The Riccati differential equation and non-associative algebras[1)]

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Introduction

1 The *Riccati differential equation*, that is, systems of differential equations of the form

$$\dot{x}_i = \sum_{k,l=1}^{n} \alpha_{ikl} x_k x_l, \quad \alpha_{ikl} \in \mathbb{R}, i = 1, \ldots, n, \tag{$*$}$$

often appears in problems regarding the behaviour of closed systems in biology, genetics, ecology, chemistry etc. Aside from the fact that the corresponding initial value problem has a unique solution, in the general case very little is known about the solutions $x_i = x_i(\xi)$.

In vector notation, the system $(*)$ can be written

$$\dot{x} = p(x), \tag{$**$}$$

where $p : \mathbb{R}^n \to \mathbb{R}^n$ is a given vector-valued homogeneous polynomial of degree 2.

The homogeneous polynomial $p : \mathbb{R}^2 \to \mathbb{R}^2$ of degree 2 correspond bijectively to the commutative algebra structures on $\mathbb{R}^n$: If p is such a polynomial, we obtain via

$$xy = \frac{1}{2}(p(x+y) - p(x) - p(y))$$

an $\mathbb{R}$-bilinear and symmetric map $(x, y) \mapsto xy$ of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. Thus we assign to every p a commutative (but not necessarily associative) algebra $\mathfrak{A} = \mathfrak{A}_p$ on $\mathbb{R}^n$, and $p(x) = x^2$ holds. Conversely, if $\mathfrak{A}$ is an algebra on $\mathbb{R}^n$, then $p(x) = x^2$ is a homogeneous polynomial of degree 2.

Subsequently, we take the following "algebraic" point of view when studying the properties of the system $(*)$ and $(**)$, respectively:

[1)]Extended version of a talk given at the Festkolloquium in honour of the 60th birthday of Helene Braun on the June 12, 1974.

Let $\mathfrak{A}$ with the product $(x, y) \mapsto xy$ be a commutative $\mathbb{R}$-algebra on $\mathbb{R}^n$. The associated Riccati differential equation is the system

$$\dot{x} = x^2. \qquad (***)$$

Every solution $x = x(\xi)$ that is differentiable in a neighborhood of $\xi = 0$ is called an $\mathfrak{A}$-*solution*.

2 Define the $\mathbb{R}$-vector space $\mathfrak{R}_n$ of power series f that converge in a neighborhood of 0 in $\mathbb{R}^n$,

$$f(u) = \sum_{m=0}^{\infty} f_m(u),$$

where f_m $\mathbb{R}^n \to \mathbb{R}^n$ are homogeneous polynomials of degree m. In addition, define the subset

$$\mathfrak{G}_n = \{f \in \mathfrak{R}_n \mid f(u) = u + \text{ higher order terms}\}$$

of $\mathfrak{R}_n$. It is well-known that every $f \in \mathfrak{G}_n$ is invertible in a neighborhood of 0 and that $\mathfrak{G}_n$ is a group with respect to $(f, g) \mapsto f \circ g$. Moreover, it is clear that $\mathfrak{G}_n$ acts on $\mathfrak{R}_n$ as a group of endomorphisms via

$$\mathfrak{R}_n \times \mathfrak{G}_n \to \mathfrak{R}_n, \quad (q, f) \mapsto q \circ f.$$

For $p, q \in \mathfrak{R}_n$, define $p \bullet q \in \mathfrak{R}_n$ by

$$((p \bullet q)(u))_i = \sum_{j=1}^{n} \frac{\partial p_i(u)}{\partial u_j} q_j(u),$$

where the indices denote indicate the components of vectors in $\mathbb{R}^n$. Geometrically, $p \bullet q$ is the *directional derivative of p in the direction of q*. Clearly, $\mathfrak{R}_n$ together with the product $(p, q) \mapsto p \bullet q$ is a (non-associative) $\mathbb{R}$-algebra.

3 As in 1, let $\mathfrak{A}$ be a commutative $\mathbb{R}$-algebra on $\mathbb{R}^n$. As usual, the powers of $\mathfrak{A}$ are defined recursively by $u^{m+1} = uu^m$, $u^1 = u$. Define recursively

$$g_{m+1} = g_m \bullet p, \quad g_0(u) = u,$$

where p is given by $p(u) = u^2$, and verifies that

$$g_{\mathfrak{A}}(u) = \sum_{m=0}^{\infty} \frac{1}{m!} g_m(u)$$

defines an element $g_{\mathfrak{A}}$ of $\mathfrak{G}_n$. Verify that

$$g_{\mathfrak{A}}(u) = u + u^2 + u^3 + \frac{1}{3}(2u^4 + u^2u^2) + \frac{1}{6}(2u^5 + u(u^2u^2) + 3u^2u^3) + \dots$$

To the algebra $\mathfrak{A}$ is now assigned a subset $\mathfrak{G}(\mathfrak{A})$ of those $f \in \mathfrak{G}_n$ for which $f(x(\xi))$ is an $\mathfrak{A}$-solution if $x = x(\xi)$ is. The elements of $\mathfrak{G}(\mathfrak{A})$ preserve solutions of the associated Riccati equation of $\mathfrak{A}$. One can prove that $\mathfrak{G}(\mathfrak{A})$ is a subgroup of $\mathfrak{G}_n$ that contains $g_{\mathfrak{A}}$. Thus $\mathfrak{G}(\mathfrak{A})$ does not consist of the identity element only.

