The radical of a left-symmetric algebra

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An algebra is called a *left-symmetric algebra* if its associator is a trilinear function that is symmetric in the two variables on the left. This mean it is a vector space $\mathfrak{A}$ equipped with a bilinear product $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $(a, b) \mapsto ab$, that satisfies the following identity:

$$a(bc) - (ab)c = b(ac) - (ba)c. \tag{1}$$

If we set $[a, b] = ab - ba$, this condition implies that the map $(a, b) \mapsto [a, b]$ is a Lie algebra bracket.

If we set $ab = \mathrm{L}_a(b) = \mathrm{R}_b(a)$, condition (1) is equivalent to each of the following two conditions:

$$\mathrm{L}_{[a,b]} + [\mathrm{L}_a, \mathrm{L}_b], \tag{2}$$

$$\mathrm{R}_{bc} - \mathrm{R}_c\mathrm{R}_b = [\mathrm{L}_b, \mathrm{R}_c]. \tag{3}$$

I intend to define and study the radical of such an algebra A that is of finite dimension over $\mathbb{C}$. This assumption allows to use the elementary theory of Lie groups and also to begin with a definition of this radical due to Monsieur J.L. Koszul. This definition, whose geometric motivation I shall explain below, motivates me to study a certain subset Ω of $\mathfrak{A}$ which is complementary to an algebraic hypersurface. If the algebra $\mathfrak{A}$ is not associative, I do not know of any other way of using this definition than studying the polynomial $D(x)$ whose set of zeros is this hypersurface, which requires me to assume the base field to be algebraically closed. If this field $\mathbb{K}$ is not algebraically closed, and if $\mathbb{K}'$ is its algebraic closure, the only modification needed in order to yield the results announced below is to replace Ω by the set of points in $\mathfrak{A} \otimes \mathbb{K}'$ on which $D(x)$ vanishes. In [5] we find proofs of these results, valid for all fields $\mathbb{K}$ of characteristic 0. There we make use of formal series whose existence and usefulness is demonstrated in a natural way via the theory of Lie groups. For these reasons, I am content to consider the most convenient case where the base field is $\mathbb{C}$ in the investigations following this introduction.

The structures of left-symmetric algebras are encountered in diverse problems in geometry. I will only recount one here, as it allows me to introduce the notions used in the following. It is the study of affine representations of Lie groups with open orbits and discrete isotropy subgroup (for the points in that orbit), which implies that the Lie group acts on a vector space of the same dimension. Note that if one know the representation of the Lie algebra of this Lie group on this affine space, one knows the points with discrete isotropy subgroup and the points in the open orbit (provided the set of connected components of the group is countable). This is why, given an affine representation of a Lie algebra, I can speak of points in an open orbit and eventually say that this algebra acts transitively if all points lie in an open orbit.

We return to the left-symmetric algebra $\mathfrak{A}$, which is also a Lie algebra of finite dimension over $\mathbb{C}$ (or $\mathbb{R}$). Let G be a simply connected with Lie algebra $\mathfrak{A}$. The set of affine operators on the vector space $\mathfrak{A}$ is identified with the vector space $\mathfrak{A} \oplus \mathrm{End}(\mathfrak{A})$. If $a \in \mathfrak{A}$, $f \in \mathrm{End}(\mathfrak{A})$, then (a, f) denotes the affine operator defined by $(a, f)(x) = a + f(x)$ for all $x \in \mathfrak{A}$. The affine transformations of $\mathfrak{A}$ form a Lie group G whose Lie algebra is identified with the vector space $\mathfrak{A} \oplus \mathrm{End}(\mathfrak{A})$ equipped with the bracket

$$[(a, f), (b, g)] = (f(b) - g(a), [f, g]).$$

The identity (2) allows us to verify that the map $a \mapsto (a, \mathrm{L}_a)$ is a Lie algebra homomorphism of $\mathfrak{A}$ into this algebra $\mathfrak{A} \oplus \mathrm{End}(\mathfrak{A})$. This is an affine representation of the Lie algebra $\mathfrak{A}$, and it determines an affine representation of the Lie group G. This representation is called the *canonical affine representation* of $\mathfrak{A}$ and G on $\mathfrak{A}$.

For all $x \in \mathfrak{A}$, consider the map

$$a \mapsto (a, \mathrm{L}_a)(x) = (I + \mathrm{R}_x)(a)$$

whose kernel is the isotropy algebra of x, and whose image is the orbit through x. Let Ω be the set of $x \in \mathfrak{A}$ such that $I + \mathrm{R}_x$ is a bijective endomorphism of $\mathfrak{A}$. This Ω is not empty since it contains 0. Ω is the set of points in $\mathfrak{A}$ whose orbit is open and whose isotropy subgroup is discrete for the canonical affine representation of G on $\mathfrak{A}$. If the base field is $\mathbb{R}$, then every connected component of Ω is an orbit of G on $\mathfrak{A}$.

Conversely, let $\mathfrak{A}$ be a Lie algebra with an affine representation of $\mathfrak{A}$ on an affine space B such that some point $y \in B$ has trivial isotropy algebra and an open orbit.

Without loss of generality we can restrict to the case where B is a vector space whose origin is this point y. The affine representation of $\mathfrak{A}$ on B is the a map denoted by $a \mapsto (\varrho(a), \lambda_a)$, where $\varrho(a) \in B$ and $\lambda_a \in \operatorname{End}(B)$. Moreover, ϱ is a bijective linear map $\mathfrak{A} \to B$. We can show that for

$$\mathrm{L}_a = \varrho^{-1} \circ \lambda_a \circ \varrho$$

the product $(a, b) \mapsto \mathrm{L}_a(b)$ gives $\mathfrak{A}$ the structure of a left-symmetric algebra that is compatible with the original Lie algebra structure on $\mathfrak{A}$.

One encounters this construction in the study of convex homogeneous cones, for example as done in [9]. The base field is then $\mathbb{R}$, and we obtain left-symmetric algebras with the following property: If Ω_0 denotes the connected component of Ω that contains the element 0 of $\mathfrak{A}$, then Ω_0 is a convex homogeneous cone. In particular, there is not translation by a non-zero vector that preserves Ω_0, and the abscence of such translations is very importent in the study of everything I refer to. Returning to the general case, one can attempt to study those vectors $a \in \mathfrak{A}$ such that $a + \Omega = \Omega$, especially since this set is precisely the radical of $\mathfrak{A}$ if the left-symmetric algebra $\mathfrak{A}$ is an associative algebra.

