Homotopy-equivalent compact Lie groups

By HANS SCHEERER

§ 1 Introduction and result

In [2] it was shown, among other things, that two compact simple Lie groups are isomorphic if and only if they are homotopy equivalent. Also, an example is given there for two homeomorphic groups whose Pontryagin rings are not isomorphic. Here, we sketch a proof for the following claim:

Theorem *Compact connected Lie groups that are homotopy-equivalent are isomorphic.*

Moreover, it is easy to describe the isomorphism classes of locally isomorphic compact connected Lie groups (compare e.g. [1]). A compact connected Lie group is homeomorphic to the direct product of a torus and its commutator subgroup. The dimension of this torus equals the rank fo the fundamental group. It is thus sufficient to prove:

Corollary *Simply connected compact Lie groups that are homotopy-equivalent are isomorphic.*

In the following we consider only groups that are compact and simply connected, and we let G, G' be two such groups that are homotopy-equivalent. Then G and G', respectively, is a product of simple groups of the following types: $\mathrm{SU}(n)$ with $n \geq 2$, $\mathrm{Spin}(2n+1)$ with $n \geq 2$, $\mathrm{Sp}(n)$ (the group of symplectic $2n \times 2n$-matrices) with $n \geq 3$, $\mathrm{Spin}(2n)$ with $n \geq 4$, and G_2, F_4, E_6, E_7, E_8. For one of these simple groups H, let $N(H)$ denote the number of factors H in G and $N(H')$ the number of factors H in G'. From the information on cohomology rings and homotopy groups we wish to conclude that $N(H) = N(H')$ for all types H. The proofs of the lemmas are omitted, as they are – using the method described here – a more or less easy exercise in algebra.

§ 2 Enumeration of the distinct cohomology rings

For all prime numbers p, we have:

$$\mathrm{H}^*(\mathrm{SU}(n); \mathbb{Z}_p) = \bigwedge\nolimits_p (x_3, x_5, \ldots, x_{2n-1}).$$
$$\mathrm{H}^*(\mathrm{Sp}(n); \mathbb{Z}_p) = \bigwedge\nolimits_p (y_3, y_7, \ldots, y_{4n-1}).$$

For $p > 2$:

$$\mathrm{H}^*(\mathrm{Spin}(2n+1); \mathbb{Z}_p) = \bigwedge\nolimits_p (z_3, z_7, \ldots, z_{4n-1}).$$
$$\mathrm{H}^*(\mathrm{Spin}(2n); \mathbb{Z}_p) = \bigwedge\nolimits_p (z_3, z_7, \ldots, z_{4n-5}, u_{2n-1}).$$

Here, $\bigwedge_p$ denotes the exterior algebra over the vector subspace of H^* generated by the elements in parantheses; these are homogeneous, and their index indicates their degree. If a_{2j-1} denotes one of the generators $x_{2j-1}, y_{2j-1}, z_{2j-1}$, then the a_{2j-1} can be chosen such that for the p above, the reduced Steenrod powers P_p^k operate as follows:

$$P_p^k(a_{2j-1}) = \begin{cases} \binom{j-1}{k} a_{2j-1+2k(p-1)} & \text{if it exists,} \\ 0 \text{ else.} \end{cases}$$

$P^k(u_{2n-1}) = 0$ for $k > 0$, and in particular, $u_{2n-1} \notin P^k(\mathrm{H}^*)$ for $k > 0$.

The Bockstein operator b_p is 0.

Let t be the smallest power of 2 with $n \leq t$. Then

$$\mathrm{H}^*(\mathrm{Spin}(n); \mathbb{Z}_2) = \bigvee\nolimits_2 (v_3, v_5, \ldots, v_{n-1}, w_{t-1}).$$

The indices of the v_i are the numbers $3 \leq i \leq n-1$ that are not powers of 2. The algebra is generated with the relations

$$v_i^2 = \begin{cases} v_{2i} & \text{if } 2i \leq n-1, \\ 0 & \text{else.} \end{cases}$$
$$w_{t-1}^2 = 0.$$

The v_i can be chosen such that

$$\mathrm{Sq}^i(v_j) = \begin{cases} \binom{j}{i} & \text{if } i + j \leq n-1 \text{ and not a power of 2,} \\ 0 & \text{else.} \end{cases}$$

The generator w_{t-1} can be chosen such that

$$\mathrm{Sq}^1(w_{t-1}) = \sum v_i \cdot v_j, \text{ summation over } i+j=t, i<j, i \text{ even.}$$
$$\mathrm{Sq}^i(w_{t-1}) = 0, \text{ for } i>1$$

Thus $\mathrm{Sq}^1(w_{t-1}) = 0$ for $\mathrm{Spin}(n)$ with $n \leq 10$ or $2^s < n \leq 2^2 + 2$ for all s.