4 From the theory of Jordan algebras it is known that certain algebras derived from $\mathfrak{A}$ play an important role: For $a \in \mathfrak{A}$ define a new product on $\mathbb{R}^n$,

$$u \perp_a v = u(va) + v(ua) - a(uv), \quad u, v \in \mathbb{R}^n.$$

We call this algebra $\mathfrak{A}_a$ on $\mathbb{R}^n$ defined by the product $(u, v) \mapsto u \perp_a v$ the *mutation of* $\mathfrak{A}$ *with respect to* a. Clearly, $\mathfrak{A}_a$ is also commutative.

In studying the group $\mathfrak{G}(\mathfrak{A})$ one necessarily encounters the vector subspace

$$\mathfrak{J}(\mathfrak{A}) = \{a \in \mathbb{R}^n \mid 2u(u(ua)) + u^3a = 2u(u^2a) + u^2(ua) \text{ for } u \in \mathbb{R}^n\}$$

of $\mathbb{R}^n$. In other places it has been shown:

Theorem A *If* $\mathfrak{A}$ *has an identity element, then*

$$a \mapsto g_{\mathfrak{B}}, \quad \mathfrak{B} = \mathfrak{A}_a,$$

is an isomorphism of the additive group of $\mathfrak{J}(\mathfrak{A})$ *onto* $\mathfrak{G}(\mathfrak{A})$.

Due to this isomorphism the seemingly arbitrarily defined vector subspace $\mathfrak{J}(\mathfrak{A})$ of $\mathbb{R}^n$ has to play an exceptional role. In the present note we will show (Theorem 2.3) that $\mathfrak{J}(\mathfrak{A})$ is indeed algebraically exceptional:

Theorem B *For every commutative algebra* $\mathfrak{A}$ *on* $\mathbb{R}^n$, $\mathfrak{J}(\mathfrak{A})$ *is a Jordan subalgebra of* $\mathfrak{A}$.

The proof uses standard arguments from the theory of Jordan and Lie algebras (compare Meyberg [2, 3, 4]). The restriction to real algebras $\mathfrak{A}$ is not essential.

5 A simple special case shall be mentioned: If the commutative $\mathbb{R}$-algebra $\mathfrak{A}$ is also associative (or more generally power-associative), then the differential equation $(***)$ can be solved explicitely. With elementary arguments we can see that the power series $g_{\mathfrak{A}}$ defined in 3 is given by

$$g_{\mathfrak{A}}(u) = \sum_{m=1}^{\infty} u^m.$$

So if $\mathfrak{A}$ has an identity element e, it follows that

$$g_{\mathfrak{A}}(u) = u(e-u)^{-1}.$$

Thus $g_{\mathfrak{A}}$ is birational. The solution of the initial value problem $x(0) = u$ of the differential equation $(***)$ is then give by

$$x(\xi) = u(e - \xi u)^{-1}.$$

We obtain a global solution whose asymptotic behaviour is easy to determine.

In the special case of a commutative and associative algebra $\mathfrak{A}$ discussed here, it follows that $\mathfrak{J}(\mathfrak{A}) = \mathfrak{A}$, so that $\mathfrak{G}(\mathfrak{A})$ is an n-dimensional vector group.

§1 Lie algebras and Jordan tripel systems

In this paragraph, let $\mathbb{k}$ always be a commutative ring with identity element. All $\mathbb{k}$-modules appearing here are $\mathbb{k}$-left-modules of rings with identity ("unitary left-modules").

1 Let $\mathfrak{Q}$ be a Lie algebra over $\mathbb{k}$ with the following properties:

(Q.1) $\mathfrak{Q} = \mathfrak{J} \oplus \mathfrak{T} \oplus \mathfrak{N}$ is the direct sum of the subalgebras $\mathfrak{J}$, $\mathfrak{T}$ and $\mathfrak{N}$.

(Q.2) $\mathfrak{J}$ and $\mathfrak{N}$ are abelian.

(Q.3) $[\mathfrak{J}, \mathfrak{T}] \subset \mathfrak{J}$, $[\mathfrak{J}, \mathfrak{N}] \subset \mathfrak{T}$, $[\mathfrak{T}, \mathfrak{N}] \subset \mathfrak{N}$.

