Étale slices

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Abstract

This article studies the action of reductive algebraic groups on affine algebraic varieties, where the base field is algebraically closed and of characteristic 0. We show the existence of an "étale slice" at every point in a closed orbit. Then we draw some consequences.

Contents

Introduction

Let G be a real compact Lie group that acts differentiably on a differentiable manifold X. The way in which to analyze such an action locally is well-known: equip X with a G-invariant Riemannian metric; for every point $x \in X$, we cal then find, by virtue of the exponential map, a neighborhood of the zero section of the normal fibration of the orbit $G(x)$, and an isomorphism from this neighborhood onto a neighborhood of $G(x)$ in X that commutes with the G-action. From this we deduce the existence of a "slice" at the point x: this is a submanifold V of X, containing x, stable under the isotropy subgroup G_x, such that the action of G on X induces an isomorphism of $G \times_{G_x} V$ onto a neighborhood of $G(x)$ in X, where the isomorphism commutes with the G-action (for example, see [8]).

In this work we observe an analogy in between the preceding situation and the action of a reductive algebraic group G on an affine algebraic variety X, where the base field is algebraically closed and of characteristic 0. The "rigidity" of such an action (which in the case described above depends on the compactness of the group) then comes from the fact that G acts irreducibly on the algebra of regular functions on X. Guided by this analogy, we pay particular attention to the closed orbits of G in X, for several reasons: firstly, because their isotropy subgroup is reductive, then, because the points in the "quotient" $X /\!\!/ G$ (which we naturally define as the variety associated to the regular G-invariant functions) corresponds in bijective way to the closed orbits, and further, because we obtain the existence of an "étale slice"[1)] at every point x whose orbit $G(x)$ is closed: there is a subvariety V of X with the following properties: it is affine and contains x, the isotropy subgroup G_x preserves V, the action of G on X induces an étale G-morphism $\psi : G \times_{G_x} V \to X$, the image U of ψ is an open affine subset of X, and finally, the essential property, the morphism

$$\psi /\!\!/ G : (G \times_{G_x} V) /\!\!/ G \cong V /\!\!/ G_x \to U /\!\!/ G$$

is étale, and the morphisms ψ and $G \times_{G_x} V \to (G \times_{G_x} V) /\!\!/ G \cong V /\!\!/ G_x$ induce a G-isomorphism $G \times_{G_x} V \cong U \times_{U /\!\!/ G} V /\!\!/ G_x$.

[1)]This definition of "étale slice" is not the same as in [9].

Part I
Preliminaries

We begin by introducing, in the first paragraph, some general facts which will be used frequently in the following. Then, in the rest of the chapter, we recall and adapt to our purposes the following items: Matsushima's theorem, fibrations, and Zariski's main theorem.

The base field $\mathbb{k}$ is asummed algebraically closed and of characteristic 0.

1 Generalities

We introduce the notions and notations that we will use in the following (for details, see [2], [5], [11] and [15]).

Let X be an affine algebraic variety. Let $\mathbb{k}[X]$ denote its affine algebra, which is a finitely generated algebra over $\mathbb{k}$, not necessarily integral, nor reduced. The points of X correpsond either to the maximal ideals of $\mathbb{k}[X]$, or to the algebra homomorphisms of $\mathbb{k}[X]$ to $\mathbb{k}$. The elements of $\mathbb{k}[X]$ define the functions on X with values in $\mathbb{k}$.

Let G be a reductive group, that is, an affine algebraic group whose unipotent radical is trivial. Suppose that G acts morphically on X. The group G then also acts irreducibly on $\mathbb{k}[X]$ by algebra automorphisms (in the sense that the G-module $\mathbb{k}[X]$ is the direct sum of its finite-dimensional G-submodules). We can conclude (see [12, Chapitre 1, §2]):

(1) $\mathbb{k}[X]^G$, the algebra of elements in $\mathbb{k}[X]$ that are fixed by G, is finitely generated over $\mathbb{k}$,

(2) for every ideal $\mathfrak{b}$ of $\mathbb{k}[X]^G$ we have $(\mathbb{k}[X]\mathfrak{b})^G = \mathfrak{b}$,

(3) if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two ideals in $\mathbb{k}[X]$ that are invariant by G and such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathbb{k}[X]$, then $\mathfrak{a}_1^G + \mathfrak{a}_2^G = \mathbb{k}[X]^G$.

The first point allows us to define an affine variety $X /\!\!/ G$ by $\mathbb{k}[X /\!\!/ G] = \mathbb{k}[X]^G$. The inclusion $\mathbb{k}[X]^G \subset \mathbb{k}[X]$ gives a morphism $\pi_X : X \to X /\!\!/ G$. The second point then means that π_X is surjective. The third shows that "the invariants separate the disjoint closed G-orbits".

Let $\xi \in X/\!\!/G$. Since the fiber $\pi_X^{-1}(\xi)$ is non-empty, closed and G-invariant, every orbit of minimal dimension in $\pi_X^{-1}(\xi)$ is closed in X (see [2, p. 98]). Since the invariants separate the closed disjoint G-orbits, $\pi_X^{-1}(\xi)$ cannot contain two closed orbits. Thus, $\pi_X^{-1}(\xi)$ contains exactly one closed orbit, denoted by $T(\xi)$. In this manner the points in the "quotient" $X/\!\!/G$ parameterize the closed orbits of G in X. If G is connected, then since the invariants separate the closed G-invariant subsets, the fibers of π_X are connected. In general, the fiber $\pi_X^{-1}(\xi)$ has as many connected components as $T(\xi)$.

Let Y be another affine variety on which G acts, and let $\varphi : X \to Y$ be a morphism that commutes with the G-action (for short, we say: a G-variety, a G-morphism, etc.). Then φ induces a morphism $\varphi/\!\!/G : X/\!\!/G \to Y/\!\!/G$ such that $\pi_Y \circ \varphi = (\varphi/\!\!/G) \circ \pi_X$.

We fix the following notation: if $x \in X$, let $G(x)$ denote the orbit of G passing through x, and let G_x denote the isotropy subgroup of G at x. If $s \in G$ and $f \in \Bbbk[X]$, let f^S be the image of f under the automorphisms of $\Bbbk[X]$ associated with s. If $f \in \Bbbk[X]$, let X_f denote the subset of points in X on which the function f is not zero. This is an open affine subset of X with affine algebra $\Bbbk[X]_f$. If $f \in \Bbbk[X]^G$, then $X_f = \pi_X^{-1}((X/\!\!/G)_f)$ is G-invariant. If $\Bbbk[X]$ is integral, let $\Bbbk(X)$ denote its field of fractions.

2 Matsushima's theorem

Let G be a reductive group that acts on an affine variety X. Let $x \in X$.

Proposition (Matsushima [10])**.** *If the orbit $G(x)$ is closed, then the isotropy subgroup G_x is reductive.*

We sketch two proofs of this proposition. The first one is inspired by the one in [1]; the second one is due to J.L. Koszul. The two proofs rely on the following lemma:

Lemma. *Let G be a reductiv group and D an algebraic subgroup of G that is isomorphic to the additive group of $\Bbbk$. Then there exists an algebraic subgroup S of G that is simple, three-dimensional and contains D.*

This lemma is an easy consequence of the analogous result for Lie algebras (Jacobson-Morozov, see [7]).

