Two theorems on equations with integer coefficients

By Leopold Kronecker

I. *If two roots of an integer equation in which the first coefficient is* 1 *are all imaginary and their analytic moduli are all equal to* 1*, then these have to be roots of unity.*

Proof: Let $a, b, c, \ldots$ denote the roots of the equation

$$F(x) = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \ldots + N = 0,$$

in which $A, B, C, \ldots, N$ are integers. Since the roots $a, b, c \ldots$ are all imaginary quantities of modulus 1, set (where $\sqrt{-1}$ is denoted by i):

$$\begin{aligned} A &= \cos(\alpha) + \cos(\beta) + \cos(\gamma) + \ldots, \\ B &= \cos(\alpha + \beta) + \cos(\alpha + \gamma) + \cos(\alpha + \delta) + \ldots, \\ C &= \cos(\alpha + \beta + \gamma) + \cos(\alpha + \beta + \delta) + \ldots, \\ &\vdots \end{aligned}$$

Hence A must be a sum of n quantities, each of which is larger than -1 and smaller than $= 1$. Also, B is a sum of $\frac{n(n-1)}{1 \cdot 2}$ of such quantities, and C equals a sum of $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$ of such quantities, etc. But since $A, B, C, \ldots$ are supposed to be integers, we see that every coefficient of the quations $F(x) = 0$ can only assume a bounded number of values, and the product of all these numbers gives the number of all those systems of values that can be assumed by the coefficients $A, B, C, \ldots$ in the first place.

This implies that *for any given degree n there can only be a finite number of equations that satisfy the conditions of the above stated theorem.*

Let r denote the number of all such equations of degree n, and let further for some number k:

$$F_k(x) = (x - a^k)(x - b^k)(x - c^k) \cdots .$$

Then $F_k(x)$ satisfies all the assumptions of the theorem stated above. For firstly, the coefficients of this equations are clearly integers as symmetric functions in $a, b, c, \ldots$, and secondly the analytic moduli of its roots,

$$a^k = \cos(k\alpha) + \mathrm{i}\sin(k\alpha),\ b^k = \cos(k\beta) + \mathrm{i}\sin(k\beta),\ c^k = \cos(k\gamma) + \mathrm{i}\sin(k\gamma),\ \ldots$$

are all equal to 1. It follows that at least three among the equations

$$F(x) = 0, \quad F_2(x) = 0, \quad F_3(x) = 0, \quad \cdots, \quad F_{r+1}(x) = 0$$

must be identical, that is, there must be two distinct numbers h and k for which $F_h(x) = F_k(x)$. The roots of $F_h(x) = 0$, namely $a^h, b^h, c^h, \ldots$, must coincide with the roots of $F_k(x) = 0$, namely $a^k, b^k, c^k, \ldots$, up to order.

For any quantity in the first row, say a^h, let now b^k denote the one in the second row that coincides with it, so that $a^h = b^k$. Similary, $b^h = c^k, c^h = d^k, \ldots$. If we continue in this manner, we clearly encounter an equation that has a^h on the right hand side. Thus we obtain a system of equations of the following form:

$$a^h = b^k, \quad b^h = c^k, \quad c^h = d^k, \quad m^h = a^k.$$

If the number of equations is denote by μ, and if we eliminate from these the $\mu - 1$ quantities $b, c, d, \ldots, m$, then we obtain, as is easy to see:

$$a^{h^\mu - k^\mu} = 1.$$

As remarked above, h and k are distinct integer numbers. So this equation shows that a is indeed a root of unity, and this result clearly holds for all roots of the equation $F(x) = 0$, since a was chosen arbitrarily among these.

II. *If an equation with integer coefficients of whom the leading one equals* 1 *has only real roots contained in the interval from* -2 *to* 2*, so that they can be represented as* $2\cos(\alpha)$*,* $2\cos(\beta)$*,*$2\cos(\gamma)$*,* $\ldots$*, then the angles* $\alpha, \beta, \gamma, \ldots$ *are all in a commensurable ratio to a right angle.*

PROOF: Let $\Phi(y) = 0$ be an equation with the given properties, and let $2\cos(\alpha)$, $2\cos(\beta)$, $2\cos(\gamma)$, $\ldots$ be its roots. If now ν denotes the degree of $\Phi(y)$, and we set

$$x^\nu \Phi\left(x + \frac{1}{x}\right) = F(x),$$

then $F(x) = 0$ is clearly an equation in which all coefficients are integers and the leading coefficient is 1. Moreover, it is easy to see that the roots of this equation are

$$\cos(\alpha) \pm \mathrm{i}\sin(\alpha), \cos(\beta) \pm \mathrm{i}\sin(\beta), \cos(\gamma) \pm \mathrm{i}\sin(\gamma), \ldots,$$

that is, they are all complex quantities of absolute value 1. So equation $F(x) = 0$ satisfies the conditions for the theorem established above, and by applying the very same theorem, we find that all the roots $\cos(\alpha) \pm \mathrm{i}\sin(\alpha)$, $\cos(\beta) \pm \mathrm{i}\sin(\beta)$, $\ldots$, are roots of unity. This result immediately implies the properties of the angles $\alpha, \beta, \gamma, \ldots$.

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