ON A REMARKABLE HERMITIAN METRIC

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1.

When studying the invariants of a real $2n$-dimensional Hermitian metric[1]

$$ds^2 = \sum g_{i\overline{k}} dx_i d\overline{x}_k \tag{1}$$

with respect to the "pseudo-conformal" transformations

$$\begin{array}{rcl} x'_i & = & \varphi_i(x_1, x_2, \dots, x_n) \\ \overline{x}'_i & = & \overline{\varphi}_i(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n) \end{array} \quad (i = 1, 2, \dots, n) \tag{2}$$

it seems natural to study, aside from (1), the alternating quadratic differential form (forme extérieure)

$$\omega = \sum g_{i\overline{k}} d(x_i, \overline{x}_k),$$

in which $d(x_i, \overline{x}_k)$ denotes the so-called exterior product[2] of the differentials $dx_i, d\overline{x}_k$, that is, a differential determinant $\frac{\partial(x_i, \overline{x}_k)}{\partial(s,t)} dsdt$. This form ω which is invariantly related to (1) allows to use the elegant calculus of symbolic differential forms[3] to construct new invariants. For example, the derivative

$$\omega' = \sum d(g_{i\overline{k}}, x_i, \overline{x}_k) = \sum \frac{\partial g_{i\overline{k}}}{\partial x_l} d(x_l, x_i, \overline{x}_k) + \frac{\partial g_{i\overline{k}}}{\partial \overline{x}_l} d(\overline{x}_l, x_i, \overline{x}_k)$$

is a new invariant form, and by combining it with ω one obtains further invariants.

In this approach, the case $\omega' = 0$ appears as a remarkable exception. We find that the metric can be derived in the following way

$$ds^2 = \sum \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k \tag{3}$$

from a "potential" U, which evidently is an invariant property equivalent to $\omega' = 0$.

This type is associated to certain metrics arising in the theory of automorphic functions. In fact, if

$$x'_i = \frac{L_i(x)}{L_0(x)} = \frac{\alpha_{i0} + \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n}{\alpha_{00} + \alpha_{01}x_1 + \ldots + \alpha_{0n}x_n} \qquad (i = 1, 2, \dots, n) \tag{4}$$

is a projective transformation mapping the unit hypersphere

$$1 - x_1\overline{x}_1 - x_2\overline{x}_2 - \ldots - x_n\overline{x}_n = 0$$

to itself, then metric

$$ds^2 = \sum \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k$$

formed with

$$U = k \log \left(1 - \sum x_\nu \overline{x}_\nu\right)$$

[1] $\overline{x}$ denotes the complex conjugate of some quantity x. The indices referring to a conjugate variable $\overline{x}$ will always be overlined.

[2] To avoid confusion with the usual multiplication, the exterior products of differentials dx, dy, dz are written as $d(x, y, z)$.

[3] Cartan, Invariants intégraux, Chap. VI, VII. Paris (1922). – Goursat, Leons sur le problème de Pfaff, Chap. III. Paris (1922).

is invariant with respect to the group of "hyperfuchsian" transformations (4) because of

$$\Big(1-\sum_\nu x'_\nu \overline{x}'_\nu\Big) = \Big(1-\sum_\nu x_\nu \overline{x}_\nu\Big)\cdot \big(L_0(x)\overline{L}_0(\overline{x})\big)^{-1}.$$

In an analogous manner the "hyperabelian" transformations

$$x'_i = \frac{\alpha_i x_i + \beta_i}{\gamma_i x_i + \delta_i} \qquad (i = 1, 2, \ldots, n),$$

fixing the unit spheres

$$1 - x_i\overline{x}_i = 0 \qquad (i = 1, 2, \ldots, n)$$

leave invariant the metric derived from the potential

$$U = \sum_{i=1}^{n} k_i \log(1 - x_i\overline{x}_i) \qquad (k_i \text{ constant}),$$

and it is clear that the intermediate cases, for example the group composed from hyperfuchsian transformations in r and s (with $r + s = n$) variables, give rise to metrics of if this type.