We then have $[\mathfrak{J}, \mathfrak{Q}] \subset \mathfrak{J} \oplus \mathfrak{T}$ and $[\mathfrak{N}, \mathfrak{Q}] \subset \mathfrak{T} \oplus \mathfrak{N}$, so that

$$[\mathfrak{J}, [\mathfrak{J}, [\mathfrak{J}, \mathfrak{Q}]]] = [\mathfrak{N}, [\mathfrak{N}, [\mathfrak{N}, \mathfrak{Q}]]] = \mathbf{0}. \tag{1.1}$$

By

$$\begin{aligned} \{a, p, b\} &= [a, [b, p]] \quad \text{for } a, b \in \mathfrak{J},\ p \in \mathfrak{N}, \\ \{p, a, q\} &= [p, [q, a]] \quad \text{for } a \in \mathfrak{J},\ p, q \in \mathfrak{N}, \end{aligned}$$

two $\mathbb{k}$-trilinear maps

$$\begin{aligned} \mathfrak{J} \times \mathfrak{N} \times \mathfrak{J} \to \mathfrak{J}, \quad & (a, p, b) \mapsto \{a, p, b\}, \\ \mathfrak{N} \times \mathfrak{J} \times \mathfrak{N} \to \mathfrak{N}, \quad & (p, a, q) \mapsto \{p, a, q\}, \end{aligned} \tag{1.2}$$

are defined. From the composition rules (Q.3) we infer that the images are indeed contained in $\mathfrak{J}$ and $\mathfrak{N}$, respectively. By (Q.2) both maps are symmetrical in the first and third argument. Using the Jacobi identity we verify

$$[t, \{b, q, c\}] = \{[t, b], q, c\} + \{b, [t, q], c\} + \{b, q, [t, c]\} \tag{1.3}$$

for $t \in \mathfrak{T}$, $b, c \in \mathfrak{J}$ and $q \in \mathfrak{N}$. Analogously,

$$[t, \{q, b, r\}] = \{[t, q], b, r\} + \{q, [t, b], r\} + \{q, b, [t, r]\} \tag{1.4}$$

for $t \in \mathfrak{T}$, $b \in \mathfrak{J}$ and $q, r \in \mathfrak{N}$. As in [3, p. 19] we see:

Lemma 1.1 *For $a, b, c \in \mathfrak{J}$ and $p, q, r \in \mathfrak{N}$:*

(a) $\{b, \{p, a, q\}, c\} = \{b, q, \{a, p, c\}\} + \{c, q, \{a, p, b\}\} - \{a, p, \{b, q, c\}\}$,

(b) $\{q, \{a, p, b\}, r\} = \{q, b, \{p, a, r\}\} + \{r, b, \{p, a, q\}\} - \{p, a, \{q, b, r\}\}$.

PROOF: Set $t = [a, p]$ in (1.3) and (1.4). $\diamond$

A pair $(\mathfrak{J}, \mathfrak{N})$ of $\mathbb{k}$-modules together with two trilinear maps (1.2) that are symmetric in the first and third argument and satisfy the identities in Lemma 1.1 are called a *Jordan tripel system* (or *connected pair* in the sense of Meyberg [2]). From [2, Satz 2.2, Satz 2.3] we obtain:

Lemma 1.2 *If $\mathfrak{Q} = \mathfrak{J} \oplus \mathfrak{T} \oplus \mathfrak{N}$ is a $\mathbb{k}$-Lie algebra satisfying (Q.1), (Q.2) and (Q.3), and if the $\mathbb{k}$-module $\mathfrak{Q}$ is divisible by 2 and 3, then:*

(a) *For every $p \in \mathfrak{N}$, $\mathfrak{J}$ with the product*

$$(a, b) \mapsto \{a, p, b\} = [a, [b, p]]$$

is a $\mathbb{k}$-Jordan algebra.

(b) *For every $a \in \mathfrak{J}$, $\mathfrak{N}$ with the product*

$$(p, q) \mapsto \{p, a, q\} = [p, [q, a]]$$

is a $\mathbb{k}$-Jordan algebra.

(c) *The two "fundamental formulas"*

$$\begin{aligned} \{a, \{p, \{a, q, a\}, p\}, a\} &= \{\{a, p, a\}, q, \{a, p, a\}\}, \\ \{p, \{a, \{p, b, p\}, a\}, p\} &= \{\{p, a, p\}, b, \{p, a, p\}\} \end{aligned}$$

hold for all $a, b \in \mathfrak{J}$, $p, q \in \mathfrak{N}$.

2 We now present a class of examples to illustrate the relation between certain Lie algebras and Jordan tripel systems from 1:

Let $\mathfrak{L}$ be a $\Bbbk$-Lie algebra with the following properties:

(L.1) $\mathfrak{L} = \bigoplus_{\nu=0}^{\infty} \mathfrak{L}_\nu$ is the direct sum of submodules $\mathfrak{L}_\nu$, $\nu = 0, 1, \ldots$.

(L.2) $[\mathfrak{L}_\nu, \mathfrak{L}_\mu] \subset \mathfrak{L}_{\nu+\mu-1}$ for all $\nu, \mu \geq 0$, $\mathfrak{L}_{-1} = \mathbf{0}$.

After a change of indices $\mathfrak{L}$ is then a graded Lie algebra. In particular, $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are subalgebras, and $\mathfrak{L}_0$ is abelian.