These equations can be found in [2] with the exception of the equation $\mathrm{Sq}^i(w_{t-1})$, contained in [10]. If neither $n \leq 10$ nor $2^s < n \leq 2^2 + 2$ for some s, then w_{t-1} cannot be modified into a different generator e_{t-1} such that $\mathrm{Sq}^i(e_{t-1}) = 0$ for all $i > 0$. We will use this in Section 5, Lemma 5. (The work [10] is based on [9].)

Let Γ be a simple group of exceptional type. If the prime number p is not a torsion coefficient of $\mathrm{H}^*(\Gamma;\mathbb{Z})$, then $\mathrm{H}^*(\Gamma;\mathbb{Z}_p)$ is a Grassmann algebra over a subspace generated by homogeneous elements of degrees $(3, 11)$ for G_2, $(3, 11, 15, 23)$ for F_4, $(3, 9, 11, 15, 17, 23)$ for E_6, $(3, 11, 15, 19, 23, 27, 35)$ for E_7, $(3, 15, 23, 27, 35, 39, 47, 59)$ for E_8. The torsion coefficients are 2 for G_2, 2, 3 for F_4, E_6, E_7, and 2, 3, 5 for E_8.

$$\mathrm{H}^*(\mathrm{G}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \textstyle\bigwedge_2(x_5).$$
$$\mathrm{H}^*(\mathrm{F}_4;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \textstyle\bigwedge_2(x_5, x_{15}, x_{23}).$$
$$\mathrm{H}^*(\mathrm{E}_6;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \textstyle\bigwedge_2(x_5, x_9, x_{15}, x_{17}, x_{23}).$$
$$\mathrm{H}^*(\mathrm{E}_7;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \textstyle\bigwedge_2(x_{15}, x_{17}, x_{23}, x_{27}).$$
$$\mathrm{H}^*(\mathrm{E}_8;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \textstyle\bigwedge_2(x_{17}, x_{23}, x_{27}, x_{29}).$$

For a suitable choice of x_i, the Steenrod squares operate as follows:

$$\mathrm{Sq}^2(x_3) = x_5 \text{ in } \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^8(x_{15}) = x_{23} \text{ in } \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^4(x_5) = x_9 \text{ in } \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^8(x_9) = x_{17} \text{ in } \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^2(x_{15}) = x_{17} \text{ in } \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^4(x_{23}) = x_{27} \text{ in } \mathrm{E}_7, \mathrm{E}_8.$$
$$\mathrm{Sq}^2(x_{27}) = x_{29} \text{ in } \mathrm{E}_8.$$
$$\mathrm{Sq}^4(x_5) = 0 \text{ in } \mathrm{G}_2, \mathrm{F}_4$$

These formulas are taken from [11].

$$H^*(F_4;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_8]/(x_8^3) \otimes \bigwedge\nolimits_3(x_3, x_7, x_{11}, x_{15}).$$
$$H^*(E_6;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_8]/(x_8^3) \otimes \bigwedge\nolimits_3(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}).$$
$$H^*(E_7;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_8]/(x_8^3) \otimes \bigwedge\nolimits_3(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}).$$
$$H^*(E_8;\mathbb{Z}_3) \cong \mathbb{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \bigwedge\nolimits_3(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$$
with $b_3(x_7) = x_8$.
$$H^*(E_8;\mathbb{Z}_5) \cong \mathbb{Z}_5[x_{12}]/(x_{12}^5) \otimes \bigwedge\nolimits_3(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$$
with $b_5(x_{11}) = x_{12}$.

These formulas can be found in [3] and [13].

§ 3 The exceptional factors

If $\Gamma_1, \ldots, \Gamma_r$ are simple groups, let $N(\Gamma_1, \ldots, \Gamma_r) = N(\Gamma_1) + \ldots + N(\Gamma_r)$. Define $N'(\Gamma_1, \ldots, \Gamma_r)$ accordingly.

Lemma 1 $N(E_8) = \dim b_5(H^{11}(G;\mathbb{Z}_5))$.

Definition Let A and B be commutative algebras over $\mathbb{Z}_2$. Let $s = 2^k$, $k \geq 1$. Then $Q_s : A \to A$, $x \mapsto x^s$, is a homomorphism of algebras, compatible with homomorphisms $f : A \to B$.