First proof. It uses the existence and properties of quotients of algebraic groups (see for example [5, exposé 7]). We will prove the following result:

($*$) *Let G be a reductive group and H an algebraic subgroup of G. If G/H is affine, then H is reductive.*

Lemma. *Let G be an affine reductive algebraic group and $H_1 \supset H_2$ two algebraic subgroups of G. If G/H_1 and H_1/H_2 are affine, then G/H_2 is also affine.*

Proof. Since H_1/H_2 is affine, the canonical morphism $G/H_2 \to H/H_1$ is also affine. In fact, after extension by a faithfully flat morphism $G \to G/H_1$, it becomes isomorphic to the projection $G \times H_1/H_2 \to G$. Since G/H_1 is affine, it follows that G/H_2 is too. □

Now we prove ($*$) by contradiction. Suppose the unipotent radical R of H is non-trivial. Then there exists a characteristic subgroup D in R that is isomorphic to the additive group of $\Bbbk$. Let S be a three-dimensional simple subgroup of G that contains D. We knwo that S is either isomorphic to $\mathbf{SL}(2, \Bbbk)$ or to $\mathbf{SL}(2, \Bbbk)/\{\pm 1\}$. Denote by $\varphi : S' = \mathbf{SL}(2, \Bbbk) \to S$ an isomorphism or a two-fold covering, and by D' the identity component of $\varphi^{-1}(D)$.

We know that the quoitent of an affine algebraic group by a characteristic subgroup is again affine. Then, since G/H, H/R and R/D are affine, so is G/D. Hence S/D is also affine, and it follows that $S'/\varphi^{-1}(D) \cong S/D$ and $\varphi^{-1}(D)/D'$ are affine. Then $S'/D' \cong \Bbbk^2 \backslash \{0\}$ is affine, which is absurd. □

Second proof. If we identify X with a closed G-invariant subset of a vector space of finite dimension over $\Bbbk$ on which G acts linearly, then it is enough to prove the proposition for a linear representation $G \to \mathbf{GL}(M)$.

We argue by contradiction. Suppose the unipotent radical R_x of G_x is non-trivial. Then it contains a subgroup D that is isomorphic to the additive group of $\Bbbk$. Let S be a three-dimensional simple subgroup of G that contains D, and let $\varphi : \mathbf{SL}(2, \Bbbk) \to S$ be an isomorphism or a two-fold covering such that $\varphi(\{\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \mid \beta \in \Bbbk\}) = D$.

By the well-known theory of linear representations of $\mathbf{SL}(2, \Bbbk)$, M decomposes into a direct sum $\bigoplus M_i$, where $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ acts on M_i by homotheties with factor α^i. Let $x = \sum x_i$ be the decomposition of x with respect to the M_i. Since $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) x = x$, we have $x_i = 0$ for $i < 0$. Set $T = \varphi(\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mid \alpha \in \Bbbk^\times\})$.

We easily see that $x_0 \in \overline{T(x)}$. Since we assume $G(x)$ to be closed, we thus have $x_0 \in G(x)$.

Clearly, $T \subset G_{x_0}$. As T normalizes D and $D \subset G_x$, we also have $D \subset G_{x_0}$. It follows that $S \subset G_{x_0}$. Consequently, the set of sx_0, where $s \in G$ is such that $\dim(S \cap R_{sx_0}) = 0$, is a neighborhood of x_0 in the orbit $G(x_0) = G(x)$. Since $x_0 \in \overline{T(x)}$, there exists then a $t \in T$ such that $\dim(S \cap R_{tx}) = 0$. But $D = tDt^{-1} \subset tR_xt^{-1} = R_{tx}$, which is absurd. □

3 Fibrations

In this paragraph we collect the definitions and results concerning fibrations that we will need in the following.

Let E, B and F be three varieties, and $\pi : E \to B$ a morphism of varieties. We say that π is a *fibration* (locally trivial in the étale sense) of the *total space* E with *fiber* F and *base* B if there exists a variety B', a surjective étale morphism $B' \to B$ and an isomorphism $F \times B' \to X \times_B B'$ that commutes with the projection to B'.

Let G be a reductive group that acts on two affine varieties X and F. We say that $\pi_X : X \to X/\!\!/G$ is a *fibration with fiber type* F if there exists an affine variety Z, a surjective étale morphism $Z \to X/\!\!/G$ and a G-isomorphism $F \times Z \to X \times_{X/\!\!/G} Z$. In the case where $F = G$, we say that X is a *principal fibration* (see also [15, pp. 360-363]).

Let G be a reductive group that acts from the left on an affine variety Y. Let X be a principal fibration of the group G (this time acting from the right). Let G act on $X \times Y$ by $s(x, y) = (xs^{-1}, sy)$, and let $X \times_G Y$ denote the quotient (in the sense of §1) of $X \times Y$ by this action. We easily verify that $X \times GY$ is the total space of a fibration with fiber type Y and base $X/\!\!/G$. We call it the *associated fibration* to the principal fibration X.

We will now describe an important construction of G-varieties. Let H be a reductive group that acts on an affine variety Y. Let G be a redictive group that contains H as a subgroup. The group H acts by right-translations on G, which thus is a principal fibration (see [14, p. 12]). We can then form the associated fibration $G \times_H Y$. The action of G on itself by left-translations passes to an action of G on $G \times_H Y$, such that the projection $G \times_H Y \to G/H$ commutes with the action of G. The action of G on $X = G \times_H Y$ is completely determined by that of H on Y. So we easily see that $X /\!\!/ G$ can be identified with $Y /\!\!/ H$ and that $\pi_X^{-1}(\xi) \cong G \times_H \pi_Y^{-1}(\xi)$ for all $\xi \in X /\!\!/ G = Y /\!\!/ H$, etc.

Lemma. *Let X' be a closed G-invariant subvariety of $X = G \times_H Y$. Then there exists a closed H-invariant subvariety Y' of Y such that X' is G-isomorphic to $G \times_H Y'$.*

Proof. Let $\mathfrak{a}'$ denote the ideal in $(\mathbb{k}[G] \otimes \mathbb{k}[Y])^H$ associated to X', $\mathfrak{a}$ the ideal in $\mathbb{k}[G] \otimes \mathbb{k}[Y]$ generated by $\mathfrak{a}'$, and $\mathfrak{b}$ the ideal of those $f \in \mathbb{k}[Y]$ such that $1 \otimes f \in \mathfrak{a}$. We easily check that $\mathfrak{b}$ is H-invariant and that $\mathfrak{a} = \mathbb{k}[G] \otimes \mathfrak{b}$. Then define Y' by $\mathbb{k}[Y'] = \mathbb{k}[Y]/\mathfrak{b}$. The exact sequence

$$\mathbf{0} \to \mathfrak{a} = \mathbb{k}[G] \otimes \mathfrak{b} \to \mathbb{k}[G] \otimes \mathbb{k}[Y] \to \mathbb{k}[G] \otimes \mathbb{k}[Y'] \to \mathbf{0}$$

yields another exact sequence

$$\mathbf{0} \to \mathfrak{a}^H = \mathfrak{a}' \to (\mathbb{k}[G] \otimes \mathbb{k}[Y])^H \to (\mathbb{k}[G] \otimes \mathbb{k}[Y'])^H \to \mathbf{0}.$$

Therfore, X' is isomorphic to $G \times_H Y'$. $\square$

4 Zariski's main theorem

The classical formulation of Zariski's main theorem is the following (see for example [3, §3, exercise 7]): *Let X and Y be affine normal irreducible varieties and $\varphi : X \to Y$ a birational morphism with finite fibers. Then φ is an open immersion.*

For our purposes we have the following version:

Proposition. *Let X and Y be affine varieties (not necessarily irreducible or reduced), G a reductive group that acts on X and Y, and $\varphi : X \to Y$ a G-morphism. Assume that the fibers of φ are finite. Then there exists an affine G-variety Z, a G-morphism $i : X \to Z$ that is an open immersion, and a finite G-morphism $\psi : Z \to Y$ such that $\psi \circ i = \varphi$.*

For lack of a good reference, we will prove this version of the classical result quoted above.

Proof. By replacing $\mathbb{k}[Y]$ by its image in $\mathbb{k}[X]$ we may assume that $\mathbb{k}[Y] \subset \mathbb{k}[X]$.

(A) We prove first that there exists irreducible reduced affine G-varieties X' and Y' and a G-morphism $\varphi' : X' \to Y'$ with the following properties: X' is normal, X' and Y' contain X and Y, respectively, as closed G-subvarieties, and the morphism $\varphi' : X' \to Y'$ is dominant, its fibers are finite and it induces $\varphi : X \to Y$.