For $p \in \mathfrak{L}_2$ define

$$\begin{aligned}
\mathfrak{N}_p &= \{q \in \mathfrak{L}_2 \mid [p, q] = 0\},\\
\mathfrak{T}_p &= \{t \in \mathfrak{L}_1 \mid [p, [p, t]] = 0\},\\
\mathfrak{J}_p &= \{a \in \mathfrak{L}_0 \mid [p, [p, [p, a]]] = 0\}.
\end{aligned}$$

As $\mathfrak{N}_p$ is restricted to elements of $\mathfrak{L}_2$, it is not necessarily a subalgebra of $\mathfrak{L}$.

Lemma 1.3 *Let $p \in \mathfrak{L}_2$ and assume $\mathfrak{N}_p$ is an abelian subalgebra of $\mathfrak{L}$. Then:*

(a) *$\mathfrak{T}_p$ is a subalgebra of $\mathfrak{L}$, and for $t \in \mathfrak{L}_1$, the following are equivalent:*

$$[p, t] \in \mathfrak{N}_p, \tag{1}$$
$$[\mathfrak{N}_p, t] \subset \mathfrak{N}_p. \tag{2}$$

(b) *$\mathfrak{J}_p$ is an abelian subalgebra of $\mathfrak{L}$, and for $a \in \mathfrak{L}_0$, the following are equivalent:*

$$[p, a] \in \mathfrak{T}_p, \tag{1}$$
$$[\mathfrak{N}_p, a] \subset \mathfrak{T}_p. \tag{2}$$

(c) *$\mathfrak{Q}_p = \mathfrak{J}_p \oplus \mathfrak{T}_p \oplus \mathfrak{N}_p$ is a subalgebra of $\mathfrak{L}$ that satisfies (Q.1), (Q.2) and (Q.3).*

PROOF: (a) For $t \in \mathfrak{L}_1$, $t \in \mathfrak{T}_p$ is equivalent to (1) by (L.2). Moreover, (1) follows directly from (2) since $p \in \mathfrak{N}_p$. Now let $t \in \mathfrak{L}_1$ with $[p, t] \in \mathfrak{N}_p$ be given. For $q \in \mathfrak{N}_p$, $[p, [q, t]] = [[p, q], t] + [q, [p, t]]$. Here, $[p, q] = 0$ and $[p, t] \in \mathfrak{N}_p$ by (1). As $\mathfrak{N}_p$ was assumed to be abelian, it follows that $[q, [p, t]] = 0$ and hence $[p, [q, t]] = 0$. Then $[q, t] \in \mathfrak{N}_p$, so that (2) holds. By (2), $\mathfrak{T}_p$ is a subalgebra of $\mathfrak{L}$.

(b) As a submodule of $\mathfrak{L}_0$, $\mathfrak{J}_p$ is an abelian subalgebra of $\mathfrak{L}$. Again $a \in \mathfrak{J}_p$ is equivalent to (1), and (1) follows from (2). To prove (2), let $a \in \mathfrak{L}_0$ with $[p,a] \in \mathfrak{T}_p$ be given. By (1) of part (a), $[p,[p,a]] \in \mathfrak{N}_p$. Since $\mathfrak{N}_p$ abelian, it follows for all $q \in \mathfrak{N}_p$ that

$$0 = [q,[p,[p,a]]] = [[q,p],[p,a]] + [p,[q,[p,a]]] = [p,[q,[p,a]]].$$

As $[q,[p,a]] \in \mathfrak{L}_2$, it follows that $[q,[p,a]] = [p,[q,a]] \in \mathfrak{N}_p$, so that by (1) of part (a), $[q,a] \in \mathfrak{T}_p$. Thus (2) holds.

(c) Conditions (Q.1) and (Q.2) have already been shown. In light of parts(2) in (a) and (b), we only need to show $[\mathfrak{J}_p, \mathfrak{T}_p] \subset \mathfrak{J}_p$ for (Q.3). For $a \in \mathfrak{J}_p$ and $t \in \mathfrak{T}_p$, $[a,t] \in \mathfrak{L}_0$ and we have

$$[p,[a,t]] = [[p,a],t] + [a,[p,t]].$$

By (b), $[p,a] \in \mathfrak{T}_p$, and by (a), $[p,t] \in \mathfrak{N}_p$, so that by (b) both summands lie in $\mathfrak{T}_p$. Thus $[a,t] \in \mathfrak{J}_p$ by (1) of part (b). $\diamondsuit$

3 For later applications we summarize the results of 1 and 2:

Theorem 1.4 *Let $\mathfrak{L} = \bigoplus_{\nu=0}^{\infty} \mathfrak{L}_\nu$ be a $\Bbbk$-Lie algebra that satisfies conditions (L.1) and (L.2), and let $p \in \mathfrak{L}_2$ be given such that $\mathfrak{N}_p$ is an abelian subalgebra of $\mathfrak{L}$. If the $\Bbbk$-module $\mathfrak{L}$ is divisible by 2 and 3, then:*

(a) *$(\mathfrak{J}_p, \mathfrak{N}_p)$ together with the two maps*

$$(a,q,b) \mapsto \{a,q,b\} = [a,[b,q]],$$
$$(r,a,q) \mapsto \{r,a,q\} = [r,[q,a]]$$

is a linear Jordan tripel system.

