Then

$$\chi(Z) = \chi(X) - \chi(Y) = 0$$
$$\chi(Z^2) = \chi(X) - 2\chi(X \circ Y) + \chi(Y) = 2(1 - \varepsilon)$$

with $\varepsilon = \chi(X \circ Y)$

$$\chi(Z^3) = \chi(X) - 3\chi(X \circ Y) + 3\chi(X \circ Y) - \chi(Y) = 0.$$

So, by (5.2.4),

$$-Z^3 + (1 - \varepsilon)Z + \det(Z) \cdot 1 \cdot e_0 = 0.$$

By taking the trace we obtain $\det(Z) = 0$, so

$$Z^3 = (1 - \varepsilon)Z,$$

and thus

$$Z^{2(n+1)} = (1 - \varepsilon)^n Z^2, \quad Z^{2n+1} = (1 - \varepsilon)^n Z.$$

In particular,

$$\chi(Z^2) = 2(1 - \varepsilon), \quad \chi(Z^4) = 2(1 - \varepsilon)^2, \quad \chi(Z^6) = 2(1 - \varepsilon)^3.$$

We now set

$$W = 1 - \frac{Z^2}{1 - \varepsilon},$$

which is reasonable for $X \neq Y$, as in this case $|(X, Y)| < 1$. Then

$$\begin{aligned}
\chi(W) &= 3 - 2 = 1.\\
\chi(W^2) &= \chi(1) - 2\chi\left(\frac{Z^2}{1-\varepsilon}\right) + \chi\left(\frac{Z}{(1-\varepsilon)^2}\right) = 3 - 4 + 2 = 1.\\
\chi(W^3) &= \chi(1) + 3\chi\left(\frac{Z^2}{1-\varepsilon}\right) + 3\chi\left(\frac{Z}{(1-\varepsilon)^2}\right) - \chi\left(\frac{Z}{(1-\varepsilon)^3}\right)\\
&= 3 - 6 + 6 - 2 = 1.
\end{aligned}$$

By (5.3.3) we have

$$W \in \Pi.$$

Moreover,

$$X \circ Z^2 = X \circ (X - 2X \circ Y + Y) = X - 2X \circ (X \circ Y) + X \circ Y,$$

and by 7.3 this is identical to

$$X - (X, Y)Y = (1 - \varepsilon)X,$$

that is,

$$X \circ W = 0,$$

and analogously

$$Y \circ W = 0.$$

Thus we have found the foloowing:

7.5. For $X, Y \in \Pi$, $X \neq Y$, let

$$X \vee Y = 1 - \frac{(X - Y)^2}{1 - \chi(X \circ Y)},$$

then

$$X \vee Y \in \pi.$$
$$X \circ (X \vee Y) = Y \circ (X \vee Y) = 0.$$

But this is the *incidence axiom: For two entities of the same genus there exists at least one entity of the opposite genus which is incident with both.*

7.5.1. Moreover: $X \vee Y$ is a multiple ($\neq 0$) of $X \circ Y - \frac{1}{2}X - \frac{1}{2}Y + \frac{1}{2}(1 - \chi(X \circ Y))$.

7.6. *$X, Y \in \Pi$ with $X \circ Y = 0$ can be simultaneously transformed to a principal matrix.*
For we may assume that $X = E_1$. Then

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \overline{y}_1 & \eta_3 \end{pmatrix}.$$

Now one proceeds as in 5.1, but with respect to the subgroup P of $\mathbf{F}_4$ fixing E_1, that is, minimze $\eta_2^2 + \eta_3^2$ in the P-equivalence class of Y. The same transformation as in 5.1 (which leaves invariant E_1) shows that Y is in principal matrix form.

7.7. X, Y, Z with $X \circ Y = Y \circ Z = Z \circ X = 0$ can be simultaneously transformed to principal form. For if X and Y are already principal matrices, the neccessarily $Z = 1 - X - Y$.