In addition, if the left-symmetric algebra $\mathfrak{A}$ preserves a convex homogeneous cone, then there exists Lie algebra homomorphism s of $\mathfrak{A}$ over $\mathbb{R}$ such that the symmetric bilinear form $(a, b) \mapsto s(ab)$ is positive definite. For example, $s(a) = \operatorname{tr} \mathrm{L}_a$. These linear forms s play an important role in the study in everything I have referenced. Moreover, if $\mathfrak{A}$ is an associative algebra of finite dimension, then it is known that its radical is the kernel of the symmetric bilinear form $(a, b) \mapsto s(ab)$, where s is the linear form $a \mapsto \operatorname{tr} \mathrm{L}_a$ or $a \mapsto \operatorname{tr} \mathrm{R}_a$. These considerations, among others, brought J.L. Koszul to propose the following definition: The *radical of a left-symmetric algebra* $\mathfrak{A}$ is the set of $a \in \mathfrak{A}$ such that $\Omega + a = \Omega$. With this definition one can show directly that the radical is a left-ideal of $\mathfrak{A}$ (this result will be encountered again later). This definition is accompanied by the conjecture according to which there must be a relation between this radical and a symmetric bilinear form associated to a Lie algebra homomorphism s from $\mathfrak{A}$ to the base field. Contrasting the study of convex homogeneous domains, it is not the linear form $a \mapsto \operatorname{tr} \mathrm{L}_a$ that is prefereable in studying the general case, but the linear form $a \mapsto \operatorname{tr} \mathrm{R}_a$, which is also a Lie algebra homomorphism, since by (3) we have $\operatorname{tr} \mathrm{R}_{bc} = \operatorname{tr} \mathrm{R}_b \mathrm{R}_c$. In fact, we show that the radical is the largest ideal contained in the kernel of the linear form $a \mapsto \operatorname{tr} \mathrm{R}_a$. Unfortunately, it is not always identical to the kernel of the bilinear form $(a, b) \mapsto \operatorname{tr} \mathrm{R}_{ab}$. We can only say that it is the largest left-ideal contained in the second kernel.

Having thus defined the radical of a left-symmetric algebra, we would like to recover certain properties of the radicals of associative algebras, in particular the fact that the quotient by the radical is an algebra with zero radical. Now the radical of a left-symmetric algebra is not always a two-sided ideal. Nevertheless, if $\mathfrak{B}$ is a left-ideal of $\mathfrak{A}$, we can give the vector space quotient $\mathfrak{A}/\mathfrak{B}$ a structure of a left-symmetric algebra, provided that we can find a Lie subalgebra $\mathfrak{C}$ complementary to $\mathfrak{B}$ in $\mathfrak{A}$. In fact, let $x, y \in \mathfrak{A}/\mathfrak{B}$, and let $a, b \in \mathfrak{C}$ be preimages of x, y. By definition, the product xy is the image of ab in $\mathfrak{A}/\mathfrak{B}$. The left-symmetric product on $\mathfrak{A}/\mathfrak{B}$ gives a Lie algebra isomorphic to $\mathfrak{C}$. It seems reasonable to conjecture a kind of generalization of Levi's theorem for left-symmetric algebras, stating, among other things, that there always exists a complementary subalgebra for the radical. If this is true, then it is easy to show that the quotient by the radical has zero radical.

We return to the case where $\mathfrak{A}$ is a left-symmetric algebra of finite dimension n over $\mathbb{C}$. Set $D(x) = \det(I + \mathrm{R}_x)$. This D is a polynomial function on $\mathfrak{A}$ of degree $\leq n$. The base field being $\mathbb{C}$, the translations that preserve the open orbit Ω are also translations that preserve the polynomial D, because Ω is the zero set of D. We can thus restrict ourselves to studying this polynomial, which we shall call the *characteristic polynomial* of the algebra $\mathfrak{A}$. Note that if the algebra $\mathfrak{A}$ has a left-neutral element e (in other words, if $\mathrm{R}_e = I$), then the canonical representation fixes the point $-e$ (and is thus isomorphic to the linear representation $a \mapsto \mathrm{L}_a$), and the polynomial D is homogeneous of degree n with respect to that point (in fact, $D(x - e) = \det \mathrm{R}_x$).

The study of the polynomial D goes in two directions. Primarily, we compute the logarithmic derivative with respect to a vector field X on $\mathfrak{A}$. We find that at every point $x \in \Omega$,

$$X.(\log(D)) = \mathrm{tr}(I + \mathrm{R}_x)^{-1}\mathrm{R}_{X(x)}. \tag{4}$$

Since $D(0) = 1$, we can define $\log(D)$ at least in a neighborhood of 0. This is enough to take the series expansion of $\log(D)$ at the point 0. For many authors who have worked on formal series, a *formal series* on the vector space $\mathfrak{A}$ is a linear form on the vector space $\mathrm{S}\,\mathfrak{A}$, the symmetric algebra of $\mathfrak{A}$, that is, the free commutative associative generated by $\mathfrak{A}$, which is also a graded algebra: $\mathrm{S}\,\mathfrak{A} = \bigoplus_p \mathrm{S}^p\,\mathfrak{A}$. This is the point of view we want to adopt in the following. So we associate to the function $\sigma = \log(D)$ the linear form σ' on $\mathrm{S}\,\mathfrak{A}$ defined as follows: First, $\sigma'(1) = \sigma(0) = 0$, then the restriction of σ' to $\mathfrak{A}$ is the differential of σ at 0, then the restriction of σ' to $\mathrm{S}^2\,\mathfrak{A}$, considered as a symmetric bilinear form on $\mathfrak{A}$, is the Hessian of σ at 0, and more generally, if $a_1, \dots, a_p \in \mathfrak{A}$ and if $a_1 \times \cdots \times a_p$ is their product in the algebra $\mathrm{S}\,\mathfrak{A}$, then $\sigma'(a_1 \times \cdots \times a_p)$ is the derivative of order p of σ at 0 with respect to p constant vector fields equal to $a_1, \dots, a_p$. One can show the following formula, in which the sum is taken over all permutations w of $\{1, \dots, p\}$:

$$\sigma'(a \times b_1 \times \cdots \times b_p) = (-1)^p \operatorname{tr}(\sum_w \mathrm{R}_{b_{w(1)}} \cdots \mathrm{R}_{b_{w(p)}} \mathrm{R}_a). \tag{5}$$

This formula (5) follows from (4) in light of the fact that the derivative of $(I + \mathrm{R}_x)^{-1}$ at 0 with respect to constant vector fields $b_1, \dots, b_p$ is equal to

$$(-1)^p \sum_w \mathrm{R}_{b_{w(1)}} \cdots \mathrm{R}_{b_{w(p)}}$$

Our second direction of study of the polynomial D is the following: Given $\gamma \in G$ and $x \in \mathfrak{A}$, if x lies in the open orbit, then the same holds for γx, and vice versa. Hence the polynomial functions $x \mapsto D(x)$ and $x \mapsto D(\gamma x)$ vanish at the same points. And since γ lies in the connected set G, the second must be proportional to the first. There exists thus a map Δ of G to $\mathbb{C}$ such that

$$D(\gamma x) = \Delta(\gamma) D(x)$$

for all $\gamma \in G$ and $x \in \mathfrak{A}$. One can easily show that Δ is a Lie group homomorphism from G to the multiplicative group $\mathbb{C}^\times$. In particular, $\Delta(\gamma 0) = \Delta(\gamma)$.