Let W be a G-submodule of $\Bbbk[X]$ of finite dimension over $\Bbbk$ that generates $\Bbbk[X]$ as a $\Bbbk$-algebra. The group G acts in a natural way on the symmetric algebra $\operatorname{Sym} W$, and we have a homomorphism of algebras $\varrho : \operatorname{Sym} W \to \Bbbk[X]$ that is surjective and commutes with the G-action. Let $\mathfrak{n}$ denote the kernel of ϱ. Define a variety X'' by $\Bbbk[X''] = \operatorname{Sym} W$, and a variety Y' by choosing for $\Bbbk[Y']$ a subalgebra of $\operatorname{Sym} W$ with the following properties:

(1) $\Bbbk[Y']$ is of finite type over $\Bbbk$.

(2) $\Bbbk[Y']$ is G-invariant.

(3) $\Bbbk[Y']$ contains a system of generators of the ideal $\mathfrak{n}$ of $\operatorname{Sym} W$.

(4) The image of $\Bbbk[Y']$ under ϱ is $\Bbbk[Y]$.

The group G acts on X'' and Y', and X and Y are identified with closed subvarieties of X'' and Y'. The inclusion $\Bbbk[X''] \supset \Bbbk[Y']$ yields a G-morphism $\varphi'' : X'' \to Y'$ that induces $\varphi : X \to Y$. Let $y \in Y \subset Y'$, and let $\mathfrak{m}$ denote the ideal of y in $\Bbbk[Y]$, and $\mathfrak{m}'$ the ideal of y in $\Bbbk[Y']$. We show that $\Bbbk[X'']_{\mathfrak{m}'} = \varrho^{-1}(\Bbbk[X]_{\mathfrak{m}})$. In fact, if $f \in \varrho^{-1}(\Bbbk[X]_{\mathfrak{m}})$, then there exists $g \in \Bbbk[X'']_{\mathfrak{m}'}$ such that $f - g \in \mathfrak{n}$. Since $\Bbbk[X'']_{\mathfrak{m}'}$ contains $\mathfrak{n}$ (because $\Bbbk[Y']$, and hence $\mathfrak{m}'$, contains a system of generators of $\mathfrak{n}$), it follows that $f \in \Bbbk[X'']_{\mathfrak{m}'}$. The other inclusion is obvious. As $\varphi^{-1}(y)$ is finite, $\Bbbk[X]/\Bbbk[X]_{\mathfrak{m}'}$ is of finite dimension over $\Bbbk$. Therefore, $\Bbbk[X'']/\Bbbk[X'']_{\mathfrak{m}'} \cong \Bbbk[X]/\Bbbk[X]_{\mathfrak{m}}$ is also finite-dimensional over $\Bbbk$, and so $(\varphi'')^{-1}(y)$ is also finite. The set V of $x'' \in X''$ in whose fiber $(\varphi'')^{-1}(\varphi''(x''))$ is zero-dimensional thus contains X. We know that it is open ([11, p. 97]) and it is G-invariant. Since the invariants separate the closed disjoint G-invariant subsets, there exists $f \in \Bbbk[X'']^G$ such that $X \subset X''_f \subset V$. It is clear that $X' = X''_f$, Y' and $\varphi' = \varphi''|_{X'}$ satisfy the conditions announced at the beginning of (A).

(B) We now prove the proposition. Let $\Bbbk[Z']$ denote the integral closure of $\Bbbk[Y']$ in $\Bbbk[X']$. We know that $\Bbbk[Z']$ is an algebra of finite type over $\Bbbk$ ([3, §3, théorème 2]). Denote by $\Bbbk[Z]$ the image of $\Bbbk[Z']$ under ϱ, and by Z and Z' the varieties of $\Bbbk[Z]$ and $\Bbbk[Z']$. The group G acts on Z and Z', and the inclusions induce G-morphisms $i' : X' \to Z'$, $i : X \to Z$ and $\psi : Z \to Y$. It is clear that

$\psi \circ i = \varphi$ and that ψ is finite. We easily verify that i' is birational and has finite fiber, and that Z' is normal. By the classical version of Zariski's main theorem, i' is therefore an open immersion. We immediately conclude that the same is true for i. $\square$

Remark. Keep the notation of the proposition. Let C be the integral closure of $\Bbbk[Y]$ in $\Bbbk[X]$. If $\Bbbk[X]$ is reduced, one can show that C is finite over $\Bbbk[Y]$ and thus C is of finite type over $\Bbbk$ (if $\Bbbk[X]$ is integral, this follows from [3, §3, Théorème 2], the general case reduces to this by embedding $\Bbbk[X]$ into the product of the $\Bbbk[X]/\mathfrak{p}_i$, where the $\mathfrak{p}_i$ are the minimal prime ideals of $\Bbbk[X]$). Then one easily concludes from the proposition that we can choose the Z as the variety defined by $\Bbbk[Z] = C$. However, if $\Bbbk[X]$ is not reduced, then C is not necessarily of finite type over $\Bbbk$.

An important consequence is the following lemma:

Lemma. *Let G be a reductive group that acts on two affine varieties X and Y, and let $\varphi : X \to Y$ be a G-morphism. Suppose that φ maps the closed G-orbits in X to closed orbits in Y, that the fibers of φ are finite, and that $\varphi /\!\!/ G$ is finite. Then φ is finite.*

Proof. Keep the notation from the proposition. If $\Bbbk[X]^G$ is finite over $\Bbbk[Y]^G$, we easily see that in the construction of the proposition we can require $\Bbbk[Z]$ to contain $\Bbbk[X]^G$. Then the map $i /\!\!/ G$ is an isomorphism. Let T be a closed orbit in X. The closed set $\psi^{-1}(\varphi(T))$ is composed of a finite number of closed orbits of the same dimension as $i(T)$. Therefore, $i(T)$ is closed in Z. These two remarks imply that $i(T)$ contains all closed orbits in Z. Since $i(X)$ is a G-invariant open set in Z, $i(X) = Z$ and thus $\varphi = \psi$ is finite. $\square$

Corollary (Mumford [12], Corollary 2.5)**.** *Let G be a reductive group that acts on an affine variety X. Suppose that the isotropy subgroups of G on X are finite. Then G acts properly on X.*

Proof. We need to show that the morphism $\varphi : G \times X \to X \times X$ defined by $\varphi(s, x) = (sx, x)$ is proper, that is, finite. Let G act on $G \times X$ by $t(s, x) = (ts, x)$, and on $X \times X$ by $t(x, y) = (tx, y)$. One easily verifies the conditions of the lemma now. $\square$

Part II
Étale G-morphisms

Let G be a real compact Lie group that acts differentiably on two differentiable manifolds X and Y, and let $\varphi : X \to Y$ be a differentiable map that commutes with the action of G. If φ is étale at a point $x \in X$ (that is, $\mathrm{T}_x\varphi$ is an isomorphism of $\mathrm{T}_x X$ onto $\mathrm{T}_{\varphi(x)}Y$), and if the restriction of φ to the orbit passing through x is injective, then there exists an open subset U of X with the following properties: U contains x, U is G-invariant, and the restriction of φ to U is an isomorphism from U to an open set of Y.

In this chapter, we will obtain a result (§2, the Fundamental Lemma) that in the algebraic context is analogous to the preceding one, and will play a decisive role in chapter III.

1 A preliminary lemma

Let G reductive group that acts on two affine varieties X and Y. Let $\varphi : X \to Y$ be a G-morphism, $\xi \in X/\!\!/G$, $x \in T(\xi)$ (for the definition of $T(\xi)$ see I, §1).

