On groups and associated Lie rings

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1. In the course of his studies of groups of prime power order, W.B. Fite[1)] introduced in 1906 the following characteristic subgroup in a given group $\mathfrak{G}$: If $\mathfrak{H}$ is any subgroup of $\mathfrak{G}$, then let $[\mathfrak{H}, \mathfrak{G}]$ denote the subgroup generated by the commutators of any element of $\mathfrak{G}$ with any element of $\mathfrak{H}$, and define recursively: $\mathfrak{G}_n = [\mathfrak{G}_{n-1}, \mathfrak{G}]$ $(n = 2, 3, \ldots)$, $\mathfrak{G}_1 = \mathfrak{G}$. The groups $\mathfrak{G}_n$ $(n = 1, 2, \ldots)$ are called the groups of the *lower central series* of $\mathfrak{G}$ by P. Hall[2)]. The investigations of Reidemeister[3)] and Hall have shown that they provide an extraordinarily useful tool in general group theory. In the following we shall develop the relations of the groups $\mathfrak{G}_n$ to a Lie ring associated to the group $\mathfrak{G}$, and study further applications.

2. The groups $\mathfrak{G}_n/\mathfrak{G}_{n+1}$ are denoted by $\mathfrak{A}_n(\mathfrak{G})$, or, if there is ambiguity, by $\mathfrak{A}_n$. If $\mathfrak{G}$ is finitely generated, then the abelian groups $\mathfrak{A}_n$ have finite bases. It suggests itself to combine such group $\mathfrak{G}$ and $\mathfrak{G}^*$ in a type such that for that all values $n = 1, 2 \ldots$ the groups $\mathfrak{A}_n(\mathfrak{G})$ and $\mathfrak{A}_n(\mathfrak{G}^*)$ are isomorphic. The question arises when, for a given sequence of abelian groups $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$, we can find a group $\mathfrak{G}$, such that for $n = 1, 2, \ldots$ we have $\mathfrak{A}_n \cong \mathfrak{A}_n(\mathfrak{G})$. It is easy to see that the groups $\mathfrak{A}_n$ cannot be chosen independently from each other. The relations between them can be analyzed by the following method:

3. Consider $\mathfrak{G}$ as the factor group $\mathfrak{F}/\mathfrak{N}$ of a free group $\mathfrak{F}$ by a normal subgroup $\mathfrak{N}$. Then we have

$$\mathfrak{A}_n(\mathfrak{G}) \sim \mathfrak{F}_n/\langle \mathfrak{F}_{n+1} \cdot (\mathfrak{F}_n, \mathfrak{N}) \rangle \tag{1}$$

where $\langle\ \rangle$ denotes the product of the normal subgroups in the brackets. $\mathfrak{A}_n(\mathfrak{G})$ is thus a factor group of $\mathfrak{A}_n(\mathfrak{F})$, and this group shall be studied first. Think of $\mathfrak{G}$ as being generated by certain elements $a, b, \ldots$. At the same time, these can be considered as elements of $\mathfrak{F}$, where in general there will be more relations satisfied by the generators of $\mathfrak{G}$ than by those of $\mathfrak{F}$. Now set

$$\begin{gathered} a = 1 + x, \quad b = 1 + y, \quad \ldots \\ a^{-1} = 1 - x + x^2 \mp \ldots, \quad b^{-1} = 1 - y + y^2 \mp \ldots, \quad \ldots \end{gathered} \tag{2}$$

[1)]W.B. Fite, Transactions of the American Mathematical Scoiety 7, 61-68, 1906.
[2)]P. Hall, Proceedings of the London Mathematical Society (2) 36, 29-95, 1933.
[3)]K. Reidemeister, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität 5, 33-39, 1927.