Let J be the map $J : G \to \mathfrak{A}$, $\gamma \mapsto \gamma 0$. Its differential at the neutral element of the group G is the linear map $\mathfrak{A} \to \mathfrak{A}$, $a \mapsto (a, \mathrm{L}_a)(0)$, that is, the identity map. Thus J is a diffeomorphism in a neighborhood of the neutral element of G to a neighborhood of 0 in $\mathfrak{A}$, and we can consider J as a chart of the differentiable manifold G on a neighborhood of the neutral element. If Φ is a function defined on G, we can associate to it a function F defined on a neighborhood of 0 in $\mathfrak{A}$, and satisfying the identity $\Phi(\gamma) = F(\gamma 0)$. We say that F *represents* Φ in the chart J. The identity $\Delta(\gamma) = D(\gamma 0)$ shows then that the polynomial D represents a Lie group homomorphism $G \to \mathbb{C}^\times$ in the chart J. Regarding $\log(D)$, it represents a Lie group homomorphism $G \to \mathbb{C}$ in the chart J.

Associated to this Lie group homomorphism is a Lie algebra homomorphism from $\mathfrak{A}$ to $\mathbb{C}$. Because of what we know about the differential of J at the neutral element of G, this homomorphism of Lie algebras is the differential of D at 0, and and by (4) this is the map $a \mapsto \operatorname{tr} \mathrm{R}_a$.

Conversely, let s be any Lie algebra homomorphism of $\mathfrak{A}$ to $\mathbb{C}$. We can associate to it a Lie group homomorphism S from G to the additive group $\mathbb{C}$. In the chart J this function S can be represented by a function σ that can be expanded into a series at the point 0, which allows us to define a linear form σ' on $\mathrm{S}\,\mathfrak{A}$ as explained above. It is clear that $\sigma'(1) = \sigma(0) = 0$ and that $\sigma'(a) = s(a)$ if $a \in \mathfrak{A}$.

It is easy to understand the image under J of right invariant vector fields on G. These are the infinitesimal transformations associated to left-translations on G. Via J they correspond to affine transformations $x \mapsto \gamma x$, where $x \in \mathfrak{A}$. The infinitesimal transformations associated to them are the affine vector fields Y_a defined as follows:

$$Y_a(x) = (a, \mathrm{L}_a)(x) = (I + \mathrm{R}_x)(a), \quad \text{with } a, x \in \mathfrak{A}.$$

As the derivative of S with respect to the right-invariant vector fields on G is constant, at every point x where σ is defined, we have

$$Y_a.\sigma = s(a). \tag{6}$$

We can also let the vector field Y_a act on the formal series; so we can write

$$(Y_a.\sigma')(b_1 \times \cdots \times b_p) = \sigma'(a \times b_1 \times \cdots \times b_p) + \sum_{1 \le i \le p} \sigma'(b_1 \times \cdots \times ab_i \times \cdots \times b_p).$$

From (6) we deduce the formula below (where $p \geq 1$):

$$\sigma'(a \times b_1 \times \cdots \times b_p) = -\sum_{1 \le i \le p} \sigma'(b_1 \times \cdots \times ab_i \times \cdots \times b_p), \tag{7}$$

this formula allows to compute the values of σ' by recursion on the subspaces $\mathrm{S}^p\,\mathfrak{A}$ beginning with $\sigma'(a) = s(a)$.

Finally, formula (6) can be written

$$\mathrm{d}\sigma \circ (I + \mathrm{R}_x) = s, \quad \text{or} \quad \mathrm{d}\sigma = s \circ (I + \mathrm{R}_x)^{-1}.$$

Knowing the derivative of $(I + \mathrm{R}_x)^{-1}$ at 0 with respect to the constant vector fields $b_1, \ldots, b_p$, we can deduce

$$\sigma'(a \times b_1 \times \cdots \times b_p) = (-1)^p s(\sum_w \mathrm{R}_{b_{w(1)}} \cdots \mathrm{R}_{b_{w(p)}}(a)) \tag{8}$$

with the same notation as in (5).

The conclusion from these calculations is that σ' satisfies equations (7) and (8) in all cases, and (5) in the case where s is the Lie algebra homomorphism such that $s(a) = \operatorname{tr} \mathrm{R}_a$ for all $a \in \mathfrak{A}$. Note that one can deduce (5) from (8), or vice-versa, from the following identity, which is a consequence of (3):

$$\operatorname{tr}((I + \mathrm{R}_x)^{-1}\mathrm{R}_a) = \operatorname{tr} \mathrm{R}_{(I+\mathrm{R}_x)^{-1}(a)}.$$

We denote by N the set of those $a \in \mathfrak{A}$ such that $\sigma'(az) = 0$ for all $z \in \mathrm{S}\,\mathfrak{A}$. In other words, N is the set of those a such that every translation by a vector parallel to a preserves the function σ. As the set of translations that preserve a polynomial is a vector subspace, in the case where s is the homomorphism $a \mapsto \operatorname{tr} \mathrm{R}_a$, we can say more simply that N is the set of $a \in \mathfrak{A}$ such that the translations by the vector a preserves the characteristic polynomial $D = \exp(\sigma)$. In this case, N is thus the radical of the left-symmetric algebra. For all integers $p \geq 0$, we call M_p the set of $a \in \mathfrak{A}$ such that

$$\sigma'(a \times b_1 \times \cdots \times b_p) = 0$$

for any $b_1, \ldots, b_p \in \mathfrak{A}$. Then M_0 is the kernel of s, and M_1 is the kernel of the bilinear form $(a, b) \mapsto s(ab)$. It is clear that N is the intersection of all the M_p. We further let N_p denote the intersection of M_0, M_1,..., M_p.

