On Riemann's unpublished works in analytic number theory

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In a letter to Weierstraß from the year 1859, Riemann mentioned a new expansion of the zeta function which he had not sufficiently simplified yet to publish it in an article on the theory of prime numbers. After H. Weber published this section from Riemann's letter in his edition of Riemann's works in 1876, one could suspect that a close investigation of Riemann's unpublished works in the university library in Göttingen would reveal further hidden equations in analytic number theory.

Indeed, the librarian Distel already found the aforementioned representation of the zeta function in Riemann's papers several decades ago. It is a semiconvergent expansion that expresses the behaviour of the function $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ and more generally in every strip $\sigma_1 \leq \sigma \leq \sigma_2$ for arbitrarily large s.

In the meantime, the principal term of this expansion was rediscovered in 1920 by Hardy and Littlewood independently of Riemann, as a special case of their "approximate functional equation". For the proof, they used the same tool as Riemann, namely approximation of an integral by the saddle point method. In Riemann's work one can also find a method to obtain further terms of the semiconvergent series, and this method uses the nice properties of the integral

$$\Phi(\tau, u) = \int \frac{\mathrm{e}^{\pi \mathrm{i} \tau x^2 + 2\pi \mathrm{i} u x}}{\mathrm{e}^{2\pi \mathrm{i} x} - 1} \mathrm{d}x,$$

which moreover has led Kronecker and more recently led Mordell to the most elegant derivation of the reciprocity law of Gaußian sums.

In 1926, Bessel-Hagen noticed in a further study of Riemann's notes yet another previously unknown representation of the zeta function by means of definite integrals; Riemann has been led to these also through properties of $\Phi(\tau, u)$.

We may consider the two expansions of $\zeta(s)$ the most important part of Riemann's unpublished papers on number theory, as far as it is not already contained in his printed works. Attempts to prove the so-called "Riemann hypothesis" or even only a proof of the existence of infinitely many zeros of the zeta function on the critical line are not contained in Riemann's papers. It seems heuristic considerations led

Riemann from the semiconvergent series to claim that the interval $0 < t < T$ asymptotically contains $\frac{T}{2\pi}\log(\frac{T}{2\pi}) - \frac{T}{2\pi}$ real zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$. However, even today it is not clear how to prove or disprove this claim. Using the semiconvergent series, Riemann approximated some real zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$.

Riemann's notes on the zeta function do not contain sections that are fit to be published; occasionally unconnected equations share the same page; often just one side of an equation is written down; estimates of remainders and considerations of convergence are always missing, even in essential steps. For these reasons, a free reworking of Riemann's fragments is necessary, as shall be done in the following.

The legend that Riemann found his mathematical results by "grand general ideas" without needing the formal analytic tools is nowadays not as widespread as in Klein's time. How strong Riemann's analytical technique really was is clearly visible from his derivation and manipulation of the semiconvergent series for $\zeta(s)$.

§ 1 Computation of a definite integral

Let u be a complex variable. Form the integral

$$\Phi(u) = \int_{0\nwarrow 1} \frac{\mathrm{e}^{-\pi\mathrm{i}x^2+2\pi\mathrm{i}ux}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x \tag{1}$$

extending from ∞ to ∞ from the lower right to the upper left along a parallel of the bisector of the fourth and second quadrant that meets the real axis in the points 0 and 1. In equation (1) the path of integration is indicated by the symbol $0 \nwarrow 1$ below the integral sign.

The function $\Phi(u)$ is entire. According to Riemann, it can be expressed via the exponential function in an elementary manner. To prove this, derive two difference equations for $\Phi(u)$ using Cauchy's Theorem:

On the one hand,

$$\begin{aligned}\Phi(u+1)-\Phi(u) &= \int_{0\nwarrow 1} \mathrm{e}^{-\pi\mathrm{i}x^2}\frac{\mathrm{e}^{2\pi\mathrm{i}(u+1)x}-\mathrm{e}^{2\pi\mathrm{i}ux}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x = \int_{0\nwarrow 1}\mathrm{e}^{-\pi\mathrm{i}x^2+2\pi\mathrm{i}(u+\frac{1}{2})x}\mathrm{d}x \\ &= \mathrm{e}^{\pi\mathrm{i}(u+\frac{1}{2})^2}\int_{0\nwarrow 1}\mathrm{e}^{-\pi\mathrm{i}(x-u-\frac{1}{2})^2}\mathrm{d}x = \mathrm{e}^{\pi\mathrm{i}(u+\frac{1}{2})^2}\int_{0\nwarrow 1}\mathrm{e}^{-\pi\mathrm{i}x^2}\mathrm{d}x,\end{aligned}$$

that is,

$$\Phi(u) = \Phi(u+1) - \mathrm{e}^{\pi\mathrm{i}(u+\frac{1}{2})^2}\int_{0\nwarrow 1}\mathrm{e}^{-\pi\mathrm{i}x^2}\mathrm{d}x. \tag{2}$$

On the other hand, if the symbol $-1 \nwarrow 0$ indicates the path of integration which arises from the previous one by a translation by the vector -1, then

$$1 = \int_{0\nwarrow 1} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x} - e^{-\pi i x}} dx - \int_{-1\nwarrow 0} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x} - e^{-\pi i x}} dx, \tag{3}$$

since the integrand has residue $\frac{1}{2\pi i}$ at the pole $x = 0$. Since

$$\begin{aligned}\int_{-1\nwarrow 0} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x} - e^{-\pi i x}} dx &= \int_{0\nwarrow 1} \frac{e^{-\pi i (x-1)^2+2\pi i u (x-1)}}{e^{\pi i (x-1)} - e^{-\pi i (x-1)}} dx \\ &= e^{-2\pi i u} \int_{0\nwarrow 1} \frac{e^{-\pi i x^2+2\pi i (u+1) x}}{e^{\pi i x} - e^{-\pi i x}} dx,\end{aligned}$$

equation (3) yields

$$\Phi(u) = e^{-2\pi i u} \Phi(u+1) + 1. \tag{4}$$

From (2) and (4), we first obtain the for $u = 0$ the known equation

$$\int_{0\nwarrow 1} e^{-\pi i x^2} dx = e^{\frac{3\pi i}{3}}$$

and then by elimination of $\Phi(u+1)$ the desired result

$$\int_{0\nwarrow 1} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x} - e^{-\pi i x}} dx = \frac{1}{1 - e^{-2\pi i u}} - \frac{e^{\pi i u^2}}{e^{\pi i u} - e^{-\pi i u}}. \tag{5}$$