These relations will justify studying the metric (3) in the following; as many things in the algebra of automorphic forms depend on the properties of this metric.

At last note that

$$ds^2 = \sum \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k$$

represents a solution to Einstein's equations of gravity

$$R_{\alpha\beta} = \lambda g_{\alpha\beta}$$

if the potential U statisfies

$$\left|\frac{\partial^2 U}{\partial x_i \partial \overline{x}_k}\right| = \mathrm{e}^{\lambda U}.$$

2.

The proof that $\omega' = 0$ implies (3) is easy. By equating the coefficients of ω' to 0 we obtain

$$\frac{\partial g_{i\overline{k}}}{\partial x_l} - \frac{\partial g_{l\overline{k}}}{\partial x_i} = 0, \qquad \frac{\partial g_{i\overline{k}}}{\partial \overline{x}_l} - \frac{\partial g_{i\overline{l}}}{\partial \overline{x}_k} = 0,$$

whereafter we may set

$$g_{i\overline{k}} = \frac{\partial V_i}{\partial \overline{x}_k} = \frac{\partial V_{\overline{k}}}{\partial x_i}.$$

The integrability conditions

$$\frac{\partial^2 V_{\overline{k}}}{\partial x_i \partial x_l} = \frac{\partial^2 V_i}{\partial \overline{x}_k \partial x_l} = \frac{\partial^2 V_l}{\partial \overline{x}_k \partial x_i},$$

i.e.

$$\frac{\partial}{\partial \overline{x}_k}\left(\frac{\partial V_i}{\partial x_l} - \frac{\partial V_l}{\partial x_i}\right) = 0,$$

then show that the expressions $\frac{\partial V_i}{\partial x_l} - \frac{\partial V_l}{\partial x_i}$ do not depend on the $\overline{x}_l$:

$$\frac{\partial V_i}{\partial x_l} - \frac{\partial V_l}{\partial x_i} = \varphi_{il}(x_1, x_2, \ldots, x_n). \tag{5}$$

Now one can determine n functions $V'_1, V'_2, \ldots, V'_n$ of $x_1, x_2 \ldots, x_n$ such that

$$\frac{\partial V'_i}{\partial x_l} - \frac{\partial V'_l}{\partial x_i} = \varphi(x_1, x_2, \ldots, x_n)$$

holds; for these equations may be combined to

$$\left(-\sum V_i' dx_i\right) = \sum \varphi_{il} d(x_i, x_l).$$

A neccessary and sufficient condition for a differential form $\sum \varphi_{il}(x_1, \dots, x_n) d(x_i, x_l)$ to be representable as the differential of another one only containing the variable x is the vanishing of the differential

$$\left(\sum \varphi_{il} d(x_i, x_l)\right)' = \sum \frac{\partial \varphi_{il}}{\partial x_m} d(x_m, x_i, x_l) = 0,$$

which clearly is the case because of the left hand side of (5).

Clearly one can, without violating the equations

$$g_{i\overline{k}} = \frac{\partial V_i}{\partial \overline{x}_k},$$

replace V_i by $V_i - V_i'$ with the consequence that the new V_i satisfy the equations

$$\frac{\partial V_i}{\partial x_l} - \frac{\partial V_l}{\partial x_i} = 0.$$

Thus the V_i can be represented in the form

$$V_i = \frac{\partial W}{\partial \overline{x}_i} + \psi_i(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)$$

and for the $g_{i\overline{k}}$ one obtains

$$g_{i\overline{k}} = \frac{\partial V_i}{\partial \overline{x}_k} = \frac{\partial^2 W}{\partial x_i \partial \overline{x}_k} + \frac{\partial \psi_i}{\partial \overline{x}_k} = \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k},$$

when setting

$$U = W + x_1\psi_1 + x_2\psi_2 + \dots + x_n\psi_n.$$

One sees immediately that for any metric

$$ds^2 = \sum \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k \tag{6}$$

the corresponding exterior differential form

$$\omega = \sum \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} d(x_i, \overline{x}_k)$$

is integrable ($\omega' = 0$).