Lemma 1. *Assume the varieties X and Y are normal, φ finite and étale at x, and the restriction of φ to $T(\xi)$ is injective. Then $\varphi/\!\!/G$ is étale at ξ.*

In the proof of Lemma 1 we use the following characterization of étale morphisms:

Lemma 2. *Let B be an algebra of finite type over $\Bbbk$ that is integral and integrally closed, L its field of fractions, K' a finite Galois extension of L, $\mathcal{G}$ its Galois group, A' the integral closure of B in K', $\mathfrak{m}'$ a maximal ideal in A', and $\mathfrak{n} = B \cap \mathfrak{m}'$. Finally, let $\mathcal{H}$ be a subgroup of $\mathcal{G}$, K the subfield of K' of elements fixed by $\mathfrak{H}$, A the integral closure of B in K, and $\mathfrak{m} = A \cap \mathfrak{m}'$. Then $A_{\mathfrak{m}}$ is étale over $B_{\mathfrak{n}}$ if and only if $\mathcal{G}_{\mathfrak{m}'}$, the splitting group of $\mathfrak{m}'$, is contained in $\mathfrak{H}$.*

Proof of Lemma 2. We know that under the hypotheses of Lemma 2, "étale" is equivalent to "non-branching" (see for example [6, Exposé I, Théorème 9.5]). Then the proof of Lemma 2 can be found in [3, §2, Proposition 7]. □

To simplify the application of Lemma 2, we translate it into geometric language:

Let X, Y, Z denote the affine varietes of A, B, A' (A and A' are of finite type over $\Bbbk$ by [3, §3, Théorème 2]), and x, y, z the points associated to the ideals $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{m}'$, and $\varphi : X \to Y$ the morphisms given by the inclusion $B \subset A$. As $\mathcal{G}$ acts on Z, we have $Z /\!\!/ \mathcal{G} = Y$ and $Z /\!\!/ \mathcal{H} = X$. For φ to be étale at x, it is necessary and sufficient that $\mathcal{G}_z \subset \mathcal{H}$.

Proof of Lemma 1. (A) First, assume that the varieties X and Y are irreducible and the group G is connected.

Let K be a finite Galois extension of $\Bbbk(Y)$ containing $\Bbbk(X)$, with the Galois group $\mathcal{G}$. Let $\mathcal{H}$ denote the subgroup of those elements of $\mathcal{G}$ that fix $\Bbbk(X)$. Let C denote the integral closure of $\Bbbk[Y]$ in K, and C' that of $\Bbbk[Y]^G$ in K. Let Z and Z' denote the varieties associated to C and C', respectively. We can easily verify that $\mathcal{G}$ acts on Z and Z', and that $Z /\!\!/ \mathcal{G} = Y$, $Z' /\!\!/ \mathcal{G} = Y /\!\!/ G$, $Z /\!\!/ \mathcal{H} = X$ and $Z' /\!\!/ \mathcal{H} = X /\!\!/ G$ (for example, we show that $Z /\!\!/ \mathcal{G} = Y /\!\!/ G$: $\Bbbk[Z']^{\mathcal{G}}$ is integral over $\Bbbk[Y]^G$, hence also over $\Bbbk[Y]$. Then, since $\Bbbk[Z']^{\mathcal{G}} \subset \Bbbk(Y)$ and $\Bbbk[Y]$ is normal, $\Bbbk[Z']^{\mathcal{G}} \subset \Bbbk[Y]$. Moreover, if G is assumed to be connected, we easily see that $\Bbbk[Y]^G$ is integrally closed in $\Bbbk[Y]$, and so is $\Bbbk[Z']^{\mathcal{G}} = \Bbbk[Y]^G$). Choose a point $z \in Z$ over x and let z' denote the point in Z' over z.

We can apply Lemma 2 to the triple (Z, X, Y): The hypothesis "φ is étale at x" then translates into $\mathcal{G}_z \subset \mathcal{H}$.

Let $\sigma \in \mathcal{G}_{z'}$ and let $\overline{\sigma z}$ denote the image of σz in X. Let point $\overline{\sigma z}$ lies over ξ and y and thus lies over the orbit $T(\xi)$. As the restriction of φ to $T(\xi)$ is assumed injective, we conclude that $\overline{\sigma z} = x$. Therefore, since $\mathcal{H}$ acts transitively on the set of points in Z lying over x, there exists $\tau \in \mathcal{H}$ such that $\tau z = \sigma z$. Hence $\tau^{-1}\sigma \in \mathcal{G}_z \subset \mathcal{H}$ and $\sigma \in \mathcal{H}$.

We have shown that $\mathcal{G}_{z'} \subset \mathcal{H}$. In light of Lemma 2, $\varphi /\!\!/ G$ is étale at ξ.

(B) Assume now that the varieties X and Y are irreducible and that the group G is finite.

Let K be a finite Galois extension of $\Bbbk(Y /\!\!/ G)$ containing $\Bbbk(X)$, with the Galois group $\mathcal{G}'$. Let $\mathcal{H}'$ (respectively $\mathcal{G}$, $\mathcal{H}$) denote the subgroups of those elements of $\mathcal{G}'$ that fix $\Bbbk(X /\!\!/ G)$ (respectively $\Bbbk(Y)$, $\Bbbk(X)$). We have $\mathcal{H} \subset \mathcal{H}'$. Let C denote the integral closure of $\Bbbk[Y /\!\!/ G]$ in K, and Z the corresponding variety. The group $\mathcal{G}'$ acts on Z and we have $Z /\!\!/ \mathcal{G}' = Y /\!\!/ G$, $Z \mathcal{H}' = X /\!\!/ G$, $Z /\!\!/ \mathcal{G} = Y$ and $Z /\!\!/ \mathcal{H} = X$. Fix a point $z \in Z$ over x.

We can apply Lemma 2 to the triple (Z, X, Y): The hypothesis "φ is étale at x"

translates into $\mathscr{G}_z \subset \mathscr{H}$.

Let $\sigma \in \mathscr{G}'_z$. It follows from Galois theory that $\mathscr{G}'$ (respectively $\mathscr{H}'$) acts on Y (respectively on X), and that $\mathscr{G}'$ and G have the same image in the group of automorphisms of $\Bbbk[Y]$ (and the same holds for $\mathscr{H}'$ and G in the automorphisms of $\Bbbk[X]$). Consequently, there exists $s \in G$ and $\tau \in \mathscr{H}'$ such that $\sigma|_{\Bbbk[Y]} = s|_{\Bbbk[Y]}$ and $\tau|_{\Bbbk[X]} = s|_{\Bbbk[X]}$. We have $sy = \sigma y = y$. Since we assume that the restriction of φ to the orbit passing through x is injective, it follows that $\tau x = sx = x$. The group $\mathscr{H}$ acts transitively on the set of points of Z over x: There exists thus a $\varrho \in \mathscr{H}$ such that $\varrho z = \tau^{-1} z$. The element $\tau^{-1}\sigma$ fixes $\Bbbk[Y]$ and is thus contained in $\mathscr{G}$. Therefore, $\varrho^{-1}\tau^{-1}\sigma \in \mathscr{G}_z \subset \mathscr{H}$, and hence $\sigma \in \mathscr{H}'$.

We have shown that $\mathscr{G}'_z \subset \mathscr{H}'$. Since we can also apply Lemma 2 to the triple $(Z, X /\!\!/ G, Y /\!\!/ G)$, it follows that $\varphi /\!\!/ G$ is étale at ξ.

(C) We now consider the general case.

We may always assume that $X /\!\!/ G$ and $Y /\!\!/ G$ are irreducible. Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_l$ be the irreducible components of X and Y, respectively (where, say, $x \in X_1$ and $y \in Y_1$). The orbit $T(\xi)$ then intersects all X_i, and $\varphi(T(\xi))$ all Y_j. Since, by assumption, the restriction of φ to $T(\xi)$ is injective, it follows that $k = 1$, and that the subgroup G_1 of G that fixes X_1 is also the one that fixes Y_1. Since $X /\!\!/ G = X_1 /\!\!/ G_1$, $Y /\!\!/ G = Y_1 /\!\!/ G_1$, etc., we see that we may assume the varieties X and Y to be irreducible. The case now follows immediately from (A) and (B) (since $X /\!\!/ G = (X /\!\!/ G^\circ)/(G/G^\circ)$, etc.). $\square$

2 The fundamental lemma

Let G be a reductive group that acts on two affine varieties X and Y, and let $\varphi : X \to Y$ be a G-morphism. Let ξ be a point in $X /\!\!/ G$ and x a point in the orbit $T(\xi)$.