where $x, y, \ldots$ denote generators of a "free" ring $\mathfrak{R}$ with unit element 1. More precisely, the ring $\mathfrak{R}$ consists of all formal power series with integer coefficients in the non-commutative but associative variables $x, y, \ldots$[4)] Every "word" or power product $W(a, b, \ldots)$ in the generators of $\mathfrak{F}$ is then assigned an element of $\mathfrak{R}$ which can be thought of as being ordered by terms of equal dimension or equal degree in the variables $x, y, \ldots$, that is,

$$W(a, b, \ldots) = 1 + d_n(x, y, \ldots) + d_{n+1}(x, y, \ldots) + \ldots \tag{3}$$

where $d_n(x, y, \ldots)$ denotes the terms of degree n, the terms of degrees 1 to $n-1$ are identically 0. For example,

$$aba^{-1}b^{-1} = 1 + xy - yx + \ldots = 1 + (xy - yx) \sum_{\mu,\nu=0}^{\infty} (-1)^{\mu+\nu} x^{\nu} y^{\mu}.$$

In this case $n = 2$, $d_2 = xy - yx$.

It can be shown that in the relation (3) the terms of degree 1 to $n-1$ vanish identically if and only if W is an element of $\mathfrak{F}_n$ but not of $\mathfrak{F}_{n+1}$. By assigning to each element $W(a, b, \ldots)$ the polynomial $D(W) = d_n(x, y, \ldots)$, we assign to each element $\neq 1$ of $\mathfrak{F}$ a homogeneous polynomial of degree n in $x, y, \ldots$ whose degree indicates precisely which is the first group in the lower central series of $\mathfrak{F}$ that contains W. It is practical to also define $D(1) = 0$. We easily see that (for $D(W_1) + D(W_2) \neq 0$)

$$D(W_1 W_2) = \begin{cases} D(W_1) + D(W_2) & \text{if } \deg W_1 = \deg W_2, \\ D(W_1) & \text{if } \deg W_1 < \deg W_2. \end{cases} \tag{4}$$

In any case $D(W_1 W_2) = D(W_2 W_1)$. With the help of this relation it is easy to define a non-archimedean valuation on $\mathfrak{F}$, and topological algebra is at the heart of several conclusions that make use of the lower central series.

Through the relations given here between the grous of the lower central series and the "expansions" (3) of elements of $\mathfrak{F}$ into a power series of $\mathfrak{R}$ we can prove many group-theoretical results[4c)]. Furthermore, we can answer the question posed by P. Hall[2)] whether there exist non-regular p-groups (in the sense of Hall) for which the order of the automorphism group realizes the upper bound given by Hall. This

[4)]Compare W. Magnus: a) Über Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring. Mathematische Annalen 111, 259-280, 1935. b) Journal für die reine und angewandte Mathematik 177, 105-115, 1937. c) Monatshefte für Mathematik und Physik 47, 307-313, 1939.

is indeed the case. We obtain such a group with generators $a, b, \ldots$ by substituting for the elements $x, y, \ldots$ on the right-hand side of (2) generators of a ring $\mathfrak{R}^*$ that is obtained from the ring $\mathfrak{R}$ by introducing the relations $px = 0$, $py = 0, \ldots$, and requiring all products of at least $m \geq p$ factors $x, y, \ldots$ to be 0. The order of such a group can also be determined, according to Witt[5)].

4. It is remarkable that the polynomials $d_n(x, y, \ldots)$ cannot be arbitrary polynomials of degree n. On the contrary, it holds that they are images of elements of a Lie ring Λ in a suitable representation of Λ by elements of $\mathfrak{R}$. A Lie ring is defined as a set of elements $\varphi, \psi, \chi, \ldots$ on which an addition and a multiplication is defined, where instead of commutativity and associativity the following laws apply:

$$\begin{gathered}[\varphi, \psi] + [\psi, \varphi] = 0 \\ \varphi[\psi, \chi] + \psi[\chi, \varphi] + \chi[\varphi, \psi] = 0.\end{gathered} \tag{5}$$

Now consider a free Lie ring Λ with generators $\xi, \eta, \ldots$, where "free" means that $\xi, \eta, \ldots$ and their products satisfy only those relations that follow from the laws above. We then obtain a faithful representation of Λ in $\mathfrak{R}$ if we assign as follows (see 4b), 5)):

$$\begin{gathered}\xi \mapsto x, \quad \eta \mapsto y, \quad \ldots, \quad \text{sums in } \Lambda \mapsto \text{ sums in } \mathfrak{R}, \\ \varphi \mapsto f(x, y, \ldots), \psi \mapsto g(x, y, \ldots) \text{ implies } [\varphi, \psi] \mapsto fg - gf.\end{gathered} \tag{6}$$