(b) *For every $q \in \mathfrak{N}_p$, $\mathfrak{J}_p$ together with the product $(a,b) \mapsto \{a,q,b\}$ is a Jordan algebra.*

(c) *For every $a \in \mathfrak{J}_p$, $\mathfrak{N}_p$ together with the product $(q,r) \mapsto \{q,a,r\}$ is a Jordan algebra.*

PROOF: By (c) of Lemma 1.3, we can apply all the results from 1. $\diamondsuit$

4 The construction of the Lie algebra $\mathfrak{Q}_p$ in 2 is based on the following: Let $\mathfrak{L}$ be a $\mathbb{k}$-Lie algebra and $\mathfrak{N}$ a subalgebra of $\mathfrak{L}$. If we put

$$\begin{aligned}\mathfrak{T} &= \{t \in \mathfrak{L} \mid [\mathfrak{N}, t] \subset \mathfrak{N}\},\\ \mathfrak{J} &= \{a \in \mathfrak{L} \mid [\mathfrak{N}, a] \subset \mathfrak{T}\},\end{aligned}$$

then $\mathfrak{T}$ and $\mathfrak{J}$ are subalgebras of $\mathfrak{L}$, and

$$\mathfrak{Q} = \mathfrak{J} + \mathfrak{T} + \mathfrak{N}$$

satisfies (Q.3). In general, this is not a direct sum.

However, if conditions (L.1) and (L.2) are satisfied by $\mathfrak{L}$ and if $\mathfrak{N}$ is an abelian subalgebra of $\mathfrak{L}_2$, then we can define

$$\begin{aligned}\mathfrak{T}(\mathfrak{N}) &= \{t \in \mathfrak{L}_1 \mid [\mathfrak{N}, t] \subset \mathfrak{N}\},\\ \mathfrak{J}(\mathfrak{N}) &= \{a \in \mathfrak{L}_0 \mid [\mathfrak{N}, a] \subset \mathfrak{T}(\mathfrak{N})\},\end{aligned}$$

and

$$\mathfrak{Q}(\mathfrak{N}) = \mathfrak{J}(\mathfrak{N}) \oplus \mathfrak{T}(\mathfrak{N}) \oplus \mathfrak{N}.$$

Then $\mathfrak{Q}(\mathfrak{N})$ satisfies (Q.2) and (Q.3), so we can again apply the results from 1.

§2 Commutative algebras

Throughout this paragraph, let $\mathbb{k}$ be an infinite field of characteristic different from 2 and 3.

1 Let $\mathfrak{V}$ be a vector space of finite dimension $n > 0$ over $\mathbb{k}$. An element x in a base field extension of $\mathfrak{V}$ is called a *generic element* of $\mathfrak{V}$ if the components of x with respect to a basis of $\mathfrak{V}$ are algebraically independent over $\mathbb{k}$. Clearly this definition does not depende on the choice of the basis.

Let us now choose n elements $\tau_1, \ldots, \tau_n$ in an extension field of $\mathbb{k}$ that are algebraically independent over $\mathbb{k}$, and form the field

$$\widetilde{\mathbb{k}} = \mathbb{k}(\tau_1, \ldots, \tau_n).$$

For an arbitrary vector space $\mathfrak{W}$ over $\mathbb{k}$ let $\widetilde{\mathfrak{W}}$ denote the $\widetilde{\mathbb{k}}$-vector space obtained from $\mathfrak{W}$ by extension of the base field $\mathbb{k}$ to $\widetilde{\mathbb{k}}$. After choosing a basis $b_1, \ldots, b_n$ of $\mathfrak{V}$,

$$x = \tau_1 b_1 + \ldots + \tau_n b_n$$

is a generic element of $\mathfrak{V}$ that is contained in $\widetilde{\mathfrak{V}}$.

If f is an element of $\widetilde{\mathfrak{W}}$, then we write $f(x)$ instead of f and call $f(x)$ a *rational function* in x. The function $f(x)$ is called a *polynomial* in x, if all components of $f(x)$ with respect to a basis of $\mathfrak{W}$ over $\mathbb{k}$ are polynomials in $\tau_1, \dots, \tau_n$.

If u is an element in a base field extension of $\mathfrak{V}$ and $f \in \widetilde{\mathfrak{V}}$, then

$$\Delta_x^u f(x) = \frac{\mathrm{d}}{\mathrm{d}\tau} f(x + \tau u)|_{\tau \to 0}$$

defines a differential operator Δ. Compare [1, Chapter II, §1].

2 For $f, g \in \widetilde{\mathfrak{V}}$,

$$[f, g](x) = \Delta_x^{g(x)} f(x) - \Delta_x^{f(x)} g(x) \tag{2.1}$$

defines an anti-commutative product $(f, g) \mapsto [f, g]$ on $\widetilde{\mathfrak{V}}$. In [3, I, §1.3] and in [4] it was shown that $\widetilde{\mathfrak{V}}$ *together with the product* $(f, g) \mapsto [f, g]$ *is a* $\mathbb{k}$*-Lie algebra* Rat $\mathfrak{V}$.