Lemma 3. *Assume the G-morphism φ is étale at x, the variety X is normal at x (or, which amounts to the same, the variety Y is normal at $\varphi(x)$), the orbit $\varphi(T(\xi))$ is closed and the restriction of φ to $T(\xi)$ is injective. Then there exists an affine open subset U' of $X /\!\!/ G$ with the following properties: The open set $U = \pi_X^{-1}(U')$ contains $T(\xi)$, the restrictions of φ to U and of $\varphi /\!\!/ G$ to U' are étale, the open sets $V = \varphi(U)$ and $V' = (\varphi /\!\!/ G)(U')$ are affine, $V = \pi_Y^{-1}(V')$, and finally, the morphism $\varphi : U \to V$ maps the closed orbits in U to closed orbits in V.*

Proof. The sets of normal points of X and Y are open, G-invariant and contain $T(\xi)$ and $T(\varphi(\xi))$. Since the invariants separate the disjoint closed G-invariant sets, there exists $f \in \Bbbk[X]^G$ and $g \in \Bbbk[Y]^G$ such that X_f and Y_g are normal and contain $T(\xi)$ and $\varphi(T(\xi))$. It is clearly sufficient to show Lemma 3 for $\varphi : X_f \cap \varphi^{-1}(Y_g) \to Y_g$. We may thus assume that the varieties X and Y are normal. By a similar argument we may also assume that the fibers of φ are finite.

Let $\Bbbk[\tilde{X}]$ denote the integral closure of $\Bbbk[Y]$ in $\Bbbk[X]$, and $\tilde{X}$ the corresponding affine variety. According to I, §4, $\tilde{X}$ is a normal G-variety, and the homomorphism of algebras $\Bbbk[Y] \to \Bbbk[\tilde{X}] \subset \Bbbk[X]$ yields an open G-immersion $i : X \to \tilde{X}$ and a finite G-morphism $\tilde{\varphi} : \tilde{X} \to Y$. By identifying X with its image under i, φ becomes the restriction of $\tilde{\varphi}$ to X. The orbit $T(\xi)$ is closed in $\tilde{X}$: In fact, $\varphi(T(\xi))$ being assumed closed, $\tilde{\varphi}^{-1}(\varphi(T(\xi)))$ is closed and composed of a finite number of orbits of the same dimension as $T(\xi)$. The set of points where $\tilde{\varphi}$ is étale is open, G-invariant and contains $T(\xi)$. By Lemma 1, $\tilde{\varphi}/\!\!/G$ is étale at $\tilde{\xi} = (i/\!\!/G)(\xi)$. Since the invariants separate the disjoint closed G-invariant subsets, we can find $f \in \Bbbk[\tilde{X}]^G \subset \Bbbk[X]^G$ with the following properties: $\tilde{X}_f = X_f \subset X$, $(\tilde{X}/\!\!/G)_f \cong (X/\!\!/G)_f$ is irreducible and contains ξ, and the restrictions of φ to X_f and $\varphi/\!\!/G$ to $(X/\!\!/G)_f$ are étale. In a similar fashion, since $\varphi(T(\xi))$ is assumed closed, we find that there exists $g \in \Bbbk[Y]^G$ such that $\varphi(T(\xi)) \subset Y_g \subset \varphi(X_f)$. Set $U' = (X/\!\!/G)_f \cap (\varphi/\!\!/G)^{-1}((Y/\!\!/G)_g)$. We see that U' satisfies the requirements of Lemma 3.

It is easily verified that $V' = (Y/\!\!/G)_g$, as $U = X_f \cap \varphi^{-1}(Y_g)$ and $V = Y_g$. All the claims of Lemma 3 follow immediately, except for the last. Here is the proof: The closed orbits in U are also closed in $\tilde{X}$ (since U contains for every point x the fiber $\pi_{\tilde{X}}^{-1}(\pi_{\tilde{X}}(x))$). As $\varphi : \tilde{X} \to Y$ is finite, φ maps the closed orbits in U to closed orbits in Y. □

Fundamental Lemma. *Let G be a reductive group that acts on two affine varieties X and Y, and let $\varphi : X \to Y$ be a G-morphism. Let $x \in X$ and assume that φ is étale at x, X is normal at x (or the variety Y is normal at $\varphi(x)$), the orbits $G(x)$ and $G(\varphi(x))$ are closed, and finally that the restriction of φ to $G(x)$ is injective. Then, there exists an affine open set U of X with the following properties: The open set U contains x and is saturated with respect to the projection π_X, the restriction of φ to U is étale, the image V of U under φ is an affine open subset of Y that is saturated with respect to the projection π_Y, the morphism $\varphi/\!\!/G : \pi_X(U) \cong U/\!\!/G \to \pi_Y(V) \cong V/\!\!/G$ is étale, and finally, the morphisms $\varphi : U \to V$ and $\pi_U : U \to U/\!\!/G$ induce a G-isomorphism*

$\chi : U \to V \times_{V/\!\!/G} U/\!\!/G$.

Proof. Let U be as in the preceding Lemma 3. It remains to prove the last assertion of the Fundamental Lemma.

We have $(V \times_{V/\!\!/G} U/\!\!/G)/\!\!/G \cong U/\!\!/G$. If $\xi \in U/\!\!/G$, then the fiber of $V \times_{V/\!\!/G} U/\!\!/G$ over ξ is canonically isomorphic to the fiber of V over $\eta = (\varphi/\!\!/G)(\xi)$. The morphism χ induces the identity on the level of quotients and maps $\pi_U^{-1}(\xi)$ to $\pi^{-1}_{V\times_{V/\!\!/G}U/\!\!/G}(\eta)$, in exactly the same fashion in which φ maps $\pi_U^{-1}(\xi)$ to $\pi_V^{-1}(\eta)$. Since φ maps the closed orbits in U onto closed ones, it follows that χ does the same. By the lemma in I, §4, it follows that χ is finite.

The morphism χ is also étale. In fact, $\mathrm{id}_V \times (\varphi/\!\!/G) : V \times_{V/\!\!/G} U/\!\!/G \to V \times_{V/\!\!/G} V/\!\!/G = V$ and $\varphi : U \to V$ are, and $\varphi = (\mathrm{id}_V \times (\varphi/\!\!/G)) \circ \chi$. Hence χ is a covering. Since $\chi/\!\!/G$ is an isomorphism, we have $\chi^{-1}(\chi(G(x))) = G(x)$. By assumption, the restriction of φ to $G(x)$is injective, and so the same is true for that of χ. Since all connected components of $V \times_{V/\!\!/G} U/\!\!/G$ meet $\chi(G(x))$ (because $(V \times_{V/\!\!/G} U/\!\!/G)/\!\!/G$ is isomorphic to $U/\!\!/G$, which is rreducible by construction), χ is a covering of one sheet, hence an isomorphism. □

Corollary. *Keep the assumptions and notations from the Fundamental Lemma. For all $\xi \in U/\!\!/G$, φ induces a G-isomorphism $\pi_X^{-1}(\xi) \to \pi_Y^{-1}((\varphi/\!\!/G)(\xi))$.*

Proof. This follows immediately from the Fundamental Lemma. □

The following lemma, technical but banal, will be useful in Chapter III, §1.