It holds that the admissible polynomials $D(W) = d_n$ are precisely those elements of $\mathfrak{R}$ that arise under the map (6) as the image of a homogeneous polynomial of degree n in $\xi, \eta, \ldots$ with integer coefficients. Thus there is precisely one polynomial $\delta_n(\xi, \eta, \ldots)$ of degree n in Λ such that

$$\delta_n(\xi, \eta, \ldots) \mapsto d_n(x, y, \ldots).$$

We write $\delta_n = \Delta(W)$ and call d_n the *value* of δ_n in $\mathfrak{R}$. In this spirit, we write for short

$$d_n = \overline{\delta_n(\xi, \eta, \ldots)}, \text{ that is, } x = \overline{\xi}, \ xy - yx = \overline{[\xi, \eta]}, \text{ etc.} \tag{7}$$

We make the following definitions: Consider all elements in $\mathfrak{F}_n$ not contained in $\mathfrak{F}_{n+1}$. There are assigned to certain polynomials d_n of degree n in $\mathfrak{R}$. Note all those d_n that equal a certain $D(W)$, where W is contained in $\langle(\mathfrak{N}, \mathfrak{F}_n), \mathfrak{F}_{n+1}\rangle$. After adding 0 they form a module $\mathfrak{M}_n$ of homogeneous polynomials of degree n in $\mathfrak{R}$. The module of all d_n is denoted by P_n. Then:

$$\mathfrak{A}_n \cong P_n/\mathfrak{M}_n, \tag{8}$$

[5)] E. Witt, Journal für die reine und angewandte Mathematik 177, 152-160, 1937.

where both modules are taken as additive abelian groups. Moreover, both $\mathfrak{M}_n$ and P_n are images of modules M_n and Λ_n, respectively, of homogeneous polynomials of degree n in Λ, where Λ_n is the module of all homogeneous polynomials of degree n in $\xi, \eta, \dots$ with integer coefficients. Now we have the theorem:

I. *The sum of all modules M_n forms an ideal M in Λ. The quotient Lie ring $\Lambda/\mathsf{M} = \Lambda^*$ is independent of the choice of generators and uniquely determined by $\mathfrak{G}$. Λ^* is nilpotent, namely, $\Lambda^{*k} = \mathbf{0}$ if $\mathfrak{G}_k = \mathfrak{G}_{k+1}$ holds. If $\mathfrak{G}$ is a p-group, then $\mathfrak{G}$ and Λ^* contain an identical number of elements.*

Λ^* is called the *Lie ring associated to* $\mathfrak{G}$.

The fact that the modules M_n form an ideal merely states that for a normal subgroup $\mathfrak{N}$ of $\mathfrak{F}$ the products of two elements of $\mathfrak{N}$ and the commutator of one element of $\mathfrak{N}$ and one element of $\mathfrak{F}$ are again contained in $\mathfrak{N}$. From this observation, the remaining claims of the theorem follow.

5. What is missing is a connection between Λ and $\mathfrak{R}$ that guarantees that for the analysis of $\mathfrak{G}$ via the map (2) indeed only those elements of $\mathfrak{R}$ are required that are images of elements of Λ under the map (6). Firstly, in the expressions for $ab = 1 + x + t + xy$ or $aba^{-1}b^{-1}$ there appear summands in the terms of higher order that cannot be composed out of images of elements of Λ (for example xy). The missing connection is provided by the so-called Hausdorff formula[6)], which can be introduced as follows: Let $\mathfrak{R}'$ be another free ring of the same kind as $\mathfrak{R}$ and with generators $x', y', \dots$. Further, let Λ' be another Lie ring of the same kind as Λ with generators $\xi', \eta', \dots$, only that this time we allow arbitrary power series (similar as in $\mathfrak{R}$) with rational coefficients as elements of Λ'. Now assign:

$$\xi' \mapsto x', \quad [\xi', \eta'] \mapsto x'y' - y'x', \qquad \text{etc. as in (6).} \tag{9}$$