3.

We now wish to compute the Riemannian curvature tensor and in particular the contracted curvature tensor $R_{\alpha\beta}$ of the metric (3). To be consistent with common notations, it is recommended to include a factor 2:

$$ds^2 = 2\frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k = g_{\alpha\beta} dx_\alpha dx_\beta.$$

Greek letters shall henceforth denote indices capable of assuming all of the $2n$ values $1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}$, whereas Latin letters denote numbers of the sequence $1, 2, \dots, n$, the overlined letters denote the numbers $\overline{1}, \overline{2}, \dots, \overline{n}$. Furthermore, by common convention the summation signs are omitted.

The terms g_{ik} and $g_{\overline{ik}}$ are missing in the quadratic form $g_{\alpha\beta} dx_\alpha dx_\beta$ and thus

$$g^{ik} = 0, \quad g^{\overline{ik}} = 0,$$

whereas

$$g^{i\overline{k}} = \frac{D^{i\overline{k}}}{D},$$

and $D^{i\bar{k}}$ is understood to be the minor associated to the element $U_{i\bar{k}}$,[4] endowed with the correct sign, of

$$D = D(U) = |U_{i\bar{k}}|.$$

The Christoffel symbols $\left[{}^{\alpha}{}_{\gamma}{}^{\beta}\right]$ are easiest computed as the coefficients of $\dot{x}_\alpha$, $\dot{x}_\beta$ in the Lagrangian expression

$$P_\gamma = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_\gamma}\right) - \frac{\partial T}{\partial x_\gamma}$$

with kinetic energy

$$T = \frac{1}{2}\dot{s}^2 = U_{i\bar{k}}\dot{x}_i\dot{\overline{x}}_k.$$

One finds

$$\begin{bmatrix} i & k \\ & \bar{l} \end{bmatrix} = U_{ik\bar{l}}, \qquad \begin{bmatrix} \bar{i} & \bar{k} \\ & l \end{bmatrix} = U_{\bar{i}\bar{k}l},$$

and all other Christoffel symbols are 0. Thereby the curvature tensor is also known; only the components of type $R_{i\bar{k},l\overline{m}}$ are different from 0,[5] more precisely

$$R_{i\bar{k},l\overline{m}} = -U_{i\bar{k}l\overline{m}} + g^{r\bar{s}}U_{il\bar{s}}U_{\bar{k}\overline{m}r}.$$

Of the contracted tensor

$$R_{\alpha\beta} = R_{\alpha}{}^{\gamma}{}_{\beta\gamma}$$

therefore only the components $R_{i\bar{k}}$ with mixed indices remain; for it is

$$R_{i\beta} = R_{i}{}^{\gamma}{}_{\beta\gamma} = g^{\gamma\delta}R_{i\delta\beta\gamma} = -g^{l\overline{m}}R_{i\overline{m}l\beta},$$

which is different from 0 only for $\beta = \bar{k}$. In

$$R_{i\bar{k}} = g^{l\overline{m}}R_{i\overline{m}l\bar{k}} = g^{l\overline{m}}U_{i\bar{k}l\overline{m}} - g^{l\overline{m}}g^{r\bar{s}}U_{il\bar{s}}U_{\bar{k}\overline{m}r}$$

the expression on the right hand side can be elegantly combined to

$$\frac{\partial^2}{\partial x_i \partial \overline{x}_k}\log(D(U)),$$

which we want to confirm by a computation. We have

$$\frac{\partial D}{\partial x_i} = D^{r\bar{s}}U_{r\bar{s}i},$$

$$\frac{\partial^2 D}{\partial x_i \partial \overline{x}_k} = D^{r\bar{s}}U_{r\bar{s}i\bar{k}} + U_{r\bar{s}i}U_{l\overline{m}\bar{k}}D^{l\overline{m}r\bar{s}},$$