7.8. Let $X, Y, Z_\nu \in \Pi$, $X \circ Z_\nu = Y \circ Z_\nu = 0$, $X \neq Y$. Then $Z_1 = Z_2$. For by 7.5 we may assume: $X = E_2$, $Z_1 = E_1$. Because of $Y \circ E_1 = E_2 \circ Z_2$ we find

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \overline{y}_1 & \eta_3 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} \xi_1 & 0 & \overline{z}_2 \\ 0 & 0 & 0 \\ z_2 & 0 & \xi_3 \end{pmatrix}.$$

$Y \circ Z_2$ has entry $\eta_3\xi_3 = 0$ in row 3, column 3. By 7.2.1, $\eta_3 = 0$ would imply $\eta_2 = 1$, $y_1 = 0$, that is, $Y = E_2 = X$, whereas $X \neq Y$ was assumed. Hence $\xi_3 = 0$ and so by (7.2.1)-(7.2.2) again $\xi_1 = 1$, $z_2 = 0$, $Z_2 = E_1 = Z_1$.

7.9. According to 7.8 the element Z among $X, Y, Z \in \Pi$ with $X \neq Y$, $X \circ Z = Y \circ Z = 0$, is uniquely determined. But this is the *incidence axiom: For two entities of the same genus there exists at most one entity of the opposite genus which is incident with both.*

7.10. By polarisation we obtain from the cubic form $\det(X)$ a trilinear form $\det(X, Y, Z)$ which is also $\mathbf{F}_4$-invariant; $\det(X, Y, Z)$ is defined as the coefficient of $6\alpha\beta\gamma$ in $\det(\alpha X + \beta Y + \gamma Z)$. From (5.2.5) it follows that for $X, Y, Z \in \Pi$

7.10.1.

$$\det(X, Y, Z) = \frac{1}{3}\chi(X \circ Y \circ Z) - \frac{1}{6}\big((\chi(X \circ Y) + \chi(Y \circ Z) + \chi(Z \circ X)\big) + \frac{1}{6}.$$

We now show: *It is neccessary and sufficient for the collinearity of three points X, Y, Z that* $\det(X, Y, Z) = 0$.

For we can write the right hand side of (7.10.1) as

$$\frac{1}{3}\Big(X \circ Y - \frac{1}{2}X - \frac{1}{2}Y + \frac{1}{2}(1 - \chi(X \circ Y)), Z\Big).$$

But by (7.5.1) this is, up to a real factor ($\neq 0$ if $X \neq Y$), identical to $(X \vee Y, Z)$.

By 7.2 this is 0 if and only if

$$X \vee Y) \circ Z = 0,$$

that is, if Z is contained in the line through X and Y.

7.11. Three points on a complex projective line determine a unique real projective line, that is, a subset which is characterised by the cross ratio of every four of its points being a real number; these real lines appear as circles in the Riemannian model of the complex projective line.

It is similar in octave geometry. If X, Y, Z are three collinear points and if one wants to determine all points

$$\alpha X + \beta Y + \gamma Z$$

on the same line with *real* α, β, γ, one finds the equations

7.11.1.

$$\chi(\alpha X + \beta Y + \gamma Z) = \alpha + \beta + \gamma = 1$$

and

7.11.2.

$$\chi((\alpha X + \beta Y + \gamma Z)^2) = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta(X, Y) + 2\beta\gamma(Y, Z) + 2\gamma\alpha(Z, X) = 1.$$

Instead of the third equation

$$\chi((\alpha X + \beta Y + \gamma Z)^3) = 1$$

one can write by (5.2.5)

$$\det(\alpha X + \beta Y + \gamma Z) = 0,$$

but this holds to begin with, as $\det(X, Y, Z)$ vanishes by assumption and $\det(X, X, Y)$ etc. vanish trivially.

As the solution to (7.11.1) and (7.11.2) one thus obtains the intersection of the octave line (8-dimensional sphere, see (7.2.2)) with the plane (in the sense of 9-dimensional space) through X, Y, Z, that is, a circle. We shall call it the *real line* through X, Y, Z.

7.12. It is obvious that the notion of the real line is $\mathbf{F}_4$-invariant. Moreover, it is invariant under perspectivities between different octave lines.