Further, assign:

$$a = 1+x = \mathrm{e}^{x'} = \sum_{\nu=0}^{\infty} \frac{x'^{\nu}}{\nu!}, \quad b = \mathrm{e}^{y'} = \sum_{\nu=0}^{\infty} \frac{y'^{\nu}}{\nu!}, \quad x' = \sum_{\nu=1}^{\infty} \frac{(-x)^{\nu+1}}{\nu}, \ \dots \text{ etc.} \tag{9'}$$

(the sign "$=$" can be used here since the assignments yield isomorphic maps, as is easy to see). Then, as Hausdorff proved,

$$ab = \mathrm{e}^{x'+y'+\frac{x'y'-y'x'}{2}+\cdots} = \mathrm{e}^{\overline{\mathsf{H}(\xi',\eta')}}, \quad \mathsf{H}(\xi', \eta') = \xi' + \eta' + \frac{[\xi', \eta']}{2} + \dots, \tag{10}$$

[6)]F. Hausdorff, Sitzungsbreichte der Sächsischen Akademie der Wissenschaften in Leipzig, Mathematisch-physikalische Klasse, 58, 19-48, 1907.

and in general

$$W(a, b \ldots) = \mathrm{e}^{\overline{\Omega(\xi', \eta', \ldots)}}, \tag{11}$$

where $\Omega(\ldots)$ is an element of Λ' and $\overline{\Omega}$, as the "value of Ω in $\mathfrak{R}'$" is defined in the same way as the value of an element of Λ in $\mathfrak{R}$ was declared above by using (6). The power series $\Omega(\ldots)$ have rational coefficients of unknown composition. It is only known that the denominators of the coefficients of degree $< p$ the prime number p cannot appear, and on the other hand that the denominators can contain arbitrary powers of arbitrary prime numbers as factors. The problem of the denominators of the Hausdorff formula (10) is related to an identity discovered by Hall[2)] which can be easily proved with the tools developed here[4b)]. In its simplest form, Hall's identity is as follows: If $\mathfrak{F}$ is the free group with generators $a, b, \ldots$, then for any prime number p,

$$(ab)^p = a^p b^p C^p C_p, \tag{12}$$

where C is an element in the commutator group $\mathfrak{F}_2$ of $\mathfrak{F}$ and C_p is an element of $\mathfrak{F}_p$. Here, C_p is unique modulo $\mathfrak{F}_{p+1}$ (independent of the choice of C) and module p-th powers of elements of $\mathfrak{F}_p$. This means $D(C_p)$ is uniquely determined module p. If $\mathsf{H}_p(\xi', \eta')$ is the homogeneous component of degree p of H in (10), then

$$p\overline{\mathsf{H}_p(\xi', \eta')} \equiv D(C_p) \bmod p. \tag{13}$$

$D(C_p)$ can be determined uniquely from those terms in H_p whose denominators contain the factor p. This explicit knowledge of C_p is very interesting for certain applications; however, so little is known on the coefficients of the Hausdorff formula that we have to rely on other methods to compute $D(C_p)$ due to O. Grün[7)] and H. Zassenhaus[8)].

6. In order to make use of the assignment of a Lie ring Λ^* to a group $\mathfrak{G}$ given by Theorem I, we first need to consider the converse problem in how far to every given Lie ring with the integers as coefficient ring there corresponds a group to which this Lie ring is assigned, provided that this Lie ring is defined by "homogeneous" relations, that is, it arises from a free Lie ring by introducing relations that are homogeneous in the generators of the ring. (For such relations we may

[7)] O. Grün, in this volume (Journal für die reine und angewandte Mathematik 182). Furthermore, additional identities in the spirit of Hall's identity (12) and more background on the importance of the computation of C_p.