where $D^{l\overline{m}r\bar{s}}$ denotes the second minor of D associated to $\left|\begin{smallmatrix} U_{r\bar{s}} & U_{r\overline{m}} \\ U_{l\bar{s}} & U_{l\overline{m}} \end{smallmatrix}\right|$. It follows that

$$\frac{\partial^2}{\partial x_i \partial \overline{x}_k}\log(D) = \frac{1}{D}\frac{\partial^2 D}{\partial x_i \partial \overline{x}_k} - \frac{1}{D^2}\frac{\partial D}{\partial x_i}\frac{\partial D}{\partial \overline{x}_k}$$

$$= \frac{D^{r\bar{s}}}{D}U_{r\bar{s}i\bar{k}} + U_{r\bar{s}i}U_{l\overline{m}\bar{k}}\left(\frac{D^{l\overline{m}r\bar{s}}}{D} - \frac{D^{r\bar{s}}D^{l\overline{m}}}{D^2}\right),$$

and because of the easily seen determinant identity[6]

$$DD^{l\overline{m}r\bar{s}} - D^{r\bar{s}}D^{l\overline{m}} + D^{r\overline{m}}D^{l\bar{s}} = 0$$

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[4]For simplicity, the derivatives of a function shall henceforth be denoted by attaching the subscript indices of the differentiation variables, e.g. $\frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} = U_{i\bar{k}}$.

[5]And of course $R_{\bar{i}k,\bar{l}m}$ etc.

[6]Fix $r, \bar{s}, l$ on the left hand side of the expression and denote it by $Z^{\overline{m}}$. One has n homogeneous equations in the immediately following equations $\sum U_{i\overline{m}}Z^{\overline{m}} = 0$ (for $i \neq r$) together with $Z^{\bar{s}} = 0$, from which $Z^{\overline{m}} = 0$ follows.

this is equivalent to

$$g^{r\bar{s}}U_{r\bar{s}i\bar{k}} - g^{r\overline{m}}g^{l\bar{s}}U_{r\bar{s}i}U_{l\overline{m}i\bar{k}} = R_{i\bar{k}}.$$

One obtains

$$R_{i\bar{k}} = \frac{\partial^2}{\partial x_i \partial \overline{x}_k} \log(D(U)). \tag{7}$$

4.

The tensor $R_{i\bar{k}} = \frac{\partial^2 V}{\partial x_i \partial \overline{x}_k}$, $V = \log(D(U))$, gives rise to a second alternating differential form

$$\Omega = R_{i\bar{k}} d(x_i, \overline{x}_k) = V_{i\bar{k}} d(x_i, \overline{x}_k),$$

whose differential Ω' vanishes, because Ω is of the same type as

$$\omega = U_{i\bar{k}} d(x_i, \overline{x}_k).$$

Moreover, both forms can be represented as differentials of forms of degree 1:

$$\begin{aligned} \omega &= -(U_i dx_i)' = (U_{\bar{k}} d\overline{x}_k)' \\ \Omega &= -(V_i dx_i)' = (V_{\bar{k}} d\overline{x}_k)'. \end{aligned} \tag{8}$$

Via exterior multiplication, these two forms ω and Ω give rise to a series of invariant differential forms, among which the forms[7] of degree 2

$$\Omega^\nu \omega^{n-\nu}, \qquad (\nu = 0, 1, 2, \ldots, n) \tag{9}$$

of the form

$$A \cdot d(x_1, \overline{x}_1, x_2, \ldots, \overline{x}_n)$$

are of some interest.

Firstly,

$$\omega^n = n! \cdot |U_{i\bar{k}}| \cdot d(x_1, \overline{x}_1, x_2, \ldots, \bar{(x)}_n)$$

equals the volume form up to a constant factor,

$$dv = \sqrt{|g_{\alpha\beta}|} d(x_1, \overline{x}_1, x_2, \ldots, \bar{(x)}_n) = |U_{i\bar{k}}| d(x_1, \overline{x}_1, x_2, \ldots, \bar{(x)}_n).$$

Further,

$$\begin{aligned} \Omega\omega &= (n-1)! \cdot R_{i\bar{k}} \cdot D^{i\bar{k}} \cdot d(x_1, \overline{x}_1, x_2, \ldots, \bar{(x)}_n) \\ &= (n-1)! \cdot R_{i\bar{k}} \cdot g^{i\bar{k}} \cdot D \cdot d(x_1, \overline{x}_1, x_2, \ldots, \bar{(x)}_n) \\ &= (n-1)! \cdot R \cdot dv, \end{aligned}$$

where R denotes the curvature scalar.