Lemma 2

(1) $N(E_7) = \dim(Q_2 \circ Sq^4)(H^5(G;\mathbb{Z}_5)) - \dim(Sq^1 \circ Sq^4 \circ Sq^6)(H^7(G;\mathbb{Z}_2)) - N(E_8)$.

(2) $N(E_6) = \dim Sq^4(H^5(G;\mathbb{Z}_2)) - \dim Sq^2(H^7(G;\mathbb{Z}_2)) - N(E_7, E_8)$.

(3) $N(F_4) = \dim b_3(H^7(G;\mathbb{Z}_3)) - N(E_6, E_7, E_8)$.

The number $N(G_2)$ will be obtained in § 7 from the homotopy groups.

§ 4 Indecomposable elements

Let J be the ideal $\sum_{i>0} \mathrm{H}^i$ of H^*. Set ${}^{\mathrm{z}}\mathrm{H}^i = \mathrm{H}^i \cap J$ and ${}^{\mathrm{uz}}\mathrm{H}^i = \mathrm{H}^i/{}^{\mathrm{z}}\mathrm{H}^i$. The elements of ${}^{\mathrm{uz}}\mathrm{H}^i$ are called *indecomposable*.[1)] By the Cartan formulas, a cohomology operation $\lambda : \mathrm{H}^i \to \mathrm{H}^{i+k}$ (here, we mean an Sq or P) induces an operation $\overline{\lambda} : {}^{\mathrm{uz}}\mathrm{H}^i \to {}^{\mathrm{uz}}\mathrm{H}^{i+k}$. Like λ, it commutes with continuous maps. If X_1 and X_2 are connected spaces with finitely generated cohomology, and if $X = X_1 \times X_2$, then ${}^{\mathrm{uz}}\mathrm{H}^i(X) \cong {}^{\mathrm{uz}}\mathrm{H}^i(X_1) \times {}^{\mathrm{uz}}\mathrm{H}^i(X_2)$ (direct sum) for $i > 0$, and here $\overline{\lambda} : {}^{\mathrm{uz}}\mathrm{H}^i(X) \to {}^{\mathrm{uz}}\mathrm{H}^{i+k}(X)$ corresponds to the homomorphism $\overline{\lambda} \oplus \overline{\lambda} : {}^{\mathrm{uz}}\mathrm{H}^i(X_1) \oplus {}^{\mathrm{uz}}\mathrm{H}^i(X_2) \to {}^{\mathrm{uz}}\mathrm{H}^{i+k}(X_1) \oplus {}^{\mathrm{uz}}\mathrm{H}^{i+k}(X_2)$. We remark that for all simple groups Γ and all i, $\dim {}^{\mathrm{uz}}\mathrm{H}^i(\Gamma; \mathbb{Z}_p)$ is at most 1, except for $\Gamma = \mathrm{Spin}(4n)$ with $n \geq 2$, $i = 4n - 1$ and $p \geq 3$.

In the following, let $A(G)$ denote the product of exceptional factors of G, and $K(G)$ the product of the remaining factors.

§ 5 The factors of type Spin(2*n*), *n* ≥ 4, Spin(2*n* + 1), *n* ≥ 2

Lemma 3 *Let $n \geq 4$. Then there exists a prime number $p \geq 3$ such that for every (kompakt simply connected Lie) group G in cohomology modulo p, we have:*

$$\begin{aligned} N(\mathrm{Spin}(2n)) = {} & \dim({}^{\mathrm{uz}}\mathrm{H}^{2n-1}(G)) - \dim {}^{\mathrm{uz}}\mathrm{H}^{2n-1}(A(G)) \\ & - \dim \overline{P}^1({}^{\mathrm{uz}}\mathrm{H}^{2n-2p+1}(G)) \\ & + \dim \overline{P}^1({}^{\mathrm{uz}}\mathrm{H}^{2n-2p+1}(A(G))) \end{aligned}$$

Definition For a power s of 2, let $\overline{Q}_s$ denote the homomorphism $\mathrm{H}^k/{}^{\mathrm{z}}\mathrm{H}^k \to \mathrm{H}^{ks}/Q_s({}^{\mathrm{z}}\mathrm{H}^k)$ induced by Q_s.

If $X = X_1 \times X_2$ is a space as in § 4, then

$${}^{\mathrm{uz}}\mathrm{H}^k(X) \cong {}^{\mathrm{uz}}\mathrm{H}^k(X_1) \oplus {}^{\mathrm{uz}}\mathrm{H}^k(X_2)$$

for $k > 0$, and

$$\dim \mathrm{im}\, \overline{Q}_s = \dim \mathrm{im}(\overline{Q}_s \oplus \overline{Q}_s).$$

[1)] *Translator's note:* The superscripts z and uz come from the German “zerlegbar” (meaning “decomposable”) and “unzerlegbar” (meaning “indecomposable”).