Lemma 9 *For all $p \geq 1$, N_p is the set of those $a \in N_{p-1}$ such that $ba \in N_{p-1}$ for all $b \in \mathfrak{A}$.*

PROOF: Formula (7) can be written

$$\begin{aligned}\sigma'(a \times b_1 \times \cdots \times b_p) = & -\sigma'(b_p a \times \cdots \times b_1 \times \cdots \times b_{p-1}) \\ & - \sum_{1 \leq i \leq p-1} \sigma'(a \times b_1 \times \cdots \times b_{p-1}).\end{aligned}$$

If $a \in N_{p-1}$, it follows that

$$\sigma'(a \times b_1 \times \cdots \times b_p) = -\sigma'(b_p a \times b_1 \times \cdots \times b_{p-1}),$$

which shows that a is in N_p if and only if $b_p a$ is in N_{p-1} for any $b_p \in \mathfrak{A}$. $\diamond$

From the above result on easily deduces the following three corollaries.

Corollary 10 *The subspace N is the largest left-ideal that is contained in N_0, the kernel of s.*

Corollary 11 *The three following assertions are equivalent:*

$$N_p = N,$$
$$N_p = N_{p+1},$$
$$N_p \text{ is a left-ideal.}$$

Corollary 12 *There exists an integer $p < n$ such that $N = N_p$.*

We continue our investigations:

Lemma 13 *For all $p \geq 1$, M_p and N_p are left-symmetric subalgebras of $\mathfrak{A}$.*

PROOF: Let $a_1, a_2 \in M_p$. We wish to show that $a_1 a_2 \in M_p$. Applying (7) twice allows us to write:

$$\begin{aligned}
&\sigma'(a_1 \times a_2 \times b_1 \times \cdots \times b_p) \\
&= -\sigma'(a_1 a_2 \times b_1 \times \cdots \times b_p) - \sum_{1 \leq i \leq p} \sigma'(a_2 \times b_1 \times \cdots \times a_1 b_i \times \cdots \times b_p) \\
&= -\sigma'(b_p a_1 \times a_2 \times b_1 \times \cdots \times b_{p-1}) - \sigma'(a_1 \times b_p a_2 \times b_1 \times \cdots \times b_{p-1}) \\
&\quad - \sum_{1 \leq i \leq p-1} \sigma'(a_1 \times a_2 \times b_1 \times \cdots \times b_p b_i \times \cdots \times b_{p-1}).
\end{aligned}$$

The equality of the last two parts gives

$$\sigma'(a_1 a_2 \times b_1 \times \cdots \times b_p) = 0$$

as all the other terms are 0. Thus $a_1 a_2 \in M_p$.

To show that N_p is a subalgebra of $\mathfrak{A}$, take $a_1, a_2 \in N_p$ and look at the quantities $\sigma'(a_1 a_2 \times b_1 \times \cdots \times b_q)$ with $0 \leq q \leq p$. If $q \geq 1$, the preceding calculation remains valid, but if $q = 0$, we write $\sigma'(a_1 a_2) = -\sigma'(a_1 \times a_2) = 0$. ◇

Corollary 14 *If $p \geq 1$ and $q \geq 1$, then*

$$\sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_p) = 0$$

for all $a_1, \ldots, a_q \in M_p$.

PROOF: We proceed by induction on q. If $q = 1$, then there is nothing to show. To pass from q to $q + 1$, we write

$$\begin{aligned}
&\sigma'(a_1 \times \cdots \times a_{q+1} \times b_1 \times \cdots \times b_p) \\
&= -\sum_{0 \leq j \leq q} \sigma'(a_1 \times \cdots \times a_{q+1} \times b_1 \times \cdots \times b_p) - \sum_{1 \leq i \leq p} \sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_p).
\end{aligned}$$

Since $a_{q+1}a_j \in M_p$,

$$\sigma'(a_1 \times \cdots \times a_{q+1} \times b_1 \times \cdots \times b_p) = 0$$

by the induction hypothesis. $\diamondsuit$

We now anounce the results that are the goal of these studies:

Proposition 15 *For all $a, x \in \mathfrak{A}$ and integers $p \geq 1$, we have*

$$\mathrm{tr}(\mathrm{R}_x^p \mathrm{R}_a) = \mathrm{tr}\, \mathrm{R}_{\mathrm{R}_x^p(a)}.$$

Let M_p denote the set of those $a \in \mathfrak{A}$ such that the common value for two of these elements is 0 for all $x \in \mathfrak{A}$. Moreover, M_p is a left-symmetric subalgebra of $\mathfrak{A}$, and for all $a \in M_p$, the map R_a is a nilpotent endomorphism of $\mathfrak{A}$.

PROOF: Suppose s is the homomorphism $a \mapsto \mathrm{tr}\, \mathrm{R}_a$. Note that x^p is the pth power of x in the algebra $\mathrm{S}\,\mathfrak{A}$, so we can write the following equalities, which originate from (5) and (8):

$$\sigma'(a \times x^p) = (-1)^p p!\, \mathrm{tr}(\mathrm{R}_x^p \mathrm{R}_a) = (-1)^p p! s(\mathrm{R}_x^p(a)).$$

The equality claimed in the proposition now follows, and the fact the set M_p that appears in the proposition is in fact that which appears in Lemma 13. The only new result is the fact that R_a is nilpotent if $a \in M_p$. Now for all $r > p$ we can write $\sigma'(a^r) = 0$, because $\sigma'(a^r) = \sigma'(a^{r-p} \times a^p)$ and we can then apply Corollary 14. Now by (5), $\sigma'(a^r) = \pm(r-1)!\, \mathrm{tr}(\mathrm{R}_a^r)$. This results in all the powers of R_a of exponent $> p$ having trace 0. This is not possible unless R_a is nilpotent. $\diamondsuit$

It follows from Proposition 15 that if s is the homomorphism $a \mapsto \mathrm{tr}\, \mathrm{R}_a$, then all the subalgebras M_p are contained in M_0, the kernel of s, so that for all $p \geq 1$ we have

$$N_p = M_1 \cap \ldots \cap M_p.$$

This property does not always hold if s is the homomorphism $a \mapsto \mathrm{tr}\, \mathrm{L}_a$. I remark that if the algebra $\mathfrak{A}$ has a left-neutral element, on can deduce from (7) or (8) that $M_p \subset M_{p-1}$ for any homomorphism s, which implies $N_p = M_p$.

The following proposition constitutes a synthesis of the results obtained up to now on the radical of the algebra $\mathfrak{A}$, and thus a proof is not necessary.