Let Pol $\mathfrak{V}$ denote the subspace of Rat $\mathfrak{V}$ of all polynomials and $\mathfrak{P}_\nu(\mathfrak{V})$ the subspace of homogeneous polynomials of degree ν. Then

$$\operatorname{Pol} \mathfrak{V} = \bigoplus_{\nu=0}^{\infty} \mathfrak{P}_\nu(\mathfrak{V})$$

is a subalgebra of Rat $\mathfrak{V}$ that satisfies (L.1) and (L.2) for $\mathfrak{L}_\nu = \mathfrak{P}_\nu(\mathfrak{V})$. In the following we study elements of $\mathfrak{P}_2(\mathfrak{V})$, that is, homogeneous polynomials of degree 2, for which

$$\mathfrak{N}_p = \{q \in \mathfrak{P}_2(\mathfrak{V}) \mid [p, q] = 0\}$$

is an abelian subalgebra of Pol $\mathfrak{V}$, compare §1.2.

3 If $\mathfrak{A}$ together with the product $(a, b) \mapsto ab$ is a commutative algebra defined on the vector space $\mathfrak{V}$, then

$$p_{\mathfrak{A}}(x) = x^2$$

defines an element $p_{\mathfrak{A}}$ of $\mathfrak{P}_2(\mathfrak{V})$. Conversely, for every $q \in \mathfrak{P}_2(\mathfrak{V})$ there exists an algebra $\mathfrak{A}$ on $\mathfrak{V}$ with $q = p_{\mathfrak{A}}$. Therefore,

$$\mathfrak{P}_2(\mathfrak{V}) = \{p_{\mathfrak{A}} \mid \mathfrak{A} \text{ is a commutative algebra on } \mathfrak{V}\}.$$

Now fix a commutative algebra $\mathfrak{A}$ with product $(a, b) \mapsto ab$ on $\mathfrak{V}$. For $p = p_{\mathfrak{A}} \in \mathfrak{P}_2(\mathfrak{V})$, that is $p(x) = x^2$, we write $\mathfrak{N}(\mathfrak{A})$ for $\mathfrak{N}_p$, and obtain by (2.1) for $q \in \mathfrak{P}_\nu(\mathfrak{V})$, $\nu \leq 3$,

$$[p_{\mathfrak{A}}, q](x) = \Delta_x^{q(x)} x^2 - \Delta_x^{x^2} q(x).$$

This q defines a ν-linear and symmetric map $(a_1, \ldots, a_\nu) \mapsto q(a_1, \ldots, a_\nu)$ from $\mathfrak{V} \times \mathfrak{V}$ to $\mathfrak{V}$ via $q(a, \ldots, a) = q(a)$. Using the chain rule, it follows that

$$[p_{\mathfrak{A}}, q](x) = 2xq(x) - \nu q(x, \ldots, x, x^2). \tag{2.2}$$

In particular,

$$\mathfrak{N}(\mathfrak{A}) = \{q \in \mathfrak{P}_2(\mathfrak{V}) \mid xq(x) = q(x, x^2)\}. \tag{2.3}$$

Lemma 2.1 *Suppose the commutative algebra $\mathfrak{A}$ on $\mathfrak{V}$ has an identity element e. If $[p_{\mathfrak{A}}, q] = 0$ for some $q \in \mathfrak{P}_3(\mathfrak{V})$, then $q = 0$.*

PROOF: Let the symmetric trilinear map $q : \mathfrak{V} \times \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ be given by $q(x, x, x) = q(x)$. By (2.2),

$$2xq(x) = 3q(x, x, x^2).$$

In particular, for $x = e$ it follows that $q(e) = 0$. By linearization,

$$yq(x) + 3xq(x, x, y) = 3q(y, x, x^2) + 3q(x, x, xy). \tag{2.4}$$

For $x = e$, $q(e, e, y) = 0$ follows. Another linearization yields

$$2yq(x, x, y) + 2xq(x, y, y) = q(y, y, x^2) + 2q(y, x, xy) + 2q(x, y, xy) + q(x, x, y^2).$$

For $x = e$ it follows that $q(e, y, y) = 0$. Finally, let $y = e$ in (2.4) and obtain $q(x) = 0$, that is, $q = 0$. ◇

Remark 1 If $\mathbb{k}$ has characteristic 0, we can show analogously that $[p, q] = 0$, $q \in \mathfrak{P}_\nu(\mathfrak{V})$, $\nu \geq 3$, already implies $q = 0$.

Corollary 1 $\mathfrak{N}(\mathfrak{A})$ *is an abelian subalgebra of* Pol $\mathfrak{V}$.