The integrals $\int \Omega^\nu \omega^{n-\nu}$ can be transformed into boundary integrals by writing $\Omega^\nu \omega^{n-\nu}$ as differential oof a form, which is possible in several different ways. For example, one has

$$\omega^n = U_{i\bar{k}} d(x_i, \overline{x}_k) \omega^{n-1} = (U_{\bar{k}} d\overline{x}_k)' \omega^{n-1} = (U_{\bar{k}} d\overline{x}_k \omega^{n-1})',$$

because $\omega' = 0$; alternatively

$$\omega^n = -(U_i dx_i \omega^{n-1})'.$$

The integral $\int R dv$ cannot be transformed into a boundary integral for a general metric (with $n > 1$), because the variation of this integral in more than 2 dimesions yields Einstein's equations of gravity.

But for our metrics one has

$$(n-1)! \int R dv = \int \Omega \omega^{n-1} = \int (V_{\bar{k}} d\overline{x}_k)' \omega^{n-1} = \int (V_{\bar{k}} d\overline{x}_k \omega^{n-1})'$$

[7]There should be no confusion if symbolically multiplied forms are simply written next to each other. One has to observe the order and possibly change the sign when permuting factors.

so that

$$\int_C R dv = \frac{1}{(n-1)!}\int_{\Re(C)} V_{\overline{k}} d\overline{x}_k \omega^{n-1} \tag{10}$$

holds by Stokes' Theorem, where C is understood to be a $2n$-dimensional cell with boundary $\Re(C)$. Also,

$$\int_C R dv = \frac{1}{(n-1)!}\int_{\Re(C)} V_i dx_i \omega^{n-1}$$

and by adding (10) one obtains with

$$\int_C R dv = \frac{1}{2\cdot(n-1)!}\int_{\Re(C)} (V_{\overline{k}} d\overline{x}_k - V_i dx_i)\omega^{n-1} \tag{11}$$

a real expression for the integrand on the right hand side.

This formula is reminiscent of the Gauss-Bonnett-Theorem, but for a proper analogy the invariance of the boundary integrand is missing. Under a pseudo-conformal transformation (2) $V = \log(D(U))$ becomes

$$V' = V + \log(\Delta) + \log(\overline{\Delta}), \qquad \Delta = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

that is, $V_{\overline{k}} d\overline{x}_k - V_i dx_i$ becomes

$$\frac{\partial V}{\partial \overline{x}'_k} d\overline{x}'_k - \frac{\partial V}{\partial x'_i} dx'_i + d\log\left(\frac{\overline{\Delta}}{\Delta}\right)$$

because of

$$\frac{\partial \overline{\Delta}}{\partial x'_i} = \frac{\partial \Delta}{\partial \overline{x}'_k} = 0.$$

The whole integrand thus changes by

$$\left(d\log\left(\frac{\overline{\Delta}}{\Delta}\right)\right)\omega^{n-1},$$

because

$$\omega = \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} d(x_i, \overline{x}_k) = \frac{\partial^2 U}{\partial x'_i \partial \overline{x}'_k} d(x'_i, \overline{x}'_k)$$

remains unchanged.

If one were to find a covariant A associated to a $2n-1$-dimensional hypersurface in such a way that A changes by

$$\log\left(\frac{\overline{\Delta}}{\Delta}\right)$$

under a pseudo-conformal transformation, then the integrand

$$(V_{\overline{k}} d\overline{x}_k - V_i dx_i)\omega^{n-1}$$

can be made invariant by adding

$$dA \cdot \omega^{n-1}$$

without essentially destroying equation (11). Because $dA\cdot\omega^{n-1}$ is integrable ($(dA \cdot \omega^{n-1})' = 0$), the integral $\int dA \cdot \omega^{n-1}$ over the boundary-free hypersurface $\Re(C)$ is invariant under continuous deformations of $\Re(C)$.