Lemma 4 *Let $0 < i$ be even and not a power of 2. Let s be the largest power of 2 that divides i, and let $k = i/s$. Then*

$$N(\mathrm{Spin}(j), j > i) = \dim \overline{Q}_s(\mathrm{H}^k(G;\mathbb{Z}_2)) - \dim \overline{Q}_s({}^{\mathrm{uz}}\mathrm{H}^k(A(G);\mathbb{Z}_2)).$$

Corollary Let $i > 2$, even and not a power of 2. If $i + 2$ is not a power of 2, then we obtain from Lemmas 3 and 4 that $N(\mathrm{Spin}(i + 1))$. If $i + 2$ is a power of 2, then we obtain only $N(\mathrm{Spin}(i + 1), \mathrm{Spin}(i + 3))$.

By using how Sq^i acts on the different algebras, in particular what was stated in 2 on $\mathrm{Sq}^i(w_{t-1})$, we prove:

Lemma 5 *With $s \geq 4$, it holds for the homotopy-equivalent groups G, G' that $N(\mathrm{Spin}(2^s - 1)) = N'(\mathrm{Spin}(2^s - 1))$.*

In 7 we will obtain $N(\mathrm{Spin}(j)) = N'(\mathrm{Spin}(j))$ for the remaining $j = 5, 7, 9$.

§ 6 The factors of type SU(k) and Sp(k)

Lemma 6

(1) *For $i \geq 3$ and $i \neq 2^r, 2^r - 1$ and for all r, $N(\mathrm{Sp}(i)) = N'(\mathrm{Sp}(i))$.*

(2) *For $i \geq$ and $i = 2^r$, $N(\mathrm{Sp}(i - 1), \mathrm{Sp}(i)) = N'(\mathrm{Sp}(i - 1), \mathrm{Sp}(i))$.*

Lemma 7

(1) *For $i \geq 3$ and $i \neq 2^r, 2^r - 1$ and for all r, $N(\mathrm{SU}(i)) = N'(\mathrm{SU}(i))$.*

(2) *For $i \geq 3$ and $i = 2^r$, $N(\mathrm{SU}(i - 1), \mathrm{SU}(i)) = N'(\mathrm{SU}(i - 1), \mathrm{SU}(i))$.*

The proof of Lemma 6 is given by our previous results and the operations of the Sq^i on the different algebras. In particular, we use that for $\mathrm{SU}(n)$ with $2n - 1 > 2i - 1$ and i even, $\mathrm{Sq}^2(x_{2i-1}) = x_{2i+1}$ holds, and that moreover for $s = 2^k$ and $0 \leq i < 2^{k+1} - 1$ for $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, we have $\mathrm{Sq}^i(a_{2s-1}) = a_{2s+i-1}$, provided a_{2s+i-1} exists. The proof of Lemma 7 is then given by considering $\dim({}^{\mathrm{uz}}\mathrm{H}^j)$ for suitable j.

§ 7 Information on homotopy groups and consequences

Periodicity theorem [12].

(1) For $n, m \geq i + 2$, $\Pi_i(\mathrm{SO}(m)) \cong \Pi_i(\mathrm{SO}(n))$.

(2) For $n, m \geq (i + 1)/2$, $\Pi_i(\mathrm{SU}(m)) \cong \Pi_i(\mathrm{SU}(n))$.

(3) For $n, m \geq (i - 1)/4$, $\Pi_i(\mathrm{Sp}(m)) \cong \Pi_i(\mathrm{Sp}(n))$.

(4) Let SU denote a suitable $\mathrm{SU}(n)$, SO a suitable $\mathrm{SO}(n)$ and Sp a suitable $\mathrm{Sp}(n)$, then: $\Pi_k(\mathrm{SU}) \cong \Pi_{k+2}(\mathrm{SU})$, $\Pi_k(\mathrm{SO}) \cong \Pi_{k+4}(\mathrm{Sp})$, $\Pi_k(\mathrm{Sp}) \cong \Pi_{k+4}(\mathrm{SO})$. The period of Π_k (beginning with $k = 0$) is 0, $\mathbb{Z}$, for SU, $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, for SO, and 0, 0, 0, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, $\mathbb{Z}$, for Sp.

Homotopy groups Π_i for small i.