PROOF: For $q, r \in \mathfrak{N}(\mathfrak{V})$, $[p, [q, r]] = 0$ follows from the Jacobi identity. As the $[q, r]$ belong to $\mathfrak{P}_3(\mathfrak{V})$, we obtain $[q, r]$ from Lemma 2.1. ◇

4 By the corollary to Lemma 2.1 and the fact that Pol $\mathfrak{V}$ satisfies (L.1) and (L.2), we can apply the results from §1 to algebras $\mathfrak{A}$ on $\mathfrak{V}$. Write $\mathfrak{T}(\mathfrak{A})$, $\mathfrak{J}(\mathfrak{A})$ for $\mathfrak{T}_p$, $\mathfrak{J}_p$, respectively, and obtain from §1.2

$$\mathfrak{T}(\mathfrak{V}) = \{T \in \operatorname{End}\mathfrak{V} \mid [p_{\mathfrak{A}}, [p_{\mathfrak{A}}, T]] = 0\},$$
$$\mathfrak{J}(\mathfrak{V}) = \{a \in \mathfrak{V} \mid [p_{\mathfrak{A}}, [p_{\mathfrak{A}}, [p_{\mathfrak{A}}, a]]] = 0\},$$

where we identify the elements of $\mathfrak{P}_1(\mathfrak{V})$ and End $\mathfrak{V}$ and those of $\mathfrak{P}_0(\mathfrak{V})$ and $\mathfrak{V}$. Since $p_{\mathfrak{A}}(x) = x^2$,

$$[p_{\mathfrak{A}}, T](x) = 2x(Tx) - Tx^2,$$
$$[p_{\mathfrak{A}}, a](x) = 2ax.$$

As usual, left-multiplication on $\mathfrak{A}$ is denoted by $L : \mathfrak{A} \to \operatorname{End}\mathfrak{V}$, that is,

$$xy = L(x)y.$$

In particular,

$$[p_{\mathfrak{A}}, a] = 2L(a),$$

and we see that $a \in \mathfrak{J}(\mathfrak{A})$ if and only if $L(a) \in \mathfrak{T}(\mathfrak{A})$. Now verify that

$$\mathfrak{T}(\mathfrak{A}) = \{T \in \operatorname{End}\mathfrak{V} \mid 2x(x \cdot Tx) + Tx^3 = 2x \cdot Tx^2 + x^2 \cdot Tx\}, \tag{2.5}$$
$$\mathfrak{J}(\mathfrak{A}) = \{a \in \mathfrak{V} \mid 2x(x \cdot xa) + ax^3 = 2x(ax^2) + x^2(ax)\}. \tag{2.6}$$

Lemma 2.2 *If the commutative algebra $\mathfrak{A}$ on $\mathfrak{V}$ has an identity element e, then*

$$\mathfrak{N}(\mathfrak{A}) = \{q \mid q(x) = 2x(xa) - ax^2 \text{ for all } a \in \mathfrak{J}(\mathfrak{A})\}$$

and $a \mapsto 2x(ax) - ax^2$ is a linear bijection of $\mathfrak{J}(\mathfrak{A})$ onto $\mathfrak{N}(\mathfrak{A})$.

PROOF: The defining condition $[p_{\mathfrak{A}}, q] = 0$ for $\mathfrak{N}(\mathfrak{A})$ means by (2.2) the identity

$$xq(x) = q(x, x^2).$$

Linearization leads to

$$yq(x) + 2xq(x, y) = q(y, x^2) + 2q(x, xy),$$

so that $x = e$ or $y = e$ with $a = q(e)$ yields

$$q(e, y) = ay \quad \text{or} \quad q(x) = 2xq(e, x) - q(e, x^2)$$

respectively. This implies

$$q(x) = 2x(xa) - ax^2,$$

that is, $q = [p_{\mathfrak{A}}, L(a)]$. As $[p_{\mathfrak{A}}, q] = 0$, $L(a) \in \mathfrak{T}(\mathfrak{A})$, that is, $a \in \mathfrak{J}(\mathfrak{A})$. Conversely, if $a \in \mathfrak{J}(\mathfrak{A})$, then $q = [p_{\mathfrak{A}}, L(a)] \in \mathfrak{N}(\mathfrak{A})$. ◇

5 In addition to the left-multiplication L of $\mathfrak{A}$ consider the quadratic representation

$$P(x) = 2L(x)^2 - L(x^2)$$

and its linearized form $P(x, y)$. By Lemma 2.2 $\mathfrak{N}(\mathfrak{A})$ consists precisely of those polynomials q with $q(x) = P(x)a$ with $a \in \mathfrak{J}(\mathfrak{A})$.

By the corollary of Lemma 2.1, Theorem 1.4 can be applied. From part (c) we obtain for $q = p_{\mathfrak{A}}$ that $\mathfrak{J}(\mathfrak{A})$ together with the product

$$(a, b) \mapsto \{a, p_{\mathfrak{A}}, b\} = -[a, [p_{\mathfrak{A}}, b]] = 2[L(b), a] = 2ab$$

is a Jordan algebra.

Theorem 2.3 *If $\mathfrak{A}$ is a finite-dimensional commutative algebra over a field of characteristic other than* 2 *and* 3, *then $\mathfrak{J}(\mathfrak{A})$ is a Jordan subalgebra of $\mathfrak{A}$.*

PROOF: First assume that $\mathfrak{A}$ contains an identity element. Then we just saw that $\mathfrak{J}(\mathfrak{A})$ is a Jordan algebra with the product $(a, b) \mapsto 2ab$, hence also with the product $(a, b) \mapsto ab$. The general case now follows by adjunction of an identity element. ◇

Remark 2 In general, $\mathfrak{J}(\mathfrak{A}) = \mathbf{0}$.