Lemma. *Let G be a reductive group that acts on two affine varieties X and Y, Y' a closed G-subvariety of Y, and $\varphi : X \to Y$ a G-morphism. Set $X' = Y' \times_Y X$ and $\varphi' = \mathrm{id}_Y \times \varphi : X' = Y' \times_Y X \to Y' \times_Y Y = Y'$. Suppose that φ and $\varphi/\!\!/G$ are étale and that $X \to Y \times_{Y/\!\!/G} X/\!\!/G$ is a G-isomorphism. Then φ' and $\varphi'/\!\!/G$ are étale and $X' \to Y' \times_{Y'/\!\!/G} X'/\!\!/G$ is a G-isomorphism.*

Proof. Clearly φ' is étale. We have

$$X' = Y' \times_Y X \cong Y' \times_Y (Y \times_{Y/\!\!/G} X/\!\!/G) \cong Y' \times_{Y/\!\!/G} X/\!\!/G.$$

Hence, $X'/\!\!/G \cong Y'/\!\!/G \times_{Y/\!\!/G} X/\!\!/G$, and

$$\varphi'/\!\!/G = \mathrm{id}_{Y'/\!\!/G} \times (\varphi/\!\!/G) : X'/\!\!/G \cong Y'/\!\!/G \times_{Y/\!\!/G} X/\!\!/G \to Y'/\!\!/G \times_{Y/\!\!/G} Y/\!\!/G = Y'/\!\!/G$$

is étale. Finally,

$$X' \cong Y' \times_{Y/\!\!/G} X/\!\!/G \cong Y' \times_{Y'/\!\!/G} (Y'/\!\!/G \times_{Y/\!\!/G} X/\!\!/G) \cong Y' \times_{Y'/\!\!/G} X'/\!\!/G. \; \square$$

Part III
Étale slices

Let G be a reductive group that acts on an affine variety X. In the first paragraph we will show that there exists an "étale slice" at every point of a closed orbit of G in X. For this, we rely on the Fundamental Lemma of the previous chapter. The following three paragraphs are dedicated to applications: If X is smooth, then we obtain essentially the same results as for the action of a compact Lie group on a differentiable manifold. However, the situation is richer since there exist non-closed orbits. Here is a typical example of a small phenomenon that we will explain in a general framework: The well-known isomorphy of the set of unipotent elements in a reductive group and that of nilpotent elements in its Lie algebra. In the last paragraph, we recover a result by R.W. Richardson [13].

1 Étale slices

Let G be a reductive group that acts on an affine variety X. Let $x \in X$.

Lemma. *Assume the variety X is smooth at X and G_x, the isotropy subgroup of G at x, is reductive. The there exists a morphism of varieties $\varphi : X \to \mathrm{T}_x X$ with the following properties:*

(1) *φ commutes with the G_x-action.*

(2) *φ is étale at x.*

(3) $\varphi(x) = 0$.

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $\Bbbk[X]$ that corresponds to the point x. The canonical map $d : \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 = (\mathrm{T}_x X)^*$ commutes with the action of G_x. As G_x is assumed to be reductive, it acts completely reducibly on $\Bbbk[X]$. Hence there exists a G_x-submodule W of $\mathfrak{m}$ such that $d : W \to (\mathrm{T}_x X)^*$ is an isomorphism. We extend $(d|_W)^{-1}$ in a canonical way to a homomorphism of the symmetric algebra of $(\mathrm{T}_x X)^*$ to $\Bbbk[X]$. It is easily verified that the morphism $\varphi : X \to \mathrm{T}_x X$ corresponding to it has the desired properties. □

Étale Slice Theorem. *Let G be a reductive group that acts on an affine variety X. Let x be a point in X whose orbit $G(x)$ is closed. Then there exists a subvariety V of X with the following properties:*

(a) *The isotropy subgroup G_x preserves V.*

(b) *The G-action on X induces an étale G-morphism $\psi : G \times_{G_x} V \to X$.*

(c) *The image U of ψ is an affine open π_X-saturated subset of X.*

(d) *The morphism $\varphi /\!\!/ G : (G \times_{G_x} V)/\!\!/ G \cong V/\!\!/ G_x \to U/\!\!/ G$ is étale.*

(e) *The morphisms ψ and $G \times_{G_x} V \to (G \times_{G_x} V)/\!\!/ G \cong V/\!\!/ G_x$ induce a G-isomorphism*

$$G \times_{G_x} V \to U \times_{U/\!\!/ G} V/\!\!/ G_x.$$

Proof. (A) Suppose first that X is smooth at x. Since the orbit $G(x)$ is assumed closed in X, the isotropy subgroup G_x is reductive (see I, §2). So the conditions of the previous lemma are satisfied. Hence there exists a morphism $\varphi : X \to \mathrm{T}_x X$ with the three properties of this lemma. Choose a G_x-invariant complementary subspace N to $\mathrm{T}_x G(x)$ in $\mathrm{T}_x X$. Set $Y = \varphi^{-1}(N)$. This is a closed subvariety of X containing x, smooth at x and G_x-invariant. The morphism $G \times X \to X$ defining the action of G on X induces a G-morphism $G \times_{G_x} Y \to X$ that is étale at the point $\overline{(e, x)}$ (where e denotes the neutral element of G and $\overline{(e, x)}$ the canonical image of $(e, x) \in G \times Y$ in $G \times_{G_x} Y$). We easily see that the assumptions of the Fundamental Lemma in II, §2 are satisfied for $G \times_{G_x} Y \to X$. The theorem follows immediately.

(B) In the general case, identify X with a closed G-subvariety of a smooth affine G-variety. Then the theorem follows from part (A), the lemma following the Fundamental Lemma, and the remarks in I, §3. □

Remark 1. Keep the assumptions of the theorem. If X is smooth at x, after shrinking V if necessary, it can be arranged that, in addition to the properties of the theorem, it also has the following (see the Fundamental Lemma): V is smooth, the morphism $\varphi : V \to N = \mathrm{T}_x V$ is étale, the image W of V under φ is an open π_N-saturated subset of N, the morphism $\varphi/\!\!/ G_x : V/\!\!/ G_x \to W/\!\!/ G_x$ is étale, and finally, the morphisms $\varphi : V \to W$ and $\pi_V : V \to V/\!\!/ G_x$ induce a G_x-isomorphism $V \to W \times_{W/\!\!/ G_x} V/\!\!/ G_x$. If we once more come back to the analogy with the behaviour of the action of a compact Lie group on a differentiable manifold, in a neighborhood of an orbit, corresponding to the isomorphism here correspond the two G-morphisms $G \times_{G_x} V \to G \times_{G_x} W$ and $G \times_{G_x} V \to U$ ($G \times_{G_x} W$ is an open subset of the "normal fibration" of the orbit).

We call the subvarieties V of X with properties enumerated in the theorem (and which also have the properties in Remark 1 if the point x is a simple point in X) the *étale slices* at x.

Remark 2. It follows from the corollary of the Fundamental Lemma that for all $x' \in V$, ψ induces a G-isomorphism $G \times_{G_x} \pi_V^{-1}(\pi_V(x')) \to \pi_X^{-1}(\pi_X(x'))$ (if X is smooth at x, also $\pi_X^{-1}(\pi_X(x)) \cong G \times_{G_x} \pi_N^{-1}(\pi_N(0))$).

Remark 3. If the base field is the field of complex numbers, then the theorem shows the existence of an "analytic slice". Namely, of an analytic submanifold V' of X containing x, invariant under G_x and such that the action of G on X induces an analytic isomorphism from $G \times_{G_x} V'$ to an open neighborhood (in the standard topology) of $G(x)$ in X. This follows immediately from Remark 2 and the fact that $\psi /\!\!/ G|_{V /\!\!/ G_x}$ is then a locally analytic isomorphism.

Remark 4. The existence of étale slices at x implies that the isotropy subgroups of points in a neighborhood of x are conjugate in G to the subgroup G_x. If the orbit $G(x)$ is not closed, this may be false, even if G_x is reductiv. Here is an example (due to R.W. Richardson): Consider $\mathbf{SL}(2, \Bbbk)$ acting in the natural way on the cubic forms in two variables. The isotropy subgroup of $x^2 y$ is trivial, but there exists an open dense set composed of forms whose isotropy subgroup is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Corollary 1. *Let G be a reductive group that acts on an affine variety X (not necessarily reduced). Then X is a principal fibration (see I, §3) if and only if all isotropy subgroups of G in X are trivial.*

This follows immediately from the theorem.