To prove this, we drop the norm of $\mathfrak{J}$ and consider the set $\Pi^\#$ of those ρX with $X \in \Pi$ and ρ real. So a point etc. is associated to a set $\{\rho X\}$. The connecting line $X \vee Y$ is now homogeneously written as

7.12.1.

$$X \vee Y = \chi(X)\chi(Y) - \chi(X \circ Y) - \chi(X)Y - \chi(Y)X + 2X \circ Y.$$

By projecting the point X on the octave line A through the fixed point Y (not on A), we obtain a pencil $X' = X \vee Y$. Restricting to the points

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$$

on the *real* line through X_1, X_2, X_3, we obtain the same relation

$$X' = \alpha_1 X_1' + \alpha_2 X_2' + \alpha_3 X_3'$$

by the linearity of (7.12.1) for the corresponding pencil line. Intersecting the pencil with another line B we obtain points $X'' = B \vee X'$, again satisfying the relation

$$X'' = \alpha_1 X_1'' + \alpha_2 X_2'' + \alpha_3 X_3''.$$

But these relations precisely characterise the points on a real line.

7.13. Interpreted as conic sections (see the end of 7.11), the real lines are mapped projectively by perspectivities. This implies that the octave lines, as quadrics in 9-dimensional space (end of 7.11), can be mapped projectively.

7.13.1. The octave line E_1 satisfies the homogeneous equation

$$\xi_2\xi_3 - x_1\overline{x}_1 = 0.$$

Call the self-maps generated by the perspectivities of an octave line *prospectivities*, then the prospectivities of E_1 from a subgroup of the form of $\mathbf{D}_5$ belonging to (7.13.1). It will be shown later on that this subgroup coincides with the 1-component of the invariance group of (7.13.1) (see 8.2).

7.14. On the octave line, one can also study *complex projective lines*: we say that X_i $(i = 1, \ldots, 5)$ lie on a complex projective line if they are real linearly independent. This notion is $\mathbf{F}_4$-invariant as well. The complex projective lines on E_1 are nothing but the intersections of (7.13.1) with 3-dimensional projective subspaces of the 9-dimensional projective space of the ξ_2, ξ_3, x_1. The prospectivities of the line E_1 on a complex projective line from E_1 are thus projective maps of the ordinary complex projective line (Möbius group).

In particular:

If X_i $(i = 1, \ldots, 5)$ lie on an octave line, and if under a certain prospectivity X_1 and X_2 are interchanged and if X_3 and X_4 are fixed, then X_i $(i = 1, \ldots, 5)$ even lie harmonically on a real line.

For the X_i $(i = 1, \ldots, 5)$ certainly lie on a complex line, and for this the assertion holds.

7.15. We call a projective geometry *harmonic* if for an ordered triple of points on a line the fourth harmonic point (by the usual construction, see Figure 3) is uniquely determined (that is, independent of the choice of auxiliary points A, B). This is also called the (spe-

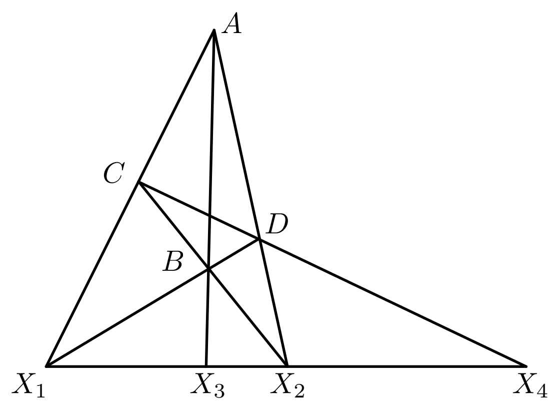

FIGURE 3.

cial) Quadrilateral Theorem. It is sufficient to require the fourth harmonic point to be independent of a translation of A along the line AB.

We show: *Octave geometry is harmonic.*

Proof. Projection from B on CD followed by projection from A back maps X_1, X_2, X_3, X_4 to X_2, X_1, X_3, X_4, and is thus a prospectivity satisfying the conditions at the end of 7.14. So X_4 is already contained in a real line X_1, X_2, X_3 and is uniquely determined by its cross ratio to X_1, X_2, X_3.

7.16. *Remark.* $\mathcal{P}(\mathfrak{C})$ is non-Desargueian, as a Desargueian geometry neccessarily is a geometry over a skew-field, which moreover has to be Euclidean in the small, meaning the real, the complex and the quaternionic field. The octave geometry does not belong to these.