Proposition 16 *In every left-symmetric algebra* $\mathfrak{A}$ *of dimension* n *over* $\mathbb{C}$*, there exists a left-ideal* $\mathfrak{R}(\mathfrak{A})$*, called the radical of* $\mathfrak{A}$*, that is defined by any one of the following properties:*

(a) *The radical is the set of those* $a \in \mathfrak{A}$ *such that the translation by the vector* a *preserves the open orbit* Ω *of the canonical affine action of* G *on* $\mathfrak{A}$*.*

(b) *The radical is the set of those* $a \in \mathfrak{A}$ *such that the translation by the vector* a *preserves the characteristic polynomial* $D(x) = \det(I + \mathrm{R}_x)$*.*

(c) *The radical is the maximal left-ideal contained in the kernel of the linear form* $a \mapsto \operatorname{tr} \mathrm{R}_a$*.*

(d) *The radical is the intersection of the subalgebras* M_p *defined in Proposition 15. More precisely, there exists an integer* $p < n$ *such that*

$$\mathfrak{R}(\mathfrak{A}) = M_1 \cap \ldots \cap M_p.$$

By analogy with the case of associative algebras, we can make the following definition:

Definition 17 A left-symmetric algebra $\mathfrak{A}$ is called *nilpotent* if for all $a \in \mathfrak{A}$, the right-multiplication R_a is nilpotent.

If $\mathfrak{B}$ is a left-ideal of $\mathfrak{A}$, for $\mathfrak{B}$ to be an ideal it is necessary and sufficient that for all $a \in \mathfrak{B}$, R_a is a nilpotent endomorphism on all of $\mathfrak{A}$, for $\mathrm{R}_a(\mathfrak{A}) \subset \mathfrak{B}$. Every nilpotent left-ideal of $\mathfrak{A}$ is thus contained in the kernel of the linear form $a \mapsto \operatorname{tr} \mathrm{R}_a$, and consequently it is contained in $\mathfrak{R}(\mathfrak{A})$. By propositions 15 and 16, $\mathfrak{R}(\mathfrak{A})$ is itself a nilpotent left-ideal. We can now state:

Proposition 18 *The radical of a left-symmetric algebra of finite dimension over* $\mathbb{C}$ *is the largest nilpotent left-ideal.*

We will now establish a property of the radical which involves the homomorphisms $s : \mathfrak{A} \to \mathbb{C}$. I recall the notations: associated to such a homomorphism s is a homomorphism S of the Lie group G to the additive group $\mathbb{C}$. By virtue of the chart J, we can represent S by an analytic function σ defined on a neighborhood of zero in $\mathfrak{A}$.

Proposition 19 *The restriction of σ to any affine subspace of $\mathfrak{A}$ that is parellel to $\mathfrak{R}(\mathfrak{A})$ (meaning of the form $x + \mathfrak{R}(\mathfrak{A})$ for some x in a neighborhood of zero) is a polynomial function of degree $\leq n$. Its degree is strictly less than n if $\mathfrak{A}$ is not nilpotent.*

PROOF: We use σ', the linear form on $\mathrm{S}\,\mathfrak{A}$ that represents the expansion into a series of σ at 0. We show that $\sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_p) = 0$ if $a_1, \ldots, a_q \in \mathfrak{R}(\mathfrak{A})$ with $q > n$ (and $q \geq n$ if $\mathfrak{A}$ is not nilpotent), and $b_1, \ldots, b_p \in \mathfrak{A}$ with $p \geq 0$. Let $r = \dim \mathfrak{R}(\mathfrak{A})$. We start by showing that $\sigma'(a^q) = 0$ for $a \in \mathfrak{R}(\mathfrak{A})$ and $q > r$, which shows that the restriction of σ to $\mathfrak{R}(\mathfrak{A})$ is a polynomial of degree $\leq r$. By (8),

$$\sigma'(a^q) = (-1)^{q-1}(q-1)!s(\mathrm{R}_a^{q-1}(a)),$$

and since R_a is nilpotent, the restriction of R_a^{q-1} to $\mathfrak{R}(\mathfrak{A})$ is zero for $q - 1 \geq r$, so that $\sigma'(a^q) = 0$.

If the algebra $\mathfrak{A}$ is nilpotent, the proof is done: σ is a polynomial of degree $\leq n$, and it is easy to verify that σ is effectively of degree n if $\mathfrak{A}$ is a nilpotent algebra of dimension 1 (thus an algebra with zero product) and s is non-zero.

If the algebra $\mathfrak{A}$ is not nilpotent, we show the equality

$$\sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_p) = 0$$

by induction on p. If $p = 0$, we deduce from the preceeding that this equality holds whenever $p > r$, and thus when $p \geq n$. If $p = 1$, it is sufficient to show that $\sigma'(a^q \times b) = 0$ for $q \geq n$ and $a \in \mathfrak{R}(\mathfrak{A})$, and for this we again use (8):

$$\sigma'(a^q \times b) = (-1)^q q! s(\mathrm{R}_a^q(b)) = 0.$$

To pass from p to $p + 1$ (with $p \geq 1$), we apply (7):

$$\begin{aligned}\sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_{p+1}) = &- \sum_{1 \leq i \leq q} \sigma'(a_1 \times \cdots \times b_{p+1}a_j \times \cdots a_q \times b_1 \times \cdots \times b_p) \\ &- \sum_{1 \leq i \leq q} \sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_{p+1}b_i \times \cdots \times b_p).\end{aligned}$$

From the induction hypothesis and the fact that $\mathfrak{R}(\mathfrak{A})$ is a left-ideal, we obtain that $\sigma'(a_1 \times \cdots \times a_q \times b_1 \times \cdots \times b_{p+1}) = 0$. ◇

Remark $\mathfrak{R}(\mathfrak{A})$ is the union of all vector subspaces B such that the functions σ have polynomial restrictions to the affine subspaces parallel to B. In fact, picture the functions σ, pick the function $\log(D)$ and for its restriction to an affine subspace to be polynomial, it is necessary that it is constant.

From now on, we will no longer need the formal series σ', and I will use this to show some examples of the theory to demonstrate firstly that the radical is not always a two-sided ideal, and then that it is not always identical to the kernel of the bilinear form $(a, b) \mapsto \operatorname{tr} \mathrm{R}_{ab}$ (this kernel is the subalgebra M_1 in the notation of Proposition 15).