For $n = 1$, that is a 2-dimensional metric, it is easy to find such a covariant A. If $x = x(t)$, $\overline{x} = \overline{x}(t)$ is the equation of a line, then

$$A = -\log\frac{\frac{dx}{dt}}{\frac{d\overline{x}}{dt}} = -2\arg\frac{dx}{dt}$$

has the desired property, and the expression $\frac{\partial V}{\partial \overline{x}}\frac{d\overline{x}}{dt} - \frac{\partial V}{\partial x}\frac{dx}{dt}$ corrected by dA, that is

$$\frac{\frac{d^2\overline{x}}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{d\overline{x}}{dt}}{\frac{dx}{dt}\frac{d\overline{x}}{dt}} + \frac{U_{x\overline{x}\overline{x}}}{U_{x\overline{x}}}\frac{d\overline{x}}{dt} - \frac{U_{xx\overline{x}}}{U_{x\overline{x}}}\frac{dx}{dt},$$

is essentially the geodesic curvature of the boundary line $\Re(C)$. The integral $\int dA$ takes the value -4π when running once through $\Re(C)$ with positive orientation. Note further that every 2-dimensional metric can be written in the form

$$ds^2 = \frac{\partial^2 U}{\partial x \partial \overline{x}} dx d\overline{x} = \Delta U(dx_1^2 + dx_2^2) \qquad (x = x_1 + ix_2).$$

It would be of major importance for the theory of automorphic functions if the curvature integral (11) or any of the other integrals $\int \Omega^\nu \omega^{n-\nu}$ could be transformed into a boundary integral invariant under pseudo-conformal transformations as described above.

5.

The simple form (7) of the contracted curvature tensor $R_{i\overline{k}}$ allows one to guess a solution to Einstein's equations

$$R_{\alpha\beta} = \lambda g_{\alpha\beta}. \tag{12}$$

For non-mixed indices $(\alpha, \beta) = (i, k)$ or $(\alpha, \beta) = (\overline{i}, \overline{k})$ these equations are always satisfied for the metric

$$ds^2 = 2\frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k, \tag{13}$$

and the equations

$$R_{i\overline{k}} = \lambda g_{i\overline{k}}$$

require:

$$\frac{\partial^2}{\partial x_i \partial \overline{x}_k} \log(D(U)) = \lambda \frac{\partial^2 U}{\partial x_i \partial \overline{x}_k}.$$

Thus, the difference $\log(D(U)) - \lambda U = \varphi$ has to be an "n-harmonic" function, that is, it has to satisfy the differential equations characterising the real parts of analytic functions,

$$\frac{\partial^2 \varphi}{\partial x_i \partial \overline{x}_k} = 0.$$

Every such function can be written in the form

$$\varphi = \psi(x_1, x_2, \ldots, x_n) + \overline{\psi}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n).$$

The equation

$$\log(D(U)) - \lambda U = \psi(x_1, \ldots, x_n) + \overline{\psi}(\overline{x}_1, \ldots, \overline{x}_n) \tag{14}$$

clearly is also sufficient for (13) to satisfy the equations of gravitation (12).

The latter equation can be simplified by a pseudo-conformal transformation. Under such a transformation $\log(D(U))$ changes by

$$\log \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1', x_2', \ldots, x_n')} + \log \frac{\partial(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)}{\partial(\overline{x}_1', \overline{x}_2', \ldots, \overline{x}_n')},$$

and one can always find a transformation

$$x_i' = x_i'(x_1, x_2, \ldots, x_n)$$

such that

$$\frac{\partial(x_1', x_2', \ldots, x_n')}{\partial(x_1, x_2, \ldots, x_n)} = \psi(x_1, x_2, \ldots, x_n)$$

holds. Thus we obtain $\psi = \overline{\psi} = 0$ in (14).