8. $\mathbf{E}_6$

8.1. We are looking for the group $\mathfrak{T}$ of linear transformations of $\mathfrak{J}$ leaving invariant

$$\det(X) = \frac{1}{3}\chi(X \circ X \circ X) - \frac{1}{2}\chi(X)\chi(X \circ X) + \frac{1}{6}\chi(X)^3$$

and thus also the trilinear form

$$\det(X,Y,Z)=\frac{1}{3}\chi(X\circ Y\circ Z)-\frac{1}{6}\Big(\chi(X)\chi(Y\circ Z)+\chi(Y)\chi(X\circ X)+\chi(Z)\chi(X\circ Y)\Big)\\ +\frac{1}{6}\chi(X)\chi(Y)\chi(Z).$$

In any case, $\mathfrak{T}$ is a group containing $\mathbf{F}_4$, which also leaves invariant (X,Y).

At first we show:

8.1.1. All

$$X\mapsto T\circ X=\frac{1}{2}(TX+X^*T^*)\qquad\text{with } T\in\mathfrak{M}_3,\ \chi(T)=0$$

belong to the infinitesimal ring $\mathfrak{T}$. We may assume $T\in\mathfrak{M}_3^+$, as for $T\in\mathfrak{M}_3^-$ the map $X\mapsto TX-XT=[T,X]$ is a transformation in $\mathbf{F}_4$.

When applying (8.1.1), X changes infinitesimally by

$$\begin{aligned}&\chi(X\circ X\circ(T\circ X))-\frac{1}{2}\chi(T\circ X)\chi(X\circ X)-\chi(X)\chi(X\circ(T\circ X))+\chi(X)^2\chi(T\circ X)\\ &=\big(X\circ X\circ X\frac{1}{2}\chi(X\circ X)X\chi(X)X\circ X+\chi(X)^2X,\ T\big)\\ &=(\det(X)\cdot 1,T)=\det(X)\cdot\chi(T)\\ &=0.\end{aligned}$$

Let an arbitrary infinitesimal $\Phi\in\mathfrak{T}$ be given. Set

$$T=\Phi 1.$$

Then $T\in\mathfrak{M}_3^+$, $\chi(T)$ is up to a positive factor identical to $\det(\Phi 1,1,1)=0$.

$$\Phi X=T\circ X,\qquad\text{for } X=1.$$

Φ_1, given by

$$\Phi_1X=\Phi X-T\circ X,$$

also belongs to $\mathfrak{T}$, and we have

$$\Phi_1 1=0.$$

But this means Φ_1 not only leaves invariant $\det(X,Y,Z)$, but also (see (7.10.1))

$$\begin{aligned}\det(X,Y,1)&=-\frac{1}{6}\big(\chi(X\circ Y)-\chi(X)\chi(Y)\big)\\ &=-\frac{1}{6}\big(\chi(X,Y)-\chi(X)\chi(Y)\big)\end{aligned}$$

and

$$\det(X,1,1)=\frac{1}{3}\chi(X),$$

and thus also $\chi(X)$, $\chi(Y)$ and (X,Y). Therefore, $\Phi_1\in\mathbf{F}_4$.

This implies:

The infinitesimal ring of linear transformations of $\mathfrak{J}$ leaving invariant $\det(X)$ is generated by the

$$X\mapsto T\circ X\qquad\text{with }\chi(T)=0;$$

each of its elements can be uniquely written as the sum of a

$$X\mapsto T\circ X\qquad\text{with } T\in\mathfrak{M}_3^+,\ \chi(T)=0$$

and an infinitesimal automorphism of $\mathfrak{J}$, or also as the sum of a $\Delta \in \mathbf{D}_4$ and a

$$X \mapsto T \circ X$$

where $T \in \mathfrak{M}_3^{\mathrm{r}}$ (that is, $T \in \mathfrak{M}_3$ with real diagonal) and $\chi(T) = 0$ (because of 4.8).

From (4.9.3) it further follows that

$$\Delta(T \circ X) - T \circ (\Delta X) = (\Delta T) \circ X,$$

so

$$\widetilde{[\Delta, T]} = \widetilde{\Delta T}.$$

These reasoning can also be applied to groups themselves. First a few preliminary remarks:

8.1.2. $X > 0$ (*positive definite*) for $X \in \mathfrak{J}$ means: all eigenvalues of X are positive.