For a left-symmetric algebra $\mathfrak{A}$ (possibly associative), put $\mathfrak{B} = \operatorname{End}(\mathfrak{A}) \oplus \mathfrak{A}$. We can give the vector space $\mathfrak{B}$ a left-symmetric algebra structure with the following product:

$$(f, a).(g.b) = (fg + [\mathrm{L}_a, g], ab + f(b)_g(a)),$$

where $a, b \in \mathfrak{A}$ and $f, g \in \operatorname{End}(\mathfrak{A})$. There is no difficulty in verifying firstly that the product is left-symmetric and secondly that the map $(f, a) \mapsto (f + \mathrm{L}_a, a)$ is a Lie algbera isomorphism of $\mathfrak{B}$ on the direct product of the Lie algebras $\operatorname{End}(\mathfrak{A})$ and $\mathfrak{A}$. The left-symmetric algebra $\mathfrak{B}$ contains a left- and right-identity element, namely $(I, 0)$. The canonical affine representation of $\mathfrak{B}$ on $\mathfrak{B}$ is thus isomorphic to a linear representation (since it fixes the point $(-I, 0)$). If we let $D_{\mathfrak{A}}$ and $D_{\mathfrak{B}}$ denote the characteristic polynomials of the algebras $\mathfrak{A}$ and $\mathfrak{B}$, it is possible to show:

$$D_{\mathfrak{B}}(f, a) = \det(I + f)^{n+1} D_{\mathfrak{A}}((I + f)^{-1}(a)).$$

We wish to understand $\mathfrak{R}(\mathfrak{B})$, and we can begin by looking for the subalgebra $M_1(\mathfrak{B})$. After some calculations we arrive at the following result: For any $a \in \mathfrak{A}$ let φ_a denote the endomorphism of $\mathfrak{A}$ defined by

$$\varphi_a(x) = -\frac{1}{n+1} \operatorname{tr}(\mathrm{R}_x) a.$$

Then $M_1(\mathfrak{B})$ is the set of (φ_a, a) with $a \in M_1(\mathfrak{A})$. If the algebra $\mathfrak{A}$ is nilpotent, then $M_1(\mathfrak{B})$ is the set of the $(0, a)$ with $a \in \mathfrak{A}$, and since this set is a left-ideal of $\mathfrak{B}$, it is the radical. Note that if the nilpotent algebra $\mathfrak{A}$ does not have the zero product, then $\mathfrak{R}(\mathfrak{B})$ is not a two-sided ideal of $\mathfrak{B}$.

Consider the case where $\mathfrak{A}$ is not nilpotent. In this case, $\mathfrak{R}(\mathfrak{B}) = \mathbf{0}$, since every left-ideal contained in $M_1(\mathfrak{B})$ is reduced to $\mathbf{0}$. In fact, if $\mathfrak{R}(\mathfrak{B})$ contains a non-zero element (g, b), then $b \neq 0$ (since $g = \varphi_b$). Since $(f, 0).(g, b) = (fg, f(b))$, there exists for all $a \in \mathfrak{A}$ an element of $\mathfrak{R}(\mathfrak{B})$ of the form $(\dots, a)$. But this is impossible (since $M_1(\mathfrak{A}) \neq \mathfrak{A}$). Therefore, $\mathfrak{R}(\mathfrak{B}) = \mathbf{0}$. Nevertheless, it can happen that $\mathbf{0} \neq M_1(\mathfrak{A}) \neq \mathfrak{A}$, which implies $\mathfrak{R}(\mathfrak{B}) \neq M_1(\mathfrak{B})$. Note, however, that $M_2(\mathfrak{B}) = \mathbf{0}$ since the algebra $\mathfrak{A}$ is nilpotent.

The fact that $\mathfrak{R}(\mathfrak{B}) = \mathbf{0}$ implies that no non-zero translation leaves the polynomial $D_{\mathfrak{B}}$ invariant. The fact that $M_1(\mathfrak{B}) \neq \mathbf{0}$ implies that the Hessian of this polynomial is degenerate everywhere. In fact, the Hessian of $\log D_{\mathfrak{B}}$ at 0 has the kernel $M_1(\mathfrak{B})$, and since $M_1(\mathfrak{B}) \subset M_0(\mathfrak{B})$, the kernel of the Hessian at this point contains $M_1(\mathfrak{B})$. As there exists a group of affine transformations of $\mathfrak{B}$ that leaves $D_{\mathfrak{B}}$ invariant up to scalar multiples, and which has an open orbit through the point 0, the Hessian of $D_{\mathfrak{B}}$ is degenerate at every point in $\mathfrak{B}$. Consequently, if $\mathbf{0} \neq M_1(\mathfrak{A}) \neq \mathfrak{A}$, then the polynomial $D_{\mathfrak{B}}(f - I, a) = \det(f)^{n+1} D_{\mathfrak{A}}(f^{-1}(a))$ is homogeneous of degree $n(n+1)$, its Hessian is degenerate at every point, and nevertheless no non-zero translation leaves a point invariant. This type of degeneracy without a doubt merits to be noticed.

I will conclude this report by investigating the nilpotent left-symmetric algebras of finite dimension over $\mathbb{C}$ (or possibly some other field of characteristic 0). Every point of such an algebra has an open orbit and trivial isotropy algebra for the canonical affine representation. The study of such algebras, if the base field is $\mathbb{C}$, amounts to the study of Lie groups that act simply transitively on by affine transformations on a vector space; in fact, if a connected Lie group acts transitively with discrete isotropy group on a simply connected manifold, its action must be simply transitive.

Proposition 20 *If $\mathfrak{A}$ is a nilpotent left-symmetric algebra, then the Lie algebra $\mathfrak{A}$ is solvable.*

PROOF: If the Lie algebra $\mathfrak{A}$ is not solvable, then it contains a non-zero semisimple subalgebra S. Every affine representations of S is equivalent to a linear representation (it has a fixed point). So for the affine representation of $\mathfrak{A}$ on $\mathfrak{A}$ there is a point on $\mathfrak{A}$ whose isotropy subgroup contains S. This is impossible, for every point in $\mathfrak{A}$ has trivial stabilizer subalgebra. $\diamond$

In Proposition 20, we cannot conclude that the Lie algebra $\mathfrak{A}$ is nilpotent. In fact, a non-abelian Lie algebras of dimension 2 (solvable, but not nilpotent) can be endowed with a nilpotent left-symmetric algebra structure.

Corollary 21 *For any finite-dimensional left-symmetric algebra $\mathfrak{A}$ it is impossible that the Lie algebra $\mathfrak{A}$ equals its derived Lie algebra* $[\mathfrak{A}, \mathfrak{A}]$.