Differentiating n times with respect to u, we obtain the more general equation

$$\int_{0\nwarrow 1} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x} - e^{-\pi i x}} x^n dx = (2\pi i)^{-n} D^n \frac{e^{\pi i u} - e^{\pi i u^2}}{e^{\pi i u} - e^{-\pi i u}}, \quad n = 0, 1, 2, \ldots \tag{6}$$

For the following it is convenient to rewrite (5). Replace u by $2u + \frac{1}{2}$ and multiply (5) by $e^{-2\pi i (u+\frac{1}{2})^2 + \frac{\pi i}{8}}$. This yields the equation found by Riemann,

$$\int_{0\nwarrow 1} \frac{e^{\pi i (x^2 - 2(u+\frac{1}{2}-x)^2 + \frac{1}{8})}}{e^{2\pi i x} - 1} dx = \frac{\cos(2\pi u^2 + \frac{3\pi}{8})}{\cos(2\pi u)},$$

which will play an important part in the following. The integral $\Phi(u)$ is a special case of the integral

$$\Phi(\tau, u) = \int_{0\nwarrow 1} \frac{e^{\pi i \tau x^2+2\pi i u x}}{e^{\pi i x} - e^{0\pi i x}} dx, \tag{7}$$

which satisfies two difference equations. It has been studied in detail by Mordell. For every negative rational value of τ there is an equation anaologous to (5), and from here we obtain the reciprocity law of Gaußian sums by specializing u. In his lectures, Riemann already based the transformation theory of the theta function on the properties of $\Phi(\tau, u)$.

§ 2 Semiconvergent expansion of the zeta function

If the real part σ of a complex variable $s = \sigma + \mathrm{i}t$ is larger than 1 and if m denotes a natural number, then

$$\zeta(s) = \sum_{n=1}^{m} n^{-s} + \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\mathrm{e}^{-mx}}{\mathrm{e}^x - 1}\mathrm{d}x,$$

or, if C_1 is a loop around the negative imaginary axis to be traversed in a positive sense, then

$$\zeta(s) = \sum_{n=1}^{m} n^{-s} + \frac{(2\pi)^s \mathrm{e}^{\frac{\pi \mathrm{i} s}{2}}}{\Gamma(s)(\mathrm{e}^{2\pi \mathrm{i} s} - 1)} \int_{C_1} \frac{x^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m x}}{\mathrm{e}^{2\pi \mathrm{i} x} - 1}\mathrm{d}x. \tag{8}$$

This formula even holds for arbitrary values of σ. Henceforth restrict σ to a fixed interval $\sigma_1 \leq \sigma \leq \sigma_2$, and let $t \geq 0$. To compute the integral appearing in (8) asymptotically for $t \to \infty$ by the saddle point method, the path of integration must lead through the zero of $D \log(x^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m x})$. For this zero we obtain from the equation

$$\frac{s-1}{x} - 2\pi \mathrm{i} m = 0$$

the value

$$\xi = \frac{s-1}{2\pi \mathrm{i} m} = \frac{t}{2\pi m} + \frac{1-\sigma}{2\pi m}\mathrm{i}. \tag{9}$$

In the circle centered at ξ with radius $|\xi|$ we now have the expansion

$$\begin{aligned} x^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m x} &= \xi^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m \xi}\mathrm{e}^{(s-1)(-\frac{1}{2}(\frac{x-\xi}{\xi})^2 + \frac{1}{3}(\frac{x-\xi}{\xi})^3 - \dots)} \\ &= \xi^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m \xi}\mathrm{e}^{-\frac{s-1}{2\xi^2}(x-\xi)^2}(c_0 + c_1(x-\xi) + c_2(x-\xi)^2 + \dots), \end{aligned}$$

and in the series

$$\xi^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m \xi} = \sum_{n=0}^{\infty} c_n \int \frac{\mathrm{e}^{-\frac{s-1}{2\xi^2}(x-\xi)^2}}{\mathrm{e}^{2\pi \mathrm{i} x} - 1}(x-\xi)^n \mathrm{d}x$$

we may suspect a semiconvergent expansion of the integral in (8). The integrals appearing in this series can now be evaluated using (6) from §1, if $\frac{s-1}{2\xi^2}$ assumes the particular value $\pi \mathrm{i}$. For fixed s, this is a condition on m that in general can only be satisfied approximately, since m is an integer. Because of this, Riemann replaces the saddle point ξ by its neighboring value η, determined by the equation

$$\frac{t\mathrm{i}}{2\eta^2} = \pi \mathrm{i}$$

as

$$\eta = +\sqrt{\frac{t}{2\pi}}, \tag{10}$$

and then determines m as in (9) as the largest integer less than $\frac{t}{2\pi\eta}$, that is,

$$m = [\eta]. \tag{11}$$

Introduce the abbreviations

$$\begin{aligned} \tau &= +\sqrt{t} = \eta\sqrt{2\pi}, \\ \varepsilon &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} = \frac{1-\mathrm{i}}{\sqrt{2}}, \\ g(x) &= x^{s-1}\frac{\mathrm{e}^{-2\pi \mathrm{i} m x}}{\mathrm{e}^{2\pi \mathrm{i} x}-1}. \end{aligned} \tag{12}$$