8.1.3. $X > 0$ if and only if $\chi(X \circ Y \circ Y) > 0$ identical in $Y \in \mathfrak{J}$. For we may assume X to be a principal matrix with diagonal elements λ_ν, and with the notation of 7.2 it then holds that

$$\begin{aligned} \chi(X \circ Y \circ Y) \quad = \quad & \lambda_1(\eta_1^2 + y_2\overline{y}_2 + y_3\overline{y}_3) \\ & + \lambda_2(\eta_2^2 + y_3\overline{y}_3 + y_1\overline{y}_1) + \lambda_3(\eta_3^2 + y_1\overline{y}_1 + y_2\overline{y}_2), \end{aligned}$$

from which the assertion follows.

8.1.4. The set $\mathfrak{J}_{\mathrm{pos}}$ of $X > 0$ is convex. This follows from 8.1.3.

8.1.5. If $\mathfrak{T}_0$ is the 1-component of the (finite) group $\mathfrak{T}$, then $\mathfrak{T}_0 1 \subset \mathfrak{J}_{\mathrm{pos}}$. This is because for $\varphi \in \mathfrak{T}_0$ the eigenvalues of $\varphi 1$ depend contiuously on φ and can never vanish because of $\det(\varphi 1) = \det(1) = 1$.

8.1.6. If $\tilde{T}$ is the infinitesimal transform $X \mapsto T \circ X$ and $\tilde{\mathfrak{J}}$ the set of $\tilde{T}$ with $T \in \mathfrak{J}$, then $\mathfrak{J}_{\mathrm{pos}} \subset \exp(\tilde{\mathfrak{J}})1$. For if $X \in \mathfrak{J}_{\mathrm{pos}}$ has the principal form with diagonal elements ξ_ν, let T be the principal matrix with diagonal elements $\log(\xi_\nu)$ and we obtain $\exp(\tilde{T})1 = X$. We can reduce the general case to this by a transformation in $\mathbf{F}_4$.

8.1.7. $\mathfrak{T}_0 1 = \mathfrak{J}_{\mathrm{pos}}$. This follow from 8.1.5-8.1.6.

8.1.8. In the same way we find that $\mathfrak{T}_0$, applied to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

yields the set of elements of $\mathfrak{J}$ with precisely two negative eigenvalues.

8.1.9. Every finite $\varphi \in \mathfrak{T}$ has to leave invariant $\mathfrak{J}_{\mathrm{pos}}$ or commute with the set in 8.1.8. But the latter is not possible, as $\mathfrak{J}_{\mathrm{pos}}$ is convex by 8.1.2 and the set 8.1.8 certainly is not convex.

So $\mathfrak{T}1 = \mathfrak{J}_{\mathrm{pos}} = \exp(\tilde{\mathfrak{J}})1$.

8.1.10. As the set of $\varphi \in \mathfrak{T}$ with $\varphi 1 = 1$ is precisely $\mathbf{F}_4$ (see 4.11 and 8.1.1), this implies

$$\mathfrak{T} = \mathbf{F}_4 \cdot \exp(\tilde{\mathfrak{J}}).$$

So $\mathfrak{T}$ has only one component.

8.2. We study those elements of $\mathfrak{T}$ leaving invariant the line E_1. They also leave invariant

$$\det(E_1, X, X) = -\frac{1}{2}\chi(X \circ X) + \frac{1}{2}\chi(X)^2 = \xi_2\xi_3 - x_1\overline{x}_1$$

and form a subring $\mathbf{G}$ of $\mathbf{D}_5$. By 8.1.1 (at the end) they can be written as the sum of a $\Delta \in \mathbf{D}_4$ and a $\tilde{T}$ with

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_2 & t_{23} \\ 0 & t_{32} & \tau_3 \end{pmatrix}.$$

A dimension count $(28 + 17 = 45)$ yields that (infinitesimally) $\mathbf{G} = \mathbf{D}_5$, so $\mathbf{G}$ contains the prospectivities of E_1 (see 7.13).

8.3. We now show that $\mathfrak{T}$ is $\mathbf{E}_6$ from Cartan's classification.

We use the decomposition of $\mathfrak{T}$ at the end of 8.1.1.