[8)] H. Zassenhaus, Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität Hamburg 13, 1-100, in particular 90-95, 1939.

choose generators of the modules M_n identical to zero.) The most important special case of this problem is that in which the given Lie ring Λ^* is nilpotent, and that its elements become 0 when multiplied by a power of the prime number p. Then Λ^* must be assigned to a p-group $\mathfrak{P}$, provided Λ^* is assigned to any group at all. So suppose

$$\Lambda^{*k} = \mathbf{0}, \quad p^m \Lambda^* = \mathbf{0}. \tag{14}$$

Then the following theorem holds:

II. *If $k \leq p$ and the relations* (14) *hold, then there exists at least one group $\mathfrak{P}$ whose order is a power of p such that Λ^* is the Lie ring assigned to $\mathfrak{P}$ according to Theorem I.*

The proof is easily done with the help of the Hausdorff formula (11) by subsituting generators of Λ^* for $\xi', \eta', \ldots$ and constructs a group with generators $a, b, \ldots$ according to (9), (9′) and (2).

Since $k \leq p$, the denominator p does not appear in the Hausdorff formula. Then all functions $\Omega(\xi', \eta', \ldots)$ in (11) are again elements of Λ^*, and one can easily show that the Lie ring assigned to this group is indeed isomorphic to Λ^*. The assumption $k \leq p$ cannot be dropped, as the following supplement to Theorem II holds:

IIa. *There is no group $\mathfrak{P}$ whose Lie ring Λ^* according to Theorem I is the ring Λ^* in two generators ξ, η with $\Lambda^{*p+2} = p\Lambda^* = \mathbf{0}$, where no relations hold in Λ^* that do not follow from these requirements.*

For, assume $\mathfrak{P}$ is a group to which the Lie ring Λ^* in IIa is assigned, then there also exists a p-group to which Λ^* is assigned. Let it also be called $\mathfrak{P}$. Then $\mathfrak{P}$ has two generators a, b, $\mathfrak{P}_{p+2} = \mathbf{1}$ and for $k = 1, \ldots, p+1$ the p-th power of any element in $\mathfrak{P}_k$ is contained in $\mathfrak{P}_{k+1}$. On the other hand, any word $W(a, b)$ in a and b is an element of $\mathfrak{P}_k$ but not of $\mathfrak{P}_{k+1}$ if $D(W)$ is of degree k not congruent 0 modulo p. Now it follows from Hall's identity that ab commutes with $a^p b^p C^p C_p$, and $a^p b^p C^p C_p$ is an element Γ in $\mathfrak{P}_2$ by the remarks above. Now, if Γ is not contained in $\mathfrak{P}_{p+1}$, but in a subgroup $\mathfrak{P}_k$ with $2 \leq k \leq p$, then Γ equals a word $W(a, b)$ such that $D(W)$ has degree k and is $\not\equiv 0 \bmod p$. Then $D(ab\Gamma b^{-1} a^{-1} \Gamma^{-1})$ has degree $k+1$ and also is $\not\equiv 0 \bmod p$, as one can easily verify, and which also follows from a theorem proved elsewhere[4b)]. Since on the other hand $ab\Gamma b^{-1} a^{-1} \Gamma^{-1}$ equals 1, it follows that $\Gamma \in \mathfrak{P}_{p+1}$. But then, as we can easily see, the p-th powers of all elements of $\mathfrak{P}$ are contained in $\mathfrak{P}_{p+1}$, that is, we have that $\mathfrak{P}/\mathfrak{P}_{p+1} = \mathfrak{G}$ is a p-group in which every p-th power is 1, and the

Lie ring Λ' assigned to $\mathfrak{G}$ is of the same kind as Λ^*, only that now $\Lambda'^{p+1} = \mathbf{0}$. But then $C_p = 1$ in $\mathfrak{G}$, contrary to the fact proven by Grün[8] and Zassenhaus[9] that $D(C_p)$ is of degree p and $\not\equiv 0 \bmod p$. So C_p must be $\neq 1$, and this shows that $\mathfrak{G}$ and thus also $\mathfrak{P}$ cannot exist.