For now, let η be a non-integral number. Replace the path of integration C_1 by the piecewise linear path C_2 consisting of the two half-lines emanating from the point $\eta - \frac{\varepsilon}{2}\eta$ and containing the points η and $-(m+\frac{1}{2})$, respectively. With regard to the poles at ± 1, $\pm 2, \ldots$, $\pm m$, the Residue Theorem yields

$$\int_{C_1} g(x)\mathrm{d}x = (\mathrm{e}^{\pi \mathrm{i} s}-1)\sum_{n=1}^{m} n^{s-1} + \int_{C_2} g(x)\mathrm{d}x$$

$$\zeta(x) = \sum_{n=1}^{m} n^{-s} + \frac{(2\pi)^s}{2\Gamma(s)\cos(\frac{\pi s}{2})}\sum_{n=1}^{m} n^{s-1} + \frac{(2\pi)^s \mathrm{e}^{\frac{\pi \mathrm{i} s}{2}}}{\Gamma(s)(\mathrm{e}^{2\pi \mathrm{i} s}-1)}\int_{C_2} g(x)\mathrm{d}x. \tag{13}$$

On the left one of the two straight line components of C_2, which will be called C_3,

$$\operatorname{arc}(x) \geq \arctan\left(\frac{1}{2\sqrt{s}-1}\right) > (2\sqrt{(2)}-1)^{-1} - \frac{1}{3}(2\sqrt{2}-1)^{-3} > \frac{1}{2\sqrt{2}} + \frac{1}{8}$$

$$\operatorname{Im}(x) \leq \frac{\eta}{2\sqrt{2}}$$

and thus by (10) and (11),

$$|x^{s-1}\mathrm{e}^{-2\pi \mathrm{i} m x}| \leq |x|^{\sigma-1}\mathrm{e}^{-t(\frac{1}{2\sqrt{2}}+\frac{1}{8})+\pi m\frac{\eta}{\sqrt{2}}} \leq |x|^{\sigma-1}\mathrm{e}^{\frac{-t}{8}}$$

$$\int_{C_3} g(x)\mathrm{d}x = \mathrm{O}(\mathrm{e}^{-\frac{t}{9}}), \tag{14}$$

uniformly in σ for $\sigma_1 \leq \sigma \leq \sigma_2$.

On the right one of the two straight line components of C_2, set, for $y \geq -\frac{\eta}{2}$,

$$x = \eta + \varepsilon y.$$

Then

$$|x^{s-1}\mathrm{e}^{-2\pi\mathrm{i}mx}| = |x|^{\sigma-1}\mathrm{e}^{t\arctan(\frac{y}{y+\eta\sqrt{2}})-\pi\sqrt{2}my}.$$

Now, if even $y \geq +\frac{\eta}{2}$ holds, then for sufficiently large t we have

$$\begin{aligned} t\arctan\left(\frac{y}{y+\eta\sqrt{2}}\right) - \pi\sqrt{2}my &\leq \frac{ty}{y+\eta\sqrt{2}} - \pi\sqrt{2}my < ty\left(\frac{1}{y+\eta\sqrt{2}} - \frac{\eta-1}{\sqrt{2}\eta^2}\right) \\ &= \frac{ty}{\eta\sqrt{2}}\left(\frac{1}{\eta} - \frac{y}{y+\eta\sqrt{2}}\right) \leq \frac{t}{2\sqrt{2}}\left(\frac{1}{\eta} - \frac{1}{1+2\sqrt{2}}\right) < -\frac{t}{11}. \end{aligned}$$

So we have the estimate

$$-\int_{\frac{\eta}{2}}^{\infty} g(x)\varepsilon\mathrm{d}y = \mathrm{O}(\mathrm{e}^{-\frac{t}{11}}), \tag{15}$$

again uniform in $\sigma_1 \leq \sigma \leq \sigma_2$. From (14) and (15) we obtain

$$\int_{C_2} g(x)\mathrm{d}x = \int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} g(x)\mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\frac{t}{11}}). \tag{16}$$

For the asymptotic expansion of the integral on the right hand side of (16) we assume the identity

$$g(x) = \eta^{s-1}\mathrm{e}^{-2\pi\mathrm{i}m\eta}\frac{\mathrm{e}^{-\pi\mathrm{i}(x-\eta)^2+2\pi\mathrm{i}(\eta-m)(x-\eta)}}{\mathrm{e}^{2\pi\mathrm{i}x}-1}\mathrm{e}^{(s-1)\log(1+\frac{x-\eta}{\eta})-2\pi\mathrm{i}\eta(x-\eta)+\pi\mathrm{i}(x-\eta)^2}. \tag{17}$$

For $|x-\eta| < \eta$, the last of the right hand side factors can be expanded into a series of powers of $x-\eta$ whose coefficients are to be studied further. With τ as defined in (12), put

$$\mathrm{e}^{(s-1)\log(1+\frac{z}{\tau})-\mathrm{i}\tau z+\frac{\mathrm{i}}{2}z^2} = \sum_{n=0}^{\infty} a_n z^n = w(z), \tag{18}$$

where $|z| < \tau$. From the differential equation

$$(z+\tau)\frac{\mathrm{d}w}{\mathrm{d}z} + (1-\sigma-\mathrm{i}z^2)w = 0$$

we obtain the recursion formula

$$(n+1)\tau a_{n+1} = -(n+1-\sigma)a_n + \mathrm{i}a_{n-2}, \quad n = 2, 3, \ldots \tag{19}$$

which is also valid for $n = 0, 1$ if we set $a_{-2} = 0$, $a_{-1} = 0$. If we include the equation $a_0 = 1$, then $a_1, a_2, \ldots$ are determined by (19), namely, a_n is a polynomial of degree n in τ^{-1} that does not contain the powers τ^{-k} for $k = 0, 1, \dots, n - 2\lfloor \frac{n}{3} \rfloor - 1$. Therefore,

$$a_n = \mathrm{O}(t^{-\frac{n}{2}+\lfloor \frac{n}{3} \rfloor})$$

uniformly for $\sigma_1 \le \sigma \le \sigma_2$, but not uniformly in n.