As a maximal abelian subring $\mathbf{H}$ we use the same one as for $\mathbf{D}_4$ and $\mathbf{F}_4$ (see 4.10) augmented by the elements $\tilde{S}$ defined by

$$S = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_1 + \xi_2 + \xi_3 = 0,$$
$$\tilde{S}X = S \cdot X$$

(so $\mathbf{H}$ is 6-dimensional). For $\delta \in \mathbf{D}_4$ we have $\delta E_\nu = 0$, so $\delta S = 0$, so (end of 8.1.1) $[\delta, \tilde{S}] = 0$, that is, every δ commutes with every S and in particular is $\mathbf{H}$ abelian. Let $F^a_{\mu\nu}$ denote the matrix with entry a in row μ and column ν, and entries 0 otherwise.

8.3.1. Then (see (4.9.2))

$$\delta F^a_{\mu\nu} = F^{\delta_{\mu\nu} a}_{\mu\nu}.$$

Moreover, as the matrix coefficients of S are real and therefore associate,

8.3.2.

$$[\tilde{S}, \tilde{F}^a_{\mu\nu}] = \frac{1}{2}\widetilde{[S, F^a_{\mu\nu}]} = \frac{1}{2}(\xi_\nu - \xi_\mu)\tilde{F}^a_{\mu\nu}.$$

Exactly as in 4.10 one determines the roots, which in addition to $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ now also depend on ξ_1, ξ_2, ξ_3 $(\xi_1 + \xi_2 + \xi_3 = 0)$. Because of (8.3.1), the dependence on the α is the same as in 4.10, the dependence on the ξ follows from (8.3.2). The roots are $\pm\alpha_\nu \mathrm{i} \pm \frac{1}{2}(\xi_2 - \xi_3)$, $\pm\alpha_\mu \mathrm{i} \pm \alpha_\nu \mathrm{i}$,

$$\frac{1}{2}(\pm\alpha_0 \mathrm{i} \pm \alpha_1 \mathrm{i} \pm \alpha_2 \mathrm{i} \pm \alpha_3 \mathrm{i}) \pm \frac{1}{2}\begin{cases} (\xi_3 - \xi_1), \\ (\xi_1 - \xi_2) \end{cases},$$

where in the last expression according to (2.2.4) in case of an even number of positive signs for the α the upper summand holds, and in case of an odd number the lower summand holds, and where moreover all sign combinations are possible.

It is not evident that this root system is isomorphic to the one of Cartan's $\mathbf{E}_6$. The transformation rules are easy to deduce, but uninteresting. But we can easily see the isomorphy without a calculation if we use Dynkin's representation [7]. One determines a system of positive irreducible roots (those which cannot be written as sums of other positive roots).

(1) $\alpha_2 \mathrm{i} - \alpha_3 \mathrm{i}$

(2±) $\alpha_3 \mathrm{i} \pm (\xi_2 - \xi_3)$

(3±) $\frac{1}{2}(\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3)\mathrm{i} \pm (\xi_1 - \xi_2)$

(4) $\alpha_1 \mathrm{i} - \alpha_2 \mathrm{i}.$

The Dynkin diagram is then

$$\begin{array}{ccccccccc} (3-) & \text{——} & (2-) & \text{——} & (1) & \text{——} & (2+) & \text{——} & (3+) \\ & & & & | & & & & \\ & & & & (4) & & & & \end{array}$$

where those roots are connected by a line whose sum is again a root. But this is precisely the diagram of $\mathbf{E}_6$.

8.4. An approach analogous to 4.13 shows that if $\chi(T_1) = \chi(T_2) = 0$, then

$$\widetilde{[T_1, T_2]} \equiv 2[\tilde{T}_1, \tilde{T}_2] + \frac{1}{3}\widetilde{\chi([T_1, T_2])}$$

modulo automorphisms of $\mathfrak{J}$ generated by an element of $\mathbf{G}_2$.

Again,

$$\begin{aligned} \dim \mathbf{E}_6 &= 78 \\ &= \dim(\text{set of the } T) + \dim \mathbf{G}_2 \\ &= 64 + 14. \end{aligned}$$

9. A

We will show: *The group of autmorphisms of $\mathcal{P}(\mathfrak{C})$ is a representation of* $\mathbf{E}_6$. As in real and quaternion geometry (but contrary to complex geometry) the continuity of the automorphisms will be automatic.