PROOF: If $\mathfrak{A} = [\mathfrak{A}, \mathfrak{A}]$, then every Lie algebra homomorphism of $\mathfrak{A}$ into the base field is zero. In particular, $\operatorname{tr} R_a = 0$ for all $a \in \mathfrak{A}$, which implies $\mathfrak{R}(\mathfrak{A}) = \mathfrak{A}$. By Proposition 20, the Lie algebra $\mathfrak{A}$ is solvable, which is impossible since $\mathfrak{A} = [\mathfrak{A}, \mathfrak{A}]$. ◇

Remark It is easy to find a left-symmetric algebra $\mathfrak{X}$ of infinite dimension such that $\mathfrak{X} = [\mathfrak{X}, \mathfrak{X}]$. Think of the algebra of vector fields on a vector space (or a differentiable manifold with torsion and curvature zero), where the left-symmetric product is the coveriant derivative.

For all nilpotent left-symmetric algebras $\mathfrak{A}$ know to the author, the matrices of the endomorphisms R_a can be trigonalize with respect to the same basis of $\mathfrak{A}$. In other words, the associative subalgebra generated by the R_a in $\operatorname{End}(\mathfrak{A})$ is nilpotent. This observations justifies the following definition:

Definition 22 For all left-symmetric algebras $\mathfrak{A}$ and for all integers $p \geq 0$, we denote by $C_p(\mathfrak{A})$ the intersection of the kernels of all the endomorphisms $R_{a_1}, \ldots, R_{a_p}$ (with $a_1, \ldots, a_p \in \mathfrak{A}$) and by $D_p(\mathfrak{A})$ the sum of their images. In particular, $C_0(\mathfrak{A}) = \mathbf{0}$ and $D_0(\mathfrak{A}) = \mathfrak{A}$. The $C_p(\mathfrak{A})$ form an ascending sequence whose union is written $C_\infty(\mathfrak{A})$, and the $D_p(\mathfrak{A})$ form a descending sequence whose intersection is written $D_\infty(\mathfrak{A})$.

It is immediately clear that the equaility $C_p(\mathfrak{A}) = C_{p+1}(\mathfrak{A})$ implies $C_p(\mathfrak{A}) = C_\infty(\mathfrak{A})$, and that the equality $D_p(\mathfrak{A}) = D_{p+1}(\mathfrak{A})$ implies $D_p(\mathfrak{A}) = D_\infty(\mathfrak{A})$. The interest in these two sequences stems from the following two propositions:

Proposition 23 *The $C_p(\mathfrak{A})$ and the $D_p(\mathfrak{A})$ are two-sided ideals of $\mathfrak{A}$.*

PROOF: Clearly they are right-ideals. To show that they are left-ideals, consider the associative subalgebra $\mathfrak{B}$ generated by the R_a, and the ideal $\mathfrak{B}^p$ of $\mathfrak{B}$ generated by the products of p elements of $\mathfrak{B}$. Clearly, $C_p(\mathfrak{A})$ is the intersection of the kernels of all elements of $\mathfrak{B}^p$ and $D_p(\mathfrak{A})$ is the sum of their images. Now equation (3) implies $[L_a, \mathfrak{B}] \subset \mathfrak{B}$ for all $a \in \mathfrak{A}$. It follows that $[L_a, \mathfrak{B}^p] \subset \mathfrak{B}^p$, and this concludes the proof. ◇

Proposition 24 *The following four statements are equivalent:*

(1) $C_\infty(\mathfrak{A}) = \mathfrak{A}$.

(2) $D_\infty(\mathfrak{A}) = \mathbf{0}$.

(3) *There exists an ascending sequence* $(F_0, \ldots, F_k)$ *of vector subspaces of* $\mathfrak{A}$ *such that* $F_0 = \mathbf{0}$, $F_k = \mathfrak{A}$ *and* $F_p\mathfrak{A} \subset F_{p-1}$ *for all* $p = 1, \ldots, k$.

(4) *The associative subalgebra generated by the* R_a *is nilpotent.*

PROOF: It is clear that (1) and (2) imply (3), and that (3) is equivalent to (4). To prove that (3) implies (1) and (2), it is enough to show, by induction on p, that there exists a sequence $(F_0, \ldots, F_k)$ that satisfies the required conditions, since $F_p \subset \mathrm{C}_p(\mathfrak{A})$ and $F_{k-p} \supset \mathrm{D}_p(\mathfrak{A})$. $\diamondsuit$

Unfortunately, not all nilpotent left-symmetric algebras satisfy the conditions of Proposition 24. Here is a counterexample, constructed after an article by Louis Auslander [1]. Consider a three-dimensional algebra that is generated, as a vector space, by three elements a, b, c which multiply in the following fashion:

$$aa = bb = cc = ba = ca = 0,$$
$$bc = cb = a, \quad ab = b, \quad ac = -c.$$

But one can prove, or hope to prove, that the properties in Proposition 24 hold under any of the following stronger hypotheses:

(25-1) The algebra $\mathfrak{A}$ is associative (or at least left- and right-symmetic) and nilpotent.

(25-2) The left-symmetric algebra $\mathfrak{A}$ is nilpotent and such that the multiplications L_a and R_a are Lie algebra derivations.

(25-3) The left-symmetric algebra $\mathfrak{A}$ is nilpotent and the subordinate Lie algebra is nilpotent.

For any of these three hypotheses, it is enough to show that $\mathrm{C}_1(\mathfrak{A}) \neq \mathbf{0}$, given that $\mathfrak{A} \neq \mathbf{0}$. In fact, $\mathrm{C}_{p+1}(\mathfrak{A})$ is the inverse image of $\mathrm{C}_1(\mathfrak{A}/\mathrm{C}_1(\mathfrak{A}))$ under the canonical homomorphism $\mathfrak{A} \to \mathfrak{A}/\mathrm{C}_p(\mathfrak{A})$. If $\mathrm{C}_p(\mathfrak{A}) \neq \mathfrak{A}$, then the algebra $\mathfrak{A}/\mathrm{C}_p(\mathfrak{A})$ is not zero and satisfies the three hypotheses (25). Thus $\mathrm{C}_1(\mathfrak{A}/\mathrm{C}_p(\mathfrak{A})) \neq \mathbf{0}$, and therefore $\mathrm{C}_p(\mathfrak{A}) \neq \mathrm{C}_{p+1}(\mathfrak{A})$. So there exists p such that $\mathrm{C}_p(\mathfrak{A}) = \mathfrak{A}$. It follows immediately from Engel's Theorem that hypothesis (25-1) implies $\mathrm{C}_1(\mathfrak{A}) \neq \mathbf{0}$ if $\mathfrak{A} \neq \mathbf{0}$.