To estimate the remainder of the power series $w(z)$, use the representation

$$r_n(z) = \sum_{k=n}^{\infty} a_k z^k = \frac{1}{2\pi \mathrm{i}} \int_C \frac{w(u) z^n}{u^n (u - z)} \mathrm{d}u \tag{20}$$

where C means a curve contained in the radius of convergence which circles each of the points 0 and z once with positive orientation. By (18),

$$\begin{aligned} \log(w(u)) &= (\sigma - 1 + \mathrm{i}\tau^2) \log\left(1 + \frac{u}{\tau}\right) - \mathrm{i}\tau u + \frac{\mathrm{i}}{2} u^2 \\ &= (\sigma - 1) \log\left(1 + \frac{u}{\tau}\right) + \mathrm{i} u^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+2} \left(\frac{u}{\tau}\right)^k, \end{aligned}$$

so that in the circle $|u| \le \frac{3}{5}\tau$ the estimate

$$\mathrm{Re}(\log(w(u))) \le |\sigma - 1| \log\left(\frac{5}{2}\right) + \frac{5}{6} \frac{|u|}{\tau} |u|^2. \tag{21}$$

holds. In (20), assume $|z| \le \frac{4}{7}\tau$ and that C is a circle about $u = 0$ of radius ϱ_n, subject to the condition

$$\frac{21}{20}|z| \le \varrho_n \le \frac{3}{5}\tau. \tag{22}$$

From (20), (21), (22), it follows that the estimate

$$r_n(z) = \mathrm{O}(|z|^n \varrho_n^{-n} \mathrm{e}^{\frac{5}{6\tau}\varrho_n^3}) \tag{23}$$

holds uniformly in σ in n. The function $\varrho^{-n} \mathrm{e}^{\frac{5}{6\tau}\varrho^3}$ of ϱ assumes its minimum $(\frac{5\mathrm{e}}{2n\tau})^{\frac{n}{3}}$ for $\varrho = (\frac{2n\tau}{5})^{\frac{1}{3}}$. By (22) the choice of $\varrho_n = \varrho$ is admissible if

$$\frac{21}{22}|z| \le \left(\frac{2n\tau}{5}\right)^{\frac{1}{3}} \le \frac{3}{5}\tau.$$

Therefore,

$$r_n(z) = \mathrm{O}\left(|z|^n \left(\frac{5\mathrm{e}}{2n\tau}\right)^{\frac{n}{3}}\right), \tag{24}$$

where $n \le \frac{27}{50}t$, $|z| \le \frac{20}{21}(\frac{2n\tau}{5})^{\frac{1}{3}}$. For $|z| \le \frac{4}{7}\tau$ the choice $\varrho_n = \frac{21}{20}|z|$ is valid by (22). Then (23) yields the relation

$$r_n(z) = \mathrm{O}\left(\left(\frac{20}{21}\right)^n \mathrm{e}^{\frac{5}{6\tau}(\frac{21}{20}|z|)^3}\right) = \mathrm{O}(\mathrm{e}^{\frac{14}{29}|z|^2}) \tag{25}$$

for $|z| \le \frac{\tau}{2}$. By (17) and (18),

$$\int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} g(x)\mathrm{d}x = \eta^{s-1}\mathrm{e}^{-2\pi\mathrm{i}m\eta}\int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} \frac{\mathrm{e}^{-\pi\mathrm{i}(x-\eta)^2+2\pi\mathrm{i}(\eta-m)(x-\eta)}}{\mathrm{e}^{2\pi\mathrm{i}x}-1} w(\sqrt{2\pi}(x-\eta))\mathrm{d}x. \tag{26}$$

To determine the error incurred by replacing $w(\sqrt{2\pi}(x-\eta))$ in this equation by the partial sum $\sum_{k=0}^{n-1} a_k(2\pi)^{\frac{k}{2}}(x-\eta)^k$, study the integral

$$J_n = \int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} \frac{\mathrm{e}^{-\pi\mathrm{i}(x-\eta)^2+2\pi\mathrm{i}(\eta-m)(x-\eta)}}{\mathrm{e}^{2\pi\mathrm{i}x}-1} r_n(\sqrt{2\pi}(x-\eta))\mathrm{d}x. \tag{27}$$

Henceforth assume $n \le \frac{5}{16}t$. Avoid the proximity of the poles $x = m$, $x = m+1$ of the integrand by replacing the part of the path of integration that is contained in the circles $|x-m| \le \frac{1}{2\sqrt{\pi}}$, $|x-m-1| \le \frac{1}{2\sqrt{\pi}}$, respectively, by the corresponding arc. By (24), integration over the arc to J_n only contributes $\mathrm{O}((\frac{5\mathrm{e}}{2n\tau})^{\frac{n}{3}})$. On the remaining path of integration, $-\mathrm{i}\pi(x-\eta)^2 = -\pi|x-\eta|^2$. Set

$$\frac{20}{21}\left(\frac{2n\tau}{5}\right)^{\frac{1}{3}} = \lambda$$

and consider (24) for $|x-\eta| \le \frac{\lambda}{\sqrt{2\pi}}$, and on the other hand (25) for $\frac{\lambda}{\sqrt{2\pi}} \le |x-\eta| \le \frac{\eta}{2}$. Then it follows that

$$\begin{aligned}
J_n &= \mathrm{O}\left(\left(\frac{5\mathrm{e}}{2n\tau}\right)^{\frac{n}{3}} \int_0^{\lambda} \mathrm{e}^{-\frac{1}{2}v^2+\sqrt{2\pi}v} v^n \mathrm{d}v + \int_0^{\frac{\tau}{2}} \mathrm{e}^{-\frac{1}{58}v^2+\sqrt{2\pi}v}\mathrm{d}v\right) \\
&= \mathrm{O}\left(\left(\frac{5\mathrm{e}}{2n\tau}\right)^{\frac{n}{3}} \mathrm{e}^{\sqrt{2\pi n}} 2^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right) + \mathrm{e}^{-\frac{1}{59}\lambda^2}\right) \\
&= \mathrm{O}\left(\left(\frac{25n}{4\mathrm{e}t}\right)^{\frac{n}{6}} \mathrm{e}^{\sqrt{2\pi n}} + \mathrm{e}^{-\frac{1}{59}\lambda^2}\right).
\end{aligned}$$

A simply computation shows that for $n \le 2\cdot 10^{-8}t$, the second O-term is dominated by the first one. Thus we obtain the estimate

$$J_n = \mathrm{O}\left(\left(\frac{3n}{t}\right)^{\frac{n}{6}}\right) \tag{28}$$

for $n \le 2 \cdot 10^{-8} t$, uniformly in σ and n.