It follows from IIa that, at least for p-groups, the assignment of a Lie ring to a group as in Theorem I is not the most practical one. Indeed it seems that the correspondence due to Zassenhaus[9] of a Lie ring of characteristic p to a p-group does not allow the situation that a given Lie ring cannot be assigned to any p-group. However, a proof of this is still missing. Anyhow, the assignment of a Lie ring to a p-group by Zassenhaus seems to be the more practical one.

7. A problem immediately related to Theorems I and II is the following:

To every relation $R(a, b, \ldots) = 1$ between the generators of the group $\mathfrak{G}$ corresponds a relation that can be placed in the ring $\mathfrak{R}$ with generators $x, y, \ldots$ or in the Lie ring Λ with generators $\xi', \eta', \ldots$. Namely, if

$$R(a, b, \ldots) = 1 + F(x, y, \ldots) = \mathrm{e}^{\overline{\Omega(\xi',\eta',\ldots)}},$$

then we can set $F = 0$ or $\Omega(\xi, \eta', \ldots) = 0$. By taking for every relation $R = 1$ the corresponding relation $\Omega = 0$, we obtain from Λ' are ring $\overline{\Lambda}$, and if we consider $\xi', \eta', \ldots$ as generators of $\overline{\Lambda}$, then the relations (10), (11), (9), (9′) yield a representation of the group $\mathfrak{G}$. The question arises when this is faithful. This can only be the case if the intersection of all groups $\mathfrak{G}_k$ is the unit element. It certainly is the case if $\mathfrak{G}$ is a p-group with $\mathfrak{G}_p = \mathbf{1}$, as can be immediately deduced from what we said about the Hausdorff formula.

For p-groups with $\mathfrak{G}_p = \mathbf{1}$ we obtain in this fashion a bijective correspondence between them and finite Lie rings $\overline{\Lambda}$ for which $\overline{\Lambda}^p = \mathbf{0}$ and $q\overline{\Lambda} = \mathbf{0}$, where q is any power of p.

Here, the relations of $\overline{\Lambda}$ are in general not homogeneous (see above). In how far such a relation can produce a larger class of p-groups is an open problem, compare also Zassenhaus[10].

8. At the end we wish to point to a problem considered elsewhere[4a], for which the assignment of a Lie ring to a group can also be useful. For both, the investigation of the automorphisms of a group given by generators and relations, and the question of fully invariant subgroups given purely by taking commutators, it would

[9] H. Zassenhaus, Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität Hamburg 13, 200-207, 1940.

be important to know: What are the irreducible representation modules over the integers within the module of polynomials $d_n(x, y, \ldots)$ from (3) (for fixed n) for the group of linear substitutions with linear coefficients in the variables $x, y, \ldots$? This is equivalent to the question of the irreducible representation modules of the same group of substitution written in the generators $\xi, \eta, \ldots$ of a free Lie ring as variables. In particular, it would be interesting to know *if the free Lie ring Λ with generators $\xi, \eta, \ldots$ contains invariants with respect to the group of all invertible linear substitutions of $\xi, \eta, \ldots$.* The assumption that all coefficients of the substitutions should be integers can apparently be dropped, as it does not pose a restriction. Here, the following is known:

Let k denote the number of generators $\xi, \eta, \zeta, \ldots$ of the Lie ring Λ. If $k = 2$, then there exist invariants with respect to the group of unimodular substitutions of $\xi, \eta, \zeta, \ldots$. More precisely, in any degree m that is twice an odd number, there exists at least one invariant. For example, for $m = 2, 6, \ldots$, the invariants

$$[\xi, \eta],\ [\xi, [\xi, \eta]],\ [\eta, [\xi, \eta]], \ldots$$

For values $m > 6$ in general there do not exist additional invariants. For $k > 2$ it can be shown that there are no invariants of degree $< 2k$. In general, only invariants exist whose degree is a multiple of k. Moreover, there is a procedure to find all invariants.