In $\mathcal{P}(\mathfrak{C})$ the group $\mathbf{F}_4$ plays the part of the elliptic group, $\mathbf{E}_6$ plays the part of the projective group.

The subgroup of index 16 in $\mathbf{E}_6$ fixing a point in $\mathcal{P}(\mathfrak{C})$ can already be found in Cartan [2, p. 152, row 3].

Here the proofs follow:

9.1. The elements $X, Y \in \Pi^\#$ (see 7.12) are characterised by: $\det(X, X, Y) = 0$ identically in Y.

Proof. Because of (7.10.1)

$$\det(X, X, Y) = \chi(X \circ X \circ X) - \frac{1}{2}\big(\chi(X \circ X)\chi(Y) + 2\chi(X \circ Y)\chi(X) + \frac{3}{2}\chi(X)^2\chi(Y)\big).$$

For $X \in \Pi^\#$ we have $X \circ X = \alpha X$, $\chi(X) = \alpha$, $\chi(X^2) = \alpha^2$, that is, $\det(X, X, Y) = 0$.

Conversely, assume $\det(X, X, Y) = 0$ in particular for $Y = 1$. Then $\chi(X \circ X) = \chi(X)^2$, and if this holds for $Y = X$, then $\chi(X \circ X \circ X) = \chi(X)^3$, and this implies $X \in \Pi^\#$.

9.2. $\Pi^\#$ is invariant under $\mathbf{E}_6$. This follows from 9.1.

9.3. As $\mathbf{E}_6$ permutes the sets $\{\rho X\}$ it can be considered a group of transformations of $\mathcal{P}(\mathfrak{C})$. As $\det(X, Y, Z) = 0$ characterises by 7.11 the collinearity of points X, Y, Z, and this equation is invariant under $\mathbf{E}_6$, it follows that $\mathbf{E}_6$ is a subgroup of the automorphism group $\mathbf{A}$ of $\mathcal{P}(\mathfrak{C})$. We now show: $\mathbf{A} = \mathbf{E}_6$. This proof is easier than the original one, which moreover contained a little gap.

9.4. We construct a (finite) element $\varphi \in \mathbf{E}_6$ with

$$\varphi E_2 \sim E_2, \quad \varphi E_3 \sim E_3, \quad \varphi A \sim E_1,$$

where A is a given point not on the line E_1. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map

$$X \mapsto PXP^*$$

is (well-defined without brackets) and a (finite) element of $\mathbf{E}_6$ leaving invariant the points E_2, E_3. If in

$$A = \begin{pmatrix} \alpha_1 & a_3 & \overline{a}_2 \\ \overline{a}_3 & \alpha_2 & a_1 \\ a_2 & \overline{a}_1 & \alpha_3 \end{pmatrix}$$

we have $a_3 \neq 0$ (so by (5.4.4) also $\alpha_1 \neq 0$), choose $p = -\alpha_1^{-1}\overline{a}_3$. If $a_3 = 0$, choose $p = 0$. Then in

$$B = PAP^*$$

the element $b_3 = 0$.

Now apply with

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q & 0 & 1 \end{pmatrix}$$

the transformation

$$X \mapsto QXQ^*,$$

which also belongs to $\mathbf{E}_6$ and fixes the points E_2, E_3. Thus, for a suitable choice of q, in

$$C - QBQ^*$$

the element c_3 also vanishes. C is the of the form

$$C = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & a_1 \\ 0 & \overline{a}_1 & \gamma_3 \end{pmatrix},$$

where (by (5.4.4)) γ_1 or a_1 vanishes. $\gamma_1 = 0$ would imply that C is contained in the line E_1 through the points E_2 and E_3; but then so would A, which was assumed not to be the case. Thus $a_1 = 0$ and (by (5.4.4)) also $\alpha_2 = \alpha_3 = 0$. So $C \sim E_1$. The map

$$X \mapsto Q(PXP^*)Q^*$$

is the desired φ.

9.5. There exists a (finite) element $\varphi \in \mathbf{E}_6$ which maps a given point A and a given line B (not incident) to the point E_1 and the line E_1, respectively.