From (16), (18), (26), (27), (28) it now follows that

$$\int_{C_2} g(x)\mathrm{d}x =$$
$$\eta^{s-1} \mathrm{e}^{-2\pi \mathrm{i} m\eta} \left(\sum_{k=0}^{n-1} a_k (2\pi)^{\frac{k}{2}} \int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} \frac{\mathrm{e}^{-\pi\mathrm{i}(x-\eta)^2 + 2\pi\mathrm{i}(\eta-m)(x-\eta)}}{\mathrm{e}^{2\pi\mathrm{i}x} - 1} (x-\eta)^k \mathrm{d}x + \mathrm{O}\left(\left(\frac{3n}{t} \right)^{\frac{n}{6}} \right) \right).$$

If we integrate the right hand side over the full line from $\eta + \varepsilon\infty$ to $\eta - \varepsilon\infty$ instead of from $\eta + \varepsilon\frac{\eta}{2}$ to $\eta - \varepsilon\frac{\eta}{2}$, then, since $n \le 2\cdot 10^{-8} t$, the value of the integral changes only by $\mathrm{O}(\mathrm{e}^{-\frac{t}{8}+\pi\eta} (\frac{\eta}{2})^k)$. On the other hand, by (24),

$$a_k = (r_k - r_{k+1}) z^{-k} = \mathrm{O}\left(\left(\frac{5\mathrm{e}}{2k\tau} \right)^{\frac{k}{3}} \right), \quad k = 1, \ldots, n-1,$$

and thus

$$\sum_{k=0}^{n-1} |a_k| \mathrm{e}^{-\frac{t}{8}+\pi\eta} \left(\frac{\tau}{2} \right)^k = \mathrm{O}\left(\mathrm{e}^{-\frac{t}{8}+\pi\eta} \left(\frac{5\mathrm{e}t}{16n} \right)^{\frac{n}{3}} \right) = \mathrm{O}\left(\left(\frac{3n}{t} \right)^{\frac{n}{6}} \right).$$

If finally we replace the integration variable x by $x + m$, then we obtain

$$\int_{C_2} g(x)\mathrm{d}x =$$
$$(-1)^m \mathrm{e}^{-\frac{\pi\mathrm{i}}{8}} \eta^{s-1} \mathrm{e}^{-\pi\mathrm{i}\eta^2} \left(\sum_{k=0}^{n-1} a_k (2\pi)^{\frac{k}{2}} \int_{0 \nwarrow 1} \frac{\mathrm{e}^{\pi\mathrm{i}(x^2 - 2(x+m-\eta)^2 + \frac{1}{8}}}{\mathrm{e}^{2\pi\mathrm{i}x} - 1} (x+m-\eta)^k \mathrm{d}x + \mathrm{O}\left(\left(\frac{3n}{t} \right)^{\frac{n}{6}} \right) \right). \tag{29}$$

By the result of §1, the integral

$$\int_{0 \nwarrow 1} \frac{\mathrm{e}^{\pi\mathrm{i}(x^2 - 2(x - \frac{u}{\sqrt{2\pi}} - \frac{1}{2})^2 + \frac{1}{8})}}{\mathrm{e}^{2\pi\mathrm{i}x} - 1} \mathrm{d}x = F(u) \tag{30}$$

has the value

$$F(u) = \frac{\cos(u^2 + \frac{3\pi}{8})}{\cos(\sqrt{2\pi}u)}.$$

To be able to form the integral appearing in (29) for values $k > 0$ as well, Riemann transforms (30) into the equation

$$F(\delta + u)\mathrm{e}^{\mathrm{i}u^2} = \int_{0 \nwarrow 1} \frac{\mathrm{e}^{\pi\mathrm{i}(x^2 - 2(x - \frac{u}{\sqrt{2\pi}} - \frac{1}{2})^2 + \frac{1}{8})}}{\mathrm{e}^{2\pi\mathrm{i}x} - 1} \mathrm{e}^{2\sqrt{2\pi}\mathrm{i}(x - \frac{\delta}{\sqrt{2\pi}} - \frac{1}{2})u} \mathrm{d}u,$$

which, by expansion into powers of u, becomes the equation

$$\int_{0\nwarrow 1} \frac{e^{\pi i(x^2 - 2(x - \frac{u}{\sqrt{2\pi}} - \frac{1}{2})^2 + \frac{1}{8})}}{e^{2\pi i x} - 1} \left(x - \frac{\delta}{\sqrt{2\pi}} - \frac{1}{2}\right)^k dx$$
$$= 2^{-k}(2\pi)^{-\frac{k}{2}} k! \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{i^{r-k}}{r!(k-2r)!} F^{(k-2r)}(\delta) \tag{31}$$

with $k = 0, 1, 2, \ldots$.

From (13), (29), (31) now follows the expansion

$$\zeta(s) = \sum_{l=1}^{m} l^{-s} + \frac{(2\pi)^s}{2\Gamma(s)\cos(\frac{\pi s}{2})} \sum_{l=1}^{m} l^{s-1} + (-1)^{m-1} \frac{(2\pi)^{\frac{s+1}{2}}}{\Gamma(s)} t^{\frac{s-1}{2}} e^{\frac{\pi i s}{2} - \frac{ti}{2} - \frac{\pi i}{8}} S \tag{32}$$

with

$$S = \sum_{0 \le 2r \le k \le n-1} \frac{2^{-k} i^{r-k} k!}{r!(k-2r)!} a_k F^{(k-2r)}(\delta) + O\left(\left(\frac{3n}{t}\right)^{\frac{n}{6}}\right) \tag{33}$$

where

$$n \le 2\cdot 10^{-8} t, \quad m = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor, \quad \delta = \sqrt{t} - \left(m + \frac{1}{2}\right)\sqrt{2\pi}, \quad F(u) = \frac{\cos(u^2 + \frac{3\pi}{8})}{\cos(\sqrt{2\pi}u)},$$

and the coefficients a_k are given by the recursion formula (19). This expansion is semiconvergent, and uniformly so for $\sigma_1 \le \sigma \le \sigma_2$, for the term (33) is of order $t^{-\frac{n}{6}}$ for every fixed n, uniformly in σ. The series (32) differs from the known semiconvergent series' in complex analysis by the appearance of the integer m, which has the effect that the different coefficients of the expansion do not all vary continuously with t. The assumption made in the proof that $\sqrt{\frac{t}{2\pi}}$ is not an integer can easily be discarded afterwards, as in (32) we can take the limit from the right of $\sqrt{\frac{t}{2\pi}}$ approaching any integer value.