Corollary 2. *Let G be a reductive group that acts on an affine variety X (not necessarily reduced). Suppose $X /\!\!/ G$ is connected and zero-dimensional. Then there exists a reductive subgroup H of G and an affine H-variety Y such that*

(1) *H as a fixed point y in Y.*

(2) *All orbits of H in Y are adherents of y.*

(3) *X is G-isomorphic to $G \times_H Y$.*

If we further assume that X is smooth at a point in a closed orbit of G, then the H-variety Y is isomorphic to a vector space of finite dimension over $\Bbbk$ on which H acts linearly.

Proof. Let x be a point in the unique closed orbit of G in X, and let V be an étale slice at x. Since we assume that $X /\!\!/ G$ is connected and of dimension 0 and since $V /\!\!/ G_x \to X /\!\!/ G$ is étale, after shrinking V if necessary, we may assume that $V /\!\!/ G_x \to X /\!\!/ G$ is an isomorphism. The open subset U of the theorem that is G-invariant and that contains the unique closed orbit of G in X is all of X. From the last claim of the theorem it then follows that $G \times_{G_x} V$ is G-isomorphic to $X \times_{X /\!\!/ G} V /\!\!/ G_x \cong X$.

Assume now that in addition X is smooth at a point in the closed orbit of G in X, and use the notation of Remark 1. Since $V /\!\!/ G_x \to W /\!\!/ G_x$ is étale, and $W /\!\!/ G_x$ is reduced and connected, $V /\!\!/ G_x$ and $W /\!\!/ G_x$ are necessarily isomorphic to reduced points. It follows that $W = \mathrm{T}_x V$ (W is open, G_x-invariant and contains the unique closed orbit of G_x in $\mathrm{T}_x V$, the origin). Hence V is G-isomorphic to $W \times_{W /\!\!/ G_x} V /\!\!/ G_x \cong \mathrm{T}_x V$. □

Remark. With the same assumptions as in the last assertion of Corollary 2, we assume further that G has a fixed point in X. It follows immediately that X is a vector space on which G acts linearly. A particular well-known case is the following result: Let $A = \bigoplus_{n \geq 0} A_n$ be a graded algebra of finite type over $\Bbbk$. Let X denote the affine variety of A. Suppose that $A_0 = \Bbbk$ and that the point of X that corresponds to a maximal ideal of $\bigoplus_{n \geq 1} A_n$ is a simple point. Then A is isomorphic to a polynomial algebra over $\Bbbk$.

2 Models and stratifications of the quotient

Let G be a reductive group. Consider the G-varieties $G \times_H N$, where H is a reductive subgroup of G and where N is a vector space of finite dimension over $\Bbbk$ on which H acts linearly. These smooth affine G-varieties – which are nothing else but the G vector bundles with affine homogeneous base – are particularly important: We will see that, in a sense to be made precise, they serve as models for G-actions on smooth affine varieties in general. Therefore we will call (the isomorphism classes of) these G-varieties the *models*. We denote by $\mathfrak{M}$ the set of models.

Let X be a smooth affine G-variety. We define a map $\mu : X/\!\!/G \to \mathfrak{M}$ by mapping $\xi \in X/\!\!/G$ to $\mu(\xi)$, the isomorphism class of the normal fibration of $T(\xi)$.

Corollary 3. *The image of $X/\!\!/G$ under μ is a finite subset of $\mathfrak{M}$.*

On can show the same result for a a real compact Lie group that acts differentiably on a compact differentiable variety (see [8, §4]). The proof of Corollary 3 is the same (where the étale slice plays the role of the differentiable slice).

For $\lambda \in \mathfrak{M}$, set $(X/\!\!/G)_\lambda = \mu^{-1}(\lambda)$.

Corollary 4. *For all $\lambda \in \mathfrak{M}$, $(X/\!\!/G)_\lambda$ is locally closed in $X/\!\!/G$ and smooth with respect to its structure of a reduced subvariety.*

We thus obtain a finite "stratification" of $X/\!\!/G$ into locally closed smooth subvarieties.

Proof. Let U and V be two smooth affine G-varieties, $\varphi : U \to V$ a surjective G-morphism, and assume that φ and $\varphi/\!\!/G$ are étale and $U \to V \times_{V/\!\!/G} U/\!\!/G$ is a G-morphism. Then one easily sees that for $(U/\!\!/G)_\lambda$ to be closed in $U/\!\!/G$ and smooth, it is necessary and sufficient that $(V/\!\!/G)_\lambda$ is closed in $V/\!\!/G$ and smooth.

Let $G \times_H N$ be a representative of λ. We can identify $(G \times_H N)/\!\!/G$ with $N/\!\!/H$ (see I, §3). It is clear that we have $((G \times_H N)/\!\!/G)_\lambda = N^H/\!\!/H \cong N^H$.

The corollary 4 follows immediately from these two remarks on étale slices. □

For $\lambda \in \mathfrak{M}$, let $X_\lambda = \pi_X^{-1}((X/\!\!/G)_\lambda)$. As preimage under π_X of $(X/\!\!/G)_\lambda$, X_λ has a structure of a closed subvariety of X, not necessarily reduced. The fibers of π_X over $(X/\!\!/G)_\lambda$ are all G-isomorphic. More precisely:

Corollary 5. *The morphism $\pi_X : X_\lambda \to (X/\!\!/G)_\lambda$ is a G-fibration (see I, §3).*

The proof is analogous to that of the preceding corollary.

3 Principal models

We keep the assumptions of the previous paragraph. If T_1 and T_2 are two homogeneous G-spaces, we say that T_1 is *larger* than T_2 if there exists a G-morphism

$T_1 \to T_2$. This way, we define an order relation on the set of homogeneous G-spaces. We say that two orbits of G in X are *neighbors* if they project to the same connected component of $X /\!\!/ G$ under π_X. For $x \in X$, we set $N_x = \mathrm{T}_x X / \mathrm{T}_x G(x)$. If M is a fully reducible G-module, we denote by M_G the canonical G-invariant complement of M^G in M.

Corollary 6. *Let $\xi \in X /\!\!/ G$. The following conditions are equivalent:*

(1) *$(X /\!\!/ G)_{\mu(\xi)}$ lies in a neighborhood of ξ in $X /\!\!/ G$.*

(2) *$T(\xi)$ is maximal among the closed orbits of G in X that are neighbors of $T(\xi)$.*

(3) *$\pi_{N_x}^{-1}(\pi_{N_x}(0)) = (N_x)_{G_x}$ for all $x \in T(\xi)$.*

(4) *π_x is smooth at the points in $T(\xi)$.*

Proof. Thanks to étale slices we easily see that it is sufficient to prove the equivalence in the case where $T(\xi)$ is a point. Then, one is immediately reduced to the case where X is a vector space M on which G acts linearly and where $\xi = \pi_M(0)$. We rewrite the conditions for this situation:

(1) $M^G /\!\!/ G (\cong M^G)$ is open in $M /\!\!/ G$ (that is, equal to $M /\!\!/ G$).

(2) The origin is a maximal orbit among the closed orbits in M.

(3) $\pi_M^{-1}(\pi_M(0)) = M_G$.

(4) π_M is smooth at 0.