To every homogeneous polynomial in $\xi, \eta, \ldots$ corresponds by (6) a uniquely determined homogeneous polynomial in the variables $x, y, \ldots$ and this again corresponds to a homogeneous polynomial of the same degree in commutative variables. It is sufficient to make such an assignment for power products of $x, y, \ldots$. Let m be the degree of the power product under consideration. Assign m commutative variables to each of the non-commutative variables $x, y, \ldots$, say, $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are assigned to x and y, respectively. Then assign to each product of $x, y, \ldots$ such a product in the variables $x_1, \ldots, x_m$, $y_1, \ldots, y_m$, ..., in which the index states the position at which the variable of the same name stands in the non-commutative product. For example, the products in the following table are assigned to one another:

$$\begin{array}{cccccc} x & y & x^n & xy^2 & yxxy & \cdots \\ x_1 & y_1 & x_1 \cdots x_n & x_1 y_2 y_3 & y_1 x_2 x_3 y_4 & \cdots \end{array}$$

To a sum, assign the corresponding sum. The question is how to characterize those homogeneous polynomials in $x_1, \ldots, x_m$, $y_1, \ldots, y_m, \ldots$ that are assigned to such homogeneous polynomials of degree m in $x, y, \ldots$ that themselves are images of polynomials of degree m in the free Lie ring generated by $\xi, \eta, \ldots$. We will call such polynomials in the variables $x_1, \ldots, x_m, y_1, \ldots, y_m, \ldots$ *Lie forms*, and simpy indicate the variables by a letter $x_\mu, y_\mu, \ldots$ $(\mu = 1, \ldots, m)$. Now: A homogeneous polynomial $P(x_\mu, y_\mu, \ldots)$ of degree m is a Lie form if and only if there is a second homogeneous polynomial $Q(x_\mu, y_\mu, \ldots)$ of degree m such that P is obtained from Q by application of the operator

$$\omega = (1 - \pi_m)(1 - \pi_{m-1}) \cdots (1 - \pi_2),$$

where ω is an element of the group ring of the symmetric group of permutations of m objects, 1 denotes the identity permutation and π_k the permutation

$$\pi_k = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & m \\ 2 & 3 & \cdots & k & 1 & k+1 & \cdots & m \end{pmatrix}.$$

The operator ω is to be applied to indices of the variables in Q and furthermore it is distributive. Now one can show:

If a Lie form $P(x_\mu, y_\mu, \ldots)$ of degree m is the image of an invariant of degree m in the generators $\xi, \eta, \ldots$ of the free Lie ring Λ, then there exists a simultaneous invariant $J(x_\mu, y_\mu, \ldots)$ of degree m in the variables

$$x_1, y_1, \ldots;\ x_2, y_2, \ldots;\ \ldots;\ x_m, y_m, \ldots,$$

such that

$$P(x_\mu, y_\mu, \ldots) = \omega J(x_\mu, y_\mu, \ldots).$$

For example, for two variables and degree 2:

$$x_1 y_2 - x_2 y_1 = (1 - \pi_2) \left(\frac{x_1 y_2 - x_2 y_1}{2} \right)$$

Now we know all forms $J(x_\mu, y_\mu, \ldots)$. They are the products of determinants

$$\begin{vmatrix} x_{\mu_1} & y_{\mu_1} & \cdots \\ x_{\mu_2} & y_{\mu_2} & \cdots \\ \vdots & \vdots & \ddots \\ x_{\mu_k} & y_{\mu_k} & \cdots \end{vmatrix}$$

where k is the number of variables $\xi, \eta, \ldots$. The difficulty is in determining whether on application of the operator ω all forms J vanish identically. This is the case for $m = k$. For example, for $m = k = 3$,

$$(1-\pi_3)(1-\pi_2)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & x_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Perhaps one can expect that this combinatorical problem can be solved by A. Young's[10) "quantitative substitutional analysis". It can be reduced to questions on the group ring of the symmetric group alone, just like some other related problems.[11)]

[10)] A. Young, Proceedings of the London Mathematical Society (1), 33, 97-146, 1901; 34, 361-397, 1902.

[11)] Compare §3 in O. Grün's work in this volume. I owe the description of the "Lie forms" given above to a personal communication by Herr Grün.

Original: *Über Gruppen und zugeordnete Liesche Ringe*, Journal für die reine und angewandte Mathematik 182, 1940, 142-149.

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