If in (33) we choose the particular value $n = 2 \cdot 10^{-8} t$, then the error term is $O(10^{-10^{-8}t})$, and thus converges exponentially to 0 for increasing t. For practical purposes, this estimate of the error term useless due to the small factor 10^{-8} in the exponent. Better estimates show that 10^{-8} can be replaced by a significantly larger number. It would be interesting to know the precise order of the error as a function in n; for it is not even trivial that it converges to 0 for fixed t and increasing n.

Because of the special importance of the case $\sigma = \frac{1}{2}$ it is useful to multiply (32) with the function

$$\mathrm{e}^{\vartheta \mathrm{i}} = \pi^{\frac{1}{4}-\frac{s}{2}} \sqrt{\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}}. \tag{34}$$

Here, understand ϑ to be the unique branch in the plane cut from 0 to $-\infty$ and from 1 to $+\infty$ that vanishes for $s = \frac{1}{2}$. Then, on the critical line $\sigma = \frac{1}{2}$, $\vartheta = \operatorname{arc}(\pi^{\frac{s}{2}}\Gamma(\frac{s}{2}))$ and $\mathrm{e}^{\vartheta \mathrm{i}}\zeta(s)$ is real. By (32), for $\sigma_1 \le \sigma \le \sigma_2$,

$$\mathrm{e}^{\vartheta \mathrm{i}}\zeta(s) = 2\sum_{l=1}^{m} \frac{\cos(\vartheta + \mathrm{i}(s-\frac{1}{2})\log(l))}{\sqrt{l}} + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{\frac{\sigma-1}{2}} \mathrm{e}^{(\frac{t}{2}\log(\frac{t}{2\pi})-\frac{t}{2}-\frac{\pi}{8}-\vartheta)\mathrm{i}} S \tag{35}$$

with S declared by (33). Every a_k is a polynomial in τ^{-1}, and so is the finite sum S. By collecting powers of τ^{-1} it follows that for every fixed n and $t \to \infty$, (33) implies the relation

$$S = \sum_{k=0}^{n-1} A_k \tau^{-k} + \mathrm{O}(\tau^{-n}),$$

where the coefficients $A_0, \ldots, A_{n-1}$ are homogeneous linear in finitely many derivatives $F(\delta)$, $F'(\delta)$,.... The explicit computation of the A_k using (33) and the recursion formula for the a_k is quite laborious. Riemann simplifies it by the following trick. Set

$$F(\delta + x)\mathrm{e}^{\mathrm{i}x^2} = \sum_{k=0}^{\infty} b_k x^k,$$

so that

$$S \sim \sum_{k=0}^{\infty} (2\mathrm{i})^{-k} k! a_k b_k \tag{36}$$

is the full semiconvergent series, and the desired quantity A_k is the coefficient of τ^{-k} obtained by collecting powers of τ^{-1}. The expression on the right hand side of (36) is the constant term in the series in positive and negative powers of x obtained by multiplying the convergent power series

$$F\left(\delta + \frac{1}{x}\right)\mathrm{e}^{\mathrm{i}x^{-2}} = \sum_{k=0}^{\infty} b_k x^{-k}$$

with the divergent series

$$y = \sum_{k=0}^{\infty} (2\mathrm{i})^{-k} k! a_k x^k. \tag{37}$$

As the fixed power τ^{-k} appears only in finitely many coefficients $a_0, a_1, a_2, \ldots$, the following procedure for the determination of A_k is valid: By formal multiplication of $e^{ix^{-2}}$ and y form the expression

$$z = e^{ix^{-2}} y = \sum_{n=-\infty}^{+\infty} d_n x^n \tag{38}$$

and out of this form the series $\sum_{k=0}^{\infty} B_k \tau^{-k}$ by collecting powers of τ^{-1}. Then A_k is the constant term in $F(\delta + \frac{1}{x}) B_k$, and since for the computation of the constant term the negative powers of x appearing in B_k are irrelevant, we only need to determine the polynomial part of B_k.

If we abbreviate

$$c_k = (2i)^{-k} k! a_k, \qquad k = 0, 1, 2, \ldots$$

then by (19)

$$\tau c_{n+1} = i \frac{n+1-\sigma}{2} c_n - \frac{n(n-1)}{8} c_{n-2}, \qquad n = 0, 1, 2, \ldots$$

with $c_{-2} = 0$, $c_{-1} = 0$, $c_0 = 1$, and thus the power series (37) formally satisfies the differential equation

$$\tau(y-1) = \frac{i}{2} x^{\sigma+1} D(x^{1-\sigma} y) - \frac{1}{8} x^3 D^2(x^2 y).$$

This implies the following differential equation for the series (38)

$$\left(\tau + \frac{1}{2x} + i\left(\frac{\sigma}{2} - \frac{1}{4}\right) x\right) z + \frac{1}{8} x^3 D^2(x^2 z) = \tau e^{ix^{-2}}. \tag{39}$$

Now if

$$z = \sum_{n=0}^{\infty} B_n \tau^{-n},$$

sorted by powers of τ^{-1}, then (39) implies

$$B_0 = eix^{-2}$$

and the recursion formula

$$B_{n+1} = \left(i \frac{1-2\sigma}{4} x - \frac{1}{2x}\right) B_n - \frac{1}{8} x^3 D^2(x^2 B_n), \qquad n = 0, 1, 2, \ldots$$