We easily see (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (4). The fiber $\pi_M^{-1}(\pi_M(0))$ is a cone in M that is smooth at the origin and G-invariant. It is then necessarily a G-submodule (see the remark following Corollary 2). For every nontrivial irreducible G°-module N we have $\pi_N^{-1}(\pi_N(0)) \neq \mathbf{0}$. It follows that $\pi_M^{-1}(\pi_M(0)) = M_{G^\circ}$. The quotient $M /\!\!/ G^\circ$ is the identified with M^{G°, and $M \to M /\!\!/ G^\circ$ is smooth. As π_M is assumed smooth, it follows that $M /\!\!/ G^\circ \cong M^{G^\circ} \to M /\!\!/ G \cong M^{G^\circ} /\!\!/ (G/G^\circ)$ is also smooth. But this only possible when $M^{G^\circ} = M^G$. Therefore, $M_G = M_{G^\circ} = \pi_M^{-1}(\pi_M(0))$. □

Definition. If one of the conditions of Corollary 6 holds, we say that the model of ξ is *principal*. If $X /\!\!/ G$ is connected, there is only one possible principal model, which we call the *principal model of X*.

Now we show that étale slices allow to recover – and thus better understand – the main result of [9].

Corollary 7. *Let G be a reductive group that acts on a smooth affine variety X. Suppose that the tangent space of every point in X is equipped with a symmetric non-degenerate bilinear form that is invariant under the isotropy subgroup. Then there exists an open dense subset of X that consists of closed orbits in X.*

Proof. Recall that a G-module is called *G-orthogonalizable* if it admits a non-degenerate symmetric G-invariant bilinear form. One easily proves the following result (see [9, Lemma 5]): Let M be a G-module, and N a G-submodule of M. If M and N are orthogonalizable, then so is M/N.

Pick $\xi \in X /\!\!/ G$ whose model is principal, and $x \in T(\xi)$. From the above it follows that $(N_x)_{G_x}$ is G_x-orthogonalizable (the Lie algebras $\mathfrak{g}$, $\mathfrak{g}_x$ of G, G_x are, then also $\mathrm{T}_x G(x) \cong \mathfrak{g}/\mathfrak{g}_x$, then also $N_x = \mathrm{T}_x / \mathrm{T}_x G(x)$, then also $(N_x)_{G_x} \cong N_x / N_x^{G_x}$, since for $N_x^{G_x}$ this is clear). Since $(N_x)_{G_x} = \pi_{N_x}^{-1}(\pi_{N_x}(0))$ (Corollary 6), it follows that $(N_x)_{G_x}$ is zero. Therefore, $\pi_X^{-1}(\xi) = T(\xi)$. Corollary 7 follows immediately. □

4 The generic isotropy subgroup

In this paragraph, all varieties are assumed to be irreducible and reduced.

Let G be a reductive group that acts on an affine variety X. We set $r = \dim X - \dim X /\!\!/ G$ and $s = \max_{x \in X} \dim G(x)$. In general, we have $s \leq r$. If $s = r$, then we say that $X /\!\!/ G$ has *good dimension*.

Lemma. *If $\Bbbk(X /\!\!/ G) = \Bbbk(X)^G$, then $X /\!\!/ G$ has good dimension.*

Proof. We may assume that $\Bbbk[X]$ is a free module over $\Bbbk[X]^G$: In fact, we know that there exist $f \neq 0$ in $\Bbbk[X]^G$ such that $\Bbbk[X]_f$ is free over $\Bbbk[X]_f^G$ (see for example [15, Lemma 2.7, p. 77]). If clearly suffices to prove the lemma for X_f.

Let $\varphi : G \times X \to X$ and $\psi : G \times X \to X \times_{X /\!\!/ G} X$ the morphisms defined by $\varphi(s, x) = sx$ and $\psi(s, x) = (sx, x)$, and let $\tilde{\varphi}$ and $\tilde{\psi}$ denote their correspondig algebra morphisms. For $s \in G$,

$$\Bbbk[X] \xrightarrow{\tilde{\varphi}} \Bbbk[G] \otimes \Bbbk[X] \xrightarrow{s \otimes 1} \Bbbk \otimes \Bbbk[X] = \Bbbk[X]$$

maps f to f^s. For all $f \otimes g \in \Bbbk[X] \otimes_{\Bbbk[X/\!\!/G]} \Bbbk[X] = \Bbbk[X \times_{X/\!\!/G} X]$, we have $\tilde{\psi}(f \otimes g) = \tilde{\varphi}(f)(1 \otimes g)$.

We show that $\tilde{\psi}$ is injective. Let e_i be a basis of $\Bbbk[X]$ over $\Bbbk[X]^G$. It is enough to see that $\sum \tilde{\varphi}(e_i)(1 \otimes f_i) = 0$ implies that all f_i are 0. But $\sum \tilde{\varphi}(e_i)(1 \otimes f_i) = 0$ implies $\sum e_i^s f_i = 0$ for any $s \in G$. Since by assumption $\Bbbk(X/\!\!/G) = \Bbbk(X)^G$, the e_i are also linearly independent over $\Bbbk(X)^G$, and we can use a theorem by Artin [4, §7, no. 1]:

Let L be a commutative field, G an automorphism group of L, K the field of those elements in L fixed by G, and $e_1, \dots, e_n$ elements of L that are linearly independent over K. Then there exist $s_1, \dots, s_n$ in G such that $\det(e_i^{s_j}) \neq 0$.

From this it follows by a linear algebraic argument that the f_i are all 0. Then $X \times_{X/\!\!/G} X$ is irreducible and the morphism $\psi : G \times X \to X \times_{X/\!\!/G} X$ is dominant. We easily see that $X \times_{X/\!\!/G} X$ is of dimension $r + \dim X$. We know that then there exists a fiber of ψ all of whose irreducible components have dimension $\dim G \times X - \dim X \times_{X/\!\!/G} X = \dim G - r$ (see [11, p. 93]). As the fibers of ψ have the same dimension as the isotropy subgroup of G on X, there exists $x \in X$ such that $\dim G_x = \dim G - r$. Therefore, $s \geq \dim G(x) = \dim G - \dim G_x = r$. □

Lemma. *Let G be a group that acts on an affine variety X. Assume that $\Bbbk[X]$ is factorial and that the only invertible elements of $\Bbbk[X]$ are the non-zero constants. Then:*

(1) *All $f \in \Bbbk(X)^G$ can be writte $f = \frac{g}{h}$, where g and h are relative invariants of G in $\Bbbk[X]$.*

(2) *There exists a G-invariant non-empty affine open set X' in X such that $\Bbbk(X')^G$ is the field of fractions of $\Bbbk[X']^G$.*

Proof. (1) Write $f = \frac{g}{h}$, where g and h are coprime elements in $\Bbbk[X]$. For all $s \in G$ we have $gh^s = g^s h$. As $\Bbbk[X]$ is factorial, we deduce that $g^s = \chi(s)g$, where $\chi(s) \in \Bbbk^\times$. We have $\chi(st)g = g^{st} = (g^s)^t = (\chi(s)g)^t = \chi(s)\chi(t)g$, hence $\chi : G \to \Bbbk^\times$ is a character.

(2) The field $\Bbbk(X)^G$ is, as a subfield of $\Bbbk(X)$, of finite type over $\Bbbk$. Let $f_1, \dots, f_n$ be a system of generators of $\Bbbk(X)^G$ over $\Bbbk$. By (1), we can write $f_i = \frac{g_i}{h_i}$, where the g_i and h_i are relative invariants of G in $\Bbbk[X]$. Let $h = h_1 \cdots h_n$ and $X' = X'_h$. We verify without difficulty that X' satisfies the conditions of the lemma. □

Corollary 8 (Richardson [13]). *Let G be a reductive group that acts on a smooth affine variety X. Then there exists a subgroup H of G, not necessarily reductive, such that the set of points in X whose isotropy subgroup is conjugate to H has non-empty interior.*

Proof. We easily see, thanks to Corollaries 5 and 6, that is is enough to show Corollary 8 in the case where X is a vector space. By the two preceding lemmas, there exists then a G-invariant non-empty affine open subset X' of X such that $X'/\!\!/G$ has good dimension. For all $\xi \in X'/\!\!/G$ whose model is principal, $\pi_{X'}^{-1}(\xi)$ then contains an orbit that is open in $\pi_{X'}^{-1}(\xi)$. The fibers of such ξ are G-isomorphic, so that Corollaries 5 and 6 yield the conclusion. □

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