If one is only looking for the essentially distinct metrics satisfying (12), then it is sufficient to restrict oneself to the solutions of the equation

$$D(U) = e^{\lambda U}.$$

The metric belonging to

$$U = -\log\left(1 - \sum x_\nu \overline{x}_\nu\right),$$

invariant under hyper-fuchsian transformations, is a solution to Einstein's equations. For one has

$$U_{i\overline{k}} = \frac{\overline{x}_i x_k}{S^2} - \frac{\delta_{ik}}{S},$$

$S = \sum_k x_k \overline{x}_k - 1$, $\delta_{ik} = 0$ $(i \neq k)$, $\delta_{ii} = 1$. The determinant in

$$D(U) = S^{-2n} \begin{vmatrix} \overline{x}_1 x_1 - S & \overline{x}_1 x_2 & \cdots & \overline{x}_1 x_n \\ \overline{x}_2 x_1 & \overline{x}_2 x_2 - S & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \overline{x}_n x_1 & \cdots & \cdots & \overline{x}_n x_n - S \end{vmatrix}$$

is computed most easily by considering it to be the determinant of the homogeneous system of equations

$$\sum_k \overline{x}_i x_k Y_k = \Lambda \cdot Y_i \tag{15}$$

with the unknowns Y_i for $\Lambda = S$. From (15) it follows that

$$\overline{x}_i \cdot \Bigl(\sum_k x_k Y_k\Bigr) = \Lambda \cdot Y_i,$$

for which the zero-dimensional solution is

$$Y_1 : Y_2 : \ldots : Y_n = \overline{x}_1 : \overline{x}_2 : \ldots : \overline{x}_n \tag{16}$$

and

$$\sum x_k Y_k = 0$$

the $n-1$-dimensional solutions. Hence the eigenvalue $\Lambda = 0$ has to be at least $n-1$-fold and as we have the additional eigenvalue $\Lambda = \sum_k x_k \overline{x}_k$ from (16), $\Lambda = 0$ is precisely $n-1$-fold. So

$$|\overline{x}_i x_k - \delta_{ik}\Lambda| = (-\Lambda)^{n-1} \cdot \Bigl(\sum_k x_k \overline{x}_k - \Lambda\Bigr),$$

and for $\Lambda = S = \sum_k x_k \overline{x}_k - 1$ we obtain as value the determinant in question:

$$\Bigl(\sum_k x_k \overline{x}_k\Bigr)^{n-1},$$

so that

$$D(U) = \Bigl(\sum_k x_k \overline{x}_k\Bigr)^{n-1} = e^{(n+1)U} \tag{17}$$

holds. For the hyper-abelian metric formed with

$$U = -\sum_i \log(1 - x_i \overline{x}_i),$$

a similar equation holds:

$$D(U) = e^{2U}. \tag{18}$$

6.

The differential expression $D(U)$ can be obtained by variation of an integral

$$\int \mathfrak{C}(U) d(x_1, \overline{x}_1, x_2, \ldots, \overline{x}_n),$$

thus reducing the problem of solving the equation

$$D(U) = e^{kU}$$

to a variational problem.

The integrand $\mathfrak{C}(U)$ is Levi's integral invariant, generalised by Wirtinger[8] to n variables, which appears in the theory of 3-dimensional singularties of analytical functions in two variables:

$$\mathfrak{C}(U) = \begin{vmatrix} 0 & U_{\overline{1}} & U_{\overline{2}} & \cdots & U_{\overline{n}} \\ U_1 & U_{1\overline{1}} & U_{1\overline{2}} & \cdots & U_{1\overline{n}} \\ U_2 & U_{2\overline{1}} & U_{2\overline{2}} & \cdots & U_{2\overline{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_n & U_{n\overline{1}} & U_{n\overline{2}} & \cdots & U_{n\overline{n}} \end{vmatrix}.$$