The left-symmetric algebras $\mathfrak{A}$ in which the multiplications L_a and R_a are Lie algebra derivations are those for which $(ab)c = (cb)a$ for all $a, b, c, \in \mathfrak{A}$. Medina [7] showed that if such an algebra is nilpotent and non-zero, then $\mathrm{C}_1(\mathfrak{A}) \neq \mathbf{0}$.

This results in the following final result: A left-symmetric algebra whose multiplications are Lie algebra derivations is the direct sum of the two-sided ideals $\mathrm{C}_\infty(\mathfrak{A})$ and $\mathrm{D}_\infty(\mathfrak{A})$. The second one is a commutative and associative algebra with identity element, and the first is clearly nilpotent.

Before discussing (25-3), I will give a geometric interpretation of $\mathrm{C}_1(\mathfrak{A})$: $\mathrm{C}_1(\mathfrak{A})$ is the kernel of the linear representation $a \mapsto \mathrm{L}_a$. Associated to the canonical affine representation $a \mapsto (a, \mathrm{L}_a)$ is an affine representation of the simply connected Lie group G whose orbit through 0 is the whole space $\mathfrak{A}$ (as the algebra $\mathfrak{A}$ is assumed to be nilpotent). This implies that the action of G on $\mathfrak{A}$ is simply transitive. $\mathrm{C}_1(\mathfrak{A})$ is the algebra of the Lie subgroup of those elements in G that act by translations on $\mathfrak{A}$. Several years ago Auslander made the following conjecture: If a nilpotent Lie acts simply transitively by affine transformations on a vector space, then its center contains a subgroup of non-zero dimension that acts by translations. This conjecture was then proved by Scheunemann [8]; unfortunately, his proof does not convince me.

It seems useful to me to synthesize certain results scattered in [1] and [8], complete them and improve the proofs:

Proposition 26 *Let $\mathfrak{A}$ be a left-symmetric algebra. The following three assertions are equivalent:*

(1) L_a *is nilpotent for all* $a \in \mathfrak{A}$.

(2) *The left-symmetric algebra* $\mathfrak{A}$ *is nilpotent and the Lie algebra* $\mathfrak{A}$ *is nilpotent.*

(3) *The left-symmetric algebra* $\mathfrak{A}$ *is nilpotent, and the group of affine transformations of* $\mathfrak{A}$ *contains an algebraic subgroup whose Lie algebra is formed by the affine operators* (a, L_a).

PROOF: We use the canonical homomorphism from the Lie algebra of affine operators (a, f) of $\mathfrak{A}$ to the Lie algebra of endomorphisms of $\Bbbk \times \mathfrak{A}$ (where $\Bbbk$ is the base field). We associate to (a, f) the map φ such that $\varphi(t, x) = (0, ta + f(x))$ for $t \in \Bbbk$ and $x \in \mathfrak{A}$. We write φ_a for the endomorphism of $\Bbbk \times \mathfrak{A}$ thus associated to (a, L_a).

We prove first that (1) implies (2) and (3). If L_a is nilpotent, the φ_a is nilpotent. Now the map $a \mapsto \varphi_a$ is a faithful linear representation of the Lie algebra $\mathfrak{A}$. This algebra is thus nilpotent. It is well-known that associated to a Lie algebra of nilpotent endomorphisms is an algebraic subgroup of the linear transformations.

These transformations preserve the affine hyperplane $1 \times \mathfrak{A}$ and determine an algebraic subgroup of affine transformations of $\mathfrak{A}$ whose Lie algebra consists of the (a, L_a). Note that the trace of L_a is zero for all $a \in \mathfrak{A}$, and so is the trace of $\mathrm{L}_a - \mathrm{R}_a$ (because the algebra is nilpotent), and therefore the trace of R_a is zero as well. It follows that the left-symmetric algebra $\mathfrak{A}$ is nilpotent.

We show that (2) implies (1). If the Lie algebra $\mathfrak{A}$ is nilpotent, we can decompose $\Bbbk \times \mathfrak{A}$ into a direct sum of two subspaces U and V that are invariant under all φ_a, such that on the one hand the restriction of every φ_a to U is nilpotent, on the other hand the restriction to V is bijective for at least one φ_a (see [2], chapitre VII, §1, 3, or [6], chapitre II, §4). As $\varphi_a(\Bbbk \times \mathfrak{A}) \subset 0 \times \mathfrak{A}$, we have $V \subset 0 \times \mathfrak{A}$. Hence the intersection of U with the hyperplane $1 \times \mathfrak{A}$ is not empty and of the form $1 \times B$, where B is an affine subspace of $\mathfrak{A}$. This B is invariant under the affine representation $a \mapsto (a, \mathrm{L}_a)$, since $1 \times B$ is invariant under the linear representation $a \mapsto \varphi_a$. As the left-symmetric algebra $\mathfrak{A}$ is nilpotent, it follows that $B = \mathfrak{A}$. Hence $U = \Bbbk \times \mathfrak{A}$. This implies that the φ_a are nilpotent, and so are the L_a.

Finally we show that (3) implies (2). If assertion (3) holds, then there exists an algebraic subgroup Γ of linear transformations of $\Bbbk \times \mathfrak{A}$, whose Lie algebra is the one formed by the φ_a. As an algebraic group, Γ is the semidirect product of a nilpotent algebraic group Γ_0 and a reductive algebraic group Γ_1. As Γ preserves the hyperplane $0 \times \mathfrak{A}$, Γ_1 perserves a complementary line of this hyperplane. This complementary line intersects the affine hyperplane $1 \times \mathfrak{A}$ in a point $(1, a)$ that is invariant under Γ_1. As the left-symmetric algebra $\mathfrak{A}$ is nilpotent, the isotropy subgroup of $(1, a)$ in Γ is discrete (and trivial if Γ is connected). Thus Γ_1 is a discrete subgroup, and the Lie algebra of Γ is nilpotent. ◇

Can the methods used here further be used if the base field is an arbitrary field of characteristic 0? It is easy to show in a purely algebraic way that there exists a formal series σ' satisfying the identities (7) and (8). The only problem is that these methods require us to define the radical by means of the translations that preserve the characteristic polynomial, without allowing us to know whether these also form the set of translations that preserve the "open orbits", that is, the set of points where this polynomial is non-zero. In particular, I am not able to say if the following two assertions are equivalent:

(a) The canonical affine representation is "simply transitive", that is, the characteristic polynomial is non-zero everywhere.

(b) The algebra is nilpotent, or in other words, the characteristic polynomial is constant.

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