Moreover, if we set

$$B_n = \sum_{k=-\infty}^{3n} a_k^{(n)} x^k, \qquad n = 0, 1, 2, \dots$$

then

$$a_k^{(0)} = 0, \qquad k = -2, -4, -6, \dots$$
$$a_{-2k}^{(0)} = \frac{\mathrm{i}^k}{k!}, \qquad k = 0, 1, 2, \dots$$
$$a_k^{(n+1)} = \mathrm{i}\frac{1-2\sigma}{4} a_{k-1}^{(n)} - \frac{1}{2} a_{k+1}^{(n)} - \frac{(k-1)(k-2)}{8} a_{k-3}^{(n)}, \qquad {\scriptstyle n=0,1,2,\dots, \atop k=0,\pm 1,\pm 2,\dots} \tag{40}$$

With the $a_k^{(n)}$ to be determined from these recursion formulas, we can explicitley state A_n, namely

$$A_n = \sum_{k=0}^{3n} \frac{a_k^{(n)}}{k!} F^{(k)}(\delta) \tag{41}$$

and then

$$S \sim \sum \frac{a_k^{(n)}}{k!} F^{(k)}(\delta) \tau^{-n}$$

where n runs through all values $0, 1, 2, \dots$ and k through all values $0, 1, \dots, 3n$. The recursion formula (40) takes its simples form for $\sigma = \frac{1}{2}$. In this case, we compute without difficulty

$$B_0 = 1 + \dots$$
$$B_1 = -\frac{1}{2^2} x^3 + \dots$$
$$B_2 = \frac{5}{2^3} x^6 + \frac{1}{2^3} x^2 + \frac{\mathrm{i}}{2^4 \cdot 3} + \dots$$
$$B_3 = -\frac{5 \cdot 7}{2^3} x^9 - \frac{1}{2} x^5 - \frac{\mathrm{i}}{2^6 \cdot 3} x^3 - \frac{1}{2^4} x + \dots$$
$$B_4 = \frac{5^2 \cdot 7 \cdot 11}{2^5} x^{12} + \frac{7 \cdot 11}{2^4} x^8 + \frac{5\mathrm{i}}{2^7 \cdot 3} x^6 + \frac{19}{2^6} x^4 + \frac{\mathrm{i}}{3 \cdot 2^7} x^2 + \frac{11 \cdot 13}{2^9 \cdot 3^2} + \dots$$

where the omitted summands contain only negative powers of x. Therefore,

$$\begin{aligned}
A_0 &= F(\delta) \\
A_1 &= -\frac{1}{2^3 \cdot 3} F^{(3)}(\delta) \\
A_2 &= \frac{1}{2^7 \cdot 3^2} F^{(6)}(\delta) + \frac{1}{2^4} F^{(2)}(\delta) + \frac{\mathrm{i}}{2^4 \cdot 3} F(\delta) \\
A_3 &= -\frac{1}{2^{10} \cdot 3^4} F^{(9)}(\delta) - \frac{1}{2^4 \cdot 3 \cdot 5} F^{(5)}(\delta) - \frac{\mathrm{i}}{2^7 \cdot 3^2} F^{(3)}(\delta) - \frac{1}{2^4} F^{(1)}(\delta) \\
A_4 &= \frac{1}{2^{15} \cdot 3^5} F^{(12)}(\delta) + \frac{11}{2^{11} \cdot 3^2 \cdot 5} F^{(8)}(\delta) + \frac{\mathrm{i}}{2^{11} \cdot 3^3} F^{(6)}(\delta) \\
&\quad + \frac{19}{2^9 \cdot 3} F^{(4)}(\delta) + \frac{\mathrm{i}}{2^8 \cdot 3} F^{(2)}(\delta) + \frac{11 \cdot 13}{2^9 \cdot 3^2} F(\delta)
\end{aligned} \tag{42}$$

for $\sigma = \frac{1}{2}$, and thus S is determined in this case up to an error of magnitude τ^{-5}.

The semiconvergent expansion (35) can be further simplified by expanding the quantity ϑ in the second term on the right hand side using Stirling's formula. To this end, Rieman considers the equation

$$\log\left(\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right) = \left(\frac{t\mathrm{i}}{2} - \frac{1}{4}\right)\log\left(\frac{t\mathrm{i}}{2}\right) - \frac{t\mathrm{i}}{2} + \log(\sqrt{2\pi}) + \frac{1}{4}\int_0^\infty \left(\frac{4\mathrm{e}^{3x}}{\mathrm{e}^{4x} - 1} - \frac{1}{x} - 1\right)\frac{\mathrm{e}^{-2t\mathrm{i}x}}{x}\mathrm{d}x$$

for $t > 0$, that arises from Binet's well-known formula for $\log(\Gamma(s))$ after an easy manipulation. With the identity

$$\frac{4\mathrm{e}^{3x}}{\mathrm{e}^{4x} - 1} = \frac{1}{\cosh(x)} + \frac{1}{\sinh(x)}$$

this implies, by splitting in real and imaginary part,

$$\begin{aligned}
\log\left|\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right| &= -\frac{\pi}{4}t - \frac{1}{4}\log\left(\frac{t}{2}\right) + \log(\sqrt{2\pi}) + \frac{1}{4}\int_0^\infty \left(\frac{1}{\cos(x)} - 1\right)\frac{\mathrm{e}^{-2tx}}{x}\mathrm{d}x \\
&\quad - \frac{1}{4}\log(1 + \mathrm{e}^{-2\pi t})
\end{aligned}$$

$$\operatorname{arc}\left(\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right) = \frac{t}{2}\log\left(\frac{t}{2}\right) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{4}\int_0^\infty \left(\frac{1}{\sin(x)} - \frac{1}{x}\right)\frac{\mathrm{e}^{-2tx}}{x}\mathrm{d}x + \frac{1}{2}\arctan(\mathrm{e}^{-\pi t}),$$

where the integrals are to be taken as Cauchy principal values due to the poles at $k\frac{\pi}{2}$, $k = 1, 2, \ldots$ Set

$$\frac{1}{\cos(x)} = \sum_{n=0}^\infty \frac{E_n}{(2n)!}x^{2n}, \qquad |x| < \frac{\pi}{2}$$

$$\frac{x}{\sin(x)} = \sum_{n=0}^\infty \frac{F_n}{(2n)!}x^{2n}, \qquad |x| < \pi.$$