For this, first map the line B to the line E_1 by some element $\varphi_1 \in \mathbf{F}_4$, and then (according to 9.4) map the point $\varphi_1 A$ to the point E_1, where the line E_1 is fixed.

9.6. This is easily understood geometrically: There exists at most one transformation in **A** fixing the point A and all points of the line B (not incident with A) and maps a point C to a point D.

9.7. The involution

$$\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}$$

leaves invariant $\det(X)$ and hence belongs to $\mathbf{E}_6$ and fixes the point E_1 and all points on the line E_1. Point and image point lie harmonically to the points E_1 and the line E_1.

9.8. According to 9.6 the involution fixing the point A and all points on the line B (not incident with A) is uniquely determined by A and B. By 9.5 and 9.7 the system S of all these involutions is contained in $\mathbf{E}_6$; as the harmonic position is **A**-invariant, S is **A**-invariant. The subgroup G of $\mathbf{E}_6$ generated by S is thus a normal subgroup of **A** and thus of $\mathbf{E}_6$, and by the simplicity of $\mathbf{E}_6$ we have $G = \mathbf{E}_6$, that is, $\mathbf{E}_6$ is a normal subgroup of **A**.

9.9. Dual to the transformations φ of $\mathscr{P}(\mathfrak{C})$ are the transformations φ^* of the lines. The relation is given by

$$(\varphi X, U) = (X, \varphi^* U).$$

For the infinitesimal elements $\Phi \in \mathbf{E}_6$ one states for $\Phi \in \mathbf{F}_4$:

$$(\Phi X, U) + (X, \Phi U) = 0$$

by 4.4.2, for $\Phi = \tilde{T}$ with $T = T^*$:

$$(\Phi X, U) = \frac{1}{2}(T, X, U) = \frac{1}{2}(X, T, U) = (X, \Phi U).$$

So

$$\Phi^* = -\Phi \quad \text{for } \Phi \in \mathbf{F}_4$$

$$\Phi^* = \Phi \quad \text{for } \Phi = \tilde{T} \text{ with } T = T^*.$$

In general,

$$\Phi : X \mapsto T \circ X$$

implies

$$\Phi^* : X \mapsto X \circ T.$$

$\Phi \mapsto -\Phi^*$ is essentially the only outer automorphism of $\mathbf{E}_6$ (see Cartan [5]).

9.10. If $\omega \in \mathbf{A} \backslash \mathbf{E}_6$ generates an outer automorphism of $\mathbf{E}_6$, then we can assume it is the one in 9.9. If ω generates an inner automorphism, then we can assume it commutes with all elements of $\mathbf{E}_6$. In any case ω commutes with all elements of $\mathbf{F}_4$. We consider the maximal subgroup G of $\mathbf{F}_4$ fixing the point X. For $\varphi \in G$ we have

$$\varphi\omega X = \omega\varphi X = \omega X.$$

So G fixes ωX. This implies $\omega X = X$ for all X, so $\omega = 1$, which contradicts $\omega \in \mathbf{A} \backslash \mathbf{E}_6$. Hence $\mathbf{A} = \mathbf{E}_6$.

10. ? ?

We collect some open problems:

10.1. Wich geometries are associated to $\mathbf{E}_7$ and $\mathbf{E}_8$?

10.2. A plane projective geometry is calle k-times differentiable or analytic if its points form a k-times differentiable or analytic manifold, respectively, and if intersection points and connecting lines depend k-times differentiably or analytically on the given quantities with a functional matrix of maximal rank.

10.3. Do there exist ∞-times differentiable non-harmonic $\mathcal{P}$ in dimensions 2, 4, 8, 16?

10.4. In which dimensions do k-times differentiable $\mathcal{P}$ exist? (For the theorem by G. Hirsch, which restricts these dimensions to powers of 2, one needs at least $k = 1$.)

10.5. Is every analytic $\mathcal{P}$ also harmonic?

10.6. Is (in dimensions 4, 8) every continuous harmonic $\mathcal{P}$ also Desargueian?

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Original: *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Geometria Dedicata 19, 1985, 7-63.

Translation by Wolfgang Globke, Version of December 2, 2013.