$i =$	6	7	8
SU(2)	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
SU(3)	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$
SU(4)	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$
SO(8)	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
SO(9)	0	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
SO(10)	0	$\mathbb{Z}$	2
G_2	$\mathbb{Z}_3$	0	$\mathbb{Z}_2$
F_4	0	0	$\mathbb{Z}_2$
$\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$	0	0	0

These results can be found in [4], those for E_6, E_7, E_8 in [5].

Lemma 8 *Let $n_k(p^r)$ denote the number of summands $\mathbb{Z}_{p^r}$ in a representation of the p-primary component of Π_k as a direct sum of cyclic groups. Then* $N(\mathrm{SU}(2)) = n_6(4)$, $N(\mathrm{SU}(3)) = n_8(4)$, $N(\mathrm{G}_2) = n_6(3) - N(\mathrm{SU}(2)) - N(\mathrm{SU}(3))$.

Lemma 9 $N(\mathrm{Spin}(5)) = N'(\mathrm{Spin}(5))$.

This follows from the study of $\dim {}^{\mathrm{uz}}\mathrm{H}^7$.

Homotopy groups of Spin(7) and Spin(9) [9].

$\Pi_i(\mathrm{Spin}(7)) = \Pi_i(\mathrm{Spin}(9))$ for $i \leq 13$.

$\Pi_{14}(\mathrm{Spin}(7)) = \mathbb{Z}_{2520} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_5$.

$\Pi_{14}(\mathrm{Spin}(9)) = \mathbb{Z}_8 \oplus \mathbb{Z}_2$.

Almost stable homotopy groups of Sp(n) [8] and SU(n) [6].

$$\Pi_{4n+2}(\mathrm{Sp}(n)) = \begin{cases} \mathbb{Z}_{(2n+1)!} & \text{for even } n. \\ \mathbb{Z}_{2(2n+1)!} & \text{for odd } n. \end{cases}$$

$$\Pi_{2n}(\mathrm{SU}(n)) = \mathbb{Z}_{n!}.$$

Corollary For Spin(7), $n_{14}(8) = 2$, for Spin(9), $n_{14}(8) = 1$, and for SU(7), SU(8), Sp(3), Sp(4), $n_{14}(8) = 0$. Hence

$$2N(\mathrm{Spin}(7)) + N(\mathrm{Spin}(9)) = 2N'(\mathrm{Spin}(7)) + N'(\mathrm{Spin}(9)).$$

Now we still have to consider the numbers $N(\mathrm{SU}(2^i))$, $N(\mathrm{SU}(2^i - 1))$ for $i \geq 3$, and $N(\mathrm{Sp}(2^i))$, $N(\mathrm{Sp}(2^i - 1))$ for $i \geq 2$. We obtain the desired result by studying the 2-primary component of $\Pi_{2(2^i-1)}$ in the order $i = 3, 4, \ldots$.

References

[1] P.F. Baum, *Local isomorphism of compact Lie groups*, Pacific Journal of Mathematics 22 (1967), 197-204.

[2] P.F. Baum, W. Browder, *The cohomology of quotients of classical groups*, Topology 3 (1965), 305-336.

[3] A. Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compactes connexes*, Tohoku Mathematical Journal 13 (1961), 216-240.

[4] A. Borel, *Topology of Lie groups and characteristic classes*, Bulletin of the American Mathematical Society 61 (1955), 397-432.

[5] R. Bott, H. Samelson, *Application of the Morse theory to symmetric spaces*, American Journal of Mathematics 80 (1958), 964-1029.

[6] M.A. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois Journal of Mathematics 4 (1960), 161-169.

[7] M. Mimura, *The homotopy groups of Lie groups of low rank*, Journal of Mathematics of Kyoto University 6 (1967), 131-176.

[8] M. Mimura, H. Toda, *Homotopy groups of symplectic groups*, Journal of Mathematics of Kyoto University 3 (1963/64), 251-273.

[9] C.E. Miller, *The topology of rotation groups*, Annals of Mathematics 57 (1953), 95-110.

[10] A.S. Schwarz, *Cohomologie der Spinorengruppe*, Doklady Akademii Nauk SSSR 104 (1955), 26-29.

[11] E. Thomas, *Exceptional Lie groups and Steenrod squares*, Michigan Journal of Mathematics 11 (1964), 151-156.

[12] R. Bott, *The stable homotopy groups of the classical groups*, Annals of Mathematics 70 (1959), 313-337.

[13] S. Araki, *Differential Hopf algebras and the cohomology mod 3 of the compact exceptional groups* E_7 *and* E_8, Annals of Mathematics 73 (1961), 404-436.

Original: *Homotopieäquivalente kompakte Liesche Gruppen*, Topology 7, 1968, 227-232

Translation by Wolfgang Globke, Version of January 19, 2019.