In the symbolism of differential forms one has

$$(n-1)!\mathfrak{C}(U) d(x_1, \overline{x}_1, x_2, \ldots, \overline{x}_n) = (U_i dx_i)(U_{\overline{k}} d\overline{x}_k)(U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1},$$

and the variation of the integral

$$I = \int \omega = \int (U_i dx_i)(U_{\overline{k}} d\overline{x}_k)(U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1}$$

yields (setting $\delta U = v$)

$$\delta I = \int \delta\omega$$

with

$$\begin{aligned} \delta\omega = {} & (v_i dx_i) \cdot (U_{\overline{k}} d\overline{x}_k) \cdot (U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1} \\ & + (U_i dx_i) \cdot (v_{\overline{k}} d\overline{x}_k) \cdot (U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1} \\ & + (n-1) \cdot (U_i dx_i) \cdot (U_{\overline{k}} d\overline{x}_k) \cdot (v_{l\overline{m}} d(x_l, \overline{x}_m)) \cdot (U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-2}. \end{aligned}$$

One convinces oneself by computations using the rules for differentials of symbolic differential forms that

$$\begin{aligned} \delta\omega = {} & -(n+1) \cdot v \cdot (U_{i\overline{k}} d(x_i, \overline{x}_k))^n \\ & +(n-1) \cdot \big((U_i dx_i) \cdot (U_{\overline{k}} d\overline{x}_k) \cdot (v_l dx_l) \cdot (U_{r\overline{s}} d(x_r, \overline{x}_s))^{n-2}\big)' \\ & -n \cdot \big(v \cdot (U_i dx_i) \cdot (U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1}\big)' \\ & +\big(v \cdot (U_{\overline{k}} d\overline{x}_k) \cdot (U_{l\overline{m}} d(x_l, \overline{x}_m))^{n-1}\big)' \end{aligned} \tag{19}$$

holds, so that δI coincides with

$$-(n+1)! \int D(U) \cdot v \cdot d(x_1 \overline{x}_1, x_2, \ldots, \overline{x}_n)$$

up to a boundary integral.

The equation

$$D(U) = e^{kU}$$

is thus obtained by the variation of

$$\int \left(\mathfrak{C}(U) + \frac{n(n+1)}{k} e^{kU} \right) d(x_1 \overline{x}_1, x_2, \ldots, \overline{x}_n). \tag{20}$$

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[8]Wirtinger, Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen. Math. Ann. 97 (1926), p. 363.

If the metric to be determined

$$ds^2 = 2\frac{\partial^2 U}{\partial x_i \partial \overline{x}_k} dx_i d\overline{x}_k$$

is positive definite, the integrand in (20) is positive; because then the quadratic form

$$g^{i\overline{k}} X_i \overline{X}_k$$

is positive definite as well, and

$$g^{i\overline{k}} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial \overline{x}_k} = \frac{D^{i\overline{k}}}{D} U_i U_{\overline{k}}$$

is identical to

$$\frac{\mathfrak{C}(U)}{D(U)}.$$

In the theory of hyper-abelian functions the equation

$$D(U) = 0 \tag{21}$$

is of importance, to be solved under the contraint

$$\mathfrak{C}(U) > 0.$$

By the above we have in

$$\int \mathfrak{C}(U) d(x_1 \overline{x}_1, x_2, \dots, \overline{x}_n)$$

an integral whose variation leads to (21).

In intend to address the application of the formal developments at hand to the theory of automorphic functions and the analogy of the equations

$$D(U) = 0, \qquad D(U) = e^{kU},$$

with the classical differential equations

$$\Delta U = 0, \qquad \Delta U = e^{kU},$$

already pointed out by G. Giraud[9] and A. Bloch[10], in a later work.

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[9] G. Giraud, Sur une équation aux dérivées partielles, non linéaires etc. Comptes Rendus 166, I (1918), p. 893.

[10] A. Bloch, Sur une nouvelle et importante généralisation de l'équation de Laplace. L'Enseignement Mathém. 26 (1926), p. 52.