§3 Some examples

Let $\mathbb{k}$ be an infinite field of characteristic other than 2 and 3, and let $\mathfrak{A}$ be a finite-dimensional commutative $\mathbb{k}$-algebra.

1 As in Theorem 2.3 we consider the Jordan subalgebra

$$\mathfrak{J}(\mathfrak{A}) = \{a \in \mathfrak{A} \mid 2x(x \cdot xa) + ax^3 = 2x(ax^2) + x^2(ax)\}$$

of $\mathfrak{A}$. As examples we study the following classes of algebras:

(a) $\mathfrak{A}$ is *power-associative*, that is, $x^m x^n = x^{m+n}$ for all $m, n \in \mathbb{N}$.

(b) $\mathfrak{A}$ is a *Lie triple algebra*, that is, for $x, y, z \in \mathfrak{A}$,

$$w(x, y, z) + y(x, w, z) = (x, yw, z),$$

where the *associator* is defined by $(x, y, z) = (xy)z - x(yz)$. Compare Osborn [5] and Petersson [6].

(c) $\mathfrak{A}$ has a non-degenerate symmetric bilinear form σ that is associative, that is, $\sigma(xy, z) = \sigma(x, yz)$ for all $x, y, z \in \mathfrak{A}$.

Lemma 3.1 *If $\mathfrak{A}$ is of type (a), (b) or (c), then*

$$\mathfrak{J}(\mathfrak{A}) = \{a \in \mathfrak{A} \mid x^2(ax) = x(ax^2)\}.$$

PROOF: Type (a): By linearization of $x^2x^2 = x^4$ we obtain

$$4(ax)x^2 = ax^3 + x(ax^2) + 2x(x \cdot ax)$$

for $a, x \in \mathfrak{A}$. Thus $a \in \mathfrak{J}(\mathfrak{A})$ is equivalent to $x^2(ax) = x(ax^2)$.

Type (b): $a \in \mathfrak{J}(\mathfrak{A})$ is equivalent to $2x(a, x, x) = x^2(ax) - ax^3$. The claim follows since $2x(a, x, x) = (a, x^2, x) = (ax^2)x - ax^3$.

Type (c): By assumption, $L(x)$ is self-adjoint with respect to σ. By linearization of the defining identity for $\mathfrak{J}(\mathfrak{A})$ we find that

$$\begin{aligned} &2L(x \cdot xa) + 2L(x)L(xa) + L(x)^2L(a) + L(a)L(x^2) + 2L(a)L(x)^2 \\ &= 2L(x^2a) + 4L(x)L(a)L(a) + 2L(xa)L(x) + L(x^2)L(a) \end{aligned}$$

holds for $a \in \mathfrak{J}(\mathfrak{A})$. Taking the adjoint with respect to σ and subtracting yields

$$2[L(x), L(xa)] + [L(a), L(x^2)] = 0$$

and application to x yields

$$2x(x \cdot ax) - 2x^2(xa) + ax^3 - x^2 \cdot xa = 0.$$

Comparing this with the definition of $\mathfrak{J}(\mathfrak{A})$, it follows that $x^2(ax) = x(ax^2)$ for all $a \in \mathfrak{J}(\mathfrak{A})$.

Conversely, if $x^2(ax) = x(ax^2)$, we obtain by linearization

$$2L(xa)L(x) + L(x^2)L(a) = L(x^2a) + 2L(x)L(a)L(x).$$

Again it follows that

$$2[L(x), L(xa)] + [L(a), L(x^2)] = 0$$

and application to x yields $a \in \mathfrak{J}(\mathfrak{A})$. $\diamond$

By (a), $\mathfrak{J}(\mathfrak{A}) = \mathfrak{A}$ is holds precisely for Jordan algebras. In the case of a Lie triple algebra, the defining relations immediately imply that $\mathfrak{J}(\mathfrak{A}) = \{a \in \mathfrak{A} \mid (x, a, x^2) = 0\}$ is a Jordan algebra.

2 Let $\mathfrak{B}$ be an arbitrary $\Bbbk$-algebra with product $(x, y) \mapsto xy$, and let $\mathfrak{B}^+$ the corresponding commutative algebra with product $(x, y) \mapsto x \cdot y = \frac{1}{2}(xy + yx)$. Verify that the defining identity for $\mathfrak{J}(\mathfrak{B}^+)$ can be written in the form

$$\begin{aligned} &2\{(xa, x, x) - (x, x, ax) + (x, x^2, a) - (a, x^2, x)\} + (ax, x, x) - (x, x, xa) \\ &+ (x, a, x^2) - (x^2, a, x) + (x, x, x)a - a(x, x, x) + x(a, x, x) - (x, x, a)x \\ &- (x, a, x)x + x(x, a, x) = 0. \end{aligned}$$

If $\mathfrak{B}$ is flexible, then Theorem 2.3 implies the curious result that the set of $a \in \mathfrak{B}$ satisfying

$$3(xa, x, x) + 3(ax, x, x) + 4(x, x^2, a) + 2(x, a, x^2) + x(a, x, x) + (a, x, x)x = 0$$

is a Jordan subalgebra of $\mathfrak{B}^+$.

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