Then $E_0 = 1$, $E_1 = 1$, $E_2 = 5$, $E_3 = 6$, $F_0 = 1$, $F_1 = \frac{1}{3}$, $F_2 = \frac{7}{15}$, $F_3 = \frac{31}{21}$, and in general

$$E_n - \binom{2n}{2} E_{n-1} + \binom{2n}{4} E_{n-2} - \ldots + (-1)^n E_0 = 0$$

$$\binom{2n+1}{1} F_n - \binom{2n+1}{3} F_{n-1} + \binom{2n+1}{5} F_{n-2} - \ldots + (-1)^n F_0 = 0$$

for $n = 1, 2, 3, \ldots$ This yields

$$\log\left|\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right| \sim -\frac{\pi}{4} t - \frac{1}{4}\log\left(\frac{t}{2}\right) + \log(\sqrt{2\pi}) + \frac{1}{8}\sum_{n=1}^{\infty} \frac{E_n}{n}(2t)^{-2n}$$

$$\operatorname{arc}\left(\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right) \sim \frac{t}{2}\log\left(\frac{t}{2}\right) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{8}\sum_{n=1}^{\infty} \frac{F_n}{n(2n-1)}(2t)^{1-2n}. \tag{43}$$

Now, $\vartheta = -\frac{t}{2}\log(\pi) + \operatorname{arc}(\Gamma(\frac{1}{4} + \frac{t\mathrm{i}}{2}))$ on $\sigma = \frac{1}{2}$, and thus

$$\frac{t}{2}\log\left(\frac{t}{2\pi}\right) - \frac{t}{2} - \frac{\pi}{8} - \vartheta \sim -\frac{1}{8}\sum_{n=1}^{\infty} \frac{F_n}{n(2n-1)}(2t)^{1-2n}$$

$$\mathrm{e}^{\frac{t}{2}\log(\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} - \vartheta)\mathrm{i}} = 1 - \frac{\mathrm{i}}{2^4 \cdot 3} t^{-1} - \frac{1}{2^9 \cdot 3^2} t^{-2} + \mathrm{O}(t^{-3}).$$

In light of (42), the definitive form of the semiconvergent series for $\zeta(s)$ on $\sigma = \frac{1}{2}$ is the equation

$$\mathrm{e}^{\vartheta\mathrm{i}}\zeta\left(\frac{1}{2} + t\mathrm{i}\right) = 2\sum_{n=1}^{m} \frac{\cos(\vartheta - t\log(n))}{\sqrt{n}} + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} R, \tag{44}$$

where

$$\vartheta = -\frac{t}{2}\log(\pi) + \operatorname{arc}\left(\Gamma\left(\frac{1}{4} + \frac{t\mathrm{i}}{2}\right)\right),$$

$$m = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor,$$

$$R \sim C_0 + C_1 t^{-\frac{1}{2}} + C_2 t^{-1} + C_3 t^{-\frac{3}{2}} + C_4 t^{-2} + \ldots$$

with

$$
\begin{aligned}
C_0 &= F(\delta) \\
C_1 &= -\frac{1}{2^3 \cdot 3} F^{(3)}(\delta) \\
C_2 &= \frac{1}{2^4} F^{(2)}(\delta) + \frac{1}{2^7 \cdot 3^2} F^{(6)}(\delta) \\
C_3 &= -\frac{1}{2^4} F^{(1)}(\delta) - \frac{1}{2^4 \cdot 3 \cdot 5} F^{(5)}(\delta) - \frac{1}{2^{10} \cdot 3^4} F^{(9)}(\delta) \\
C_4 &= \frac{1}{2^5} F(\delta) + \frac{19}{2^9 \cdot 3} F^{(4)}(\delta) + \frac{11}{2^{11} \cdot 3^2 \cdot 5} F^{(8)}(\delta) + \frac{1}{2^{15} \cdot 3^5} F^{(12)}(\delta)
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
F(x) &= \frac{\cos(x^3 + \frac{3\pi}{8})}{\cos(\sqrt{2\pi}x)} \\
\delta &= \sqrt{t} - \left(m + \frac{1}{2}\right)\sqrt{2\pi},
\end{aligned}
$$

and this is essentially how it can be found in Riemann's work. The only new item in the above is the estimate of the remainder term.

If we drop the assumption that $\sigma = \frac{1}{2}$ and restrict σ to an interval $\sigma_1 \leq \sigma \leq \sigma_2$, then we can still use the semiconvergent expansion (44); we only have to take t to be the complex number $-\mathrm{i}(s - \frac{1}{2})$ and m the integer number $\lfloor\sqrt{\frac{|t|}{2\pi}}\rfloor$, whereas ϑ is again given by (34). The necessary changes in the proof of this claim can be made without difficulty to the above derivation of (44).

The semiconvergent series R is a homogeneous linear relation of the equantities $F(\delta)$, $F'(\delta)$, $F''(\delta)$,.... By rearranging we obtain from it an expression of the form

$$
D_0^* F(\delta) + D_1^* F'(\delta) + D_2^* F''(\delta) + \ldots,
$$

in which every D_n^* is a power series in τ^{-1}. These power series are divergent. This begs the question whether they are semiconvergent expansions of certain analytic functions D_0, D_1, D_2,... and if the sequence

$$
D_0 F(\delta) + D_1 F'(\delta) + D_2 F''(\delta) + \ldots \tag{46}
$$

is also a semiconvergent expansion of R. This question was also considered by Riemann, once again without the necessary estimate of the remainder. However, since the series (46) is of lesser theoretical and practical importance than the original semiconvergent expansion, we also omit the rather tedious analysis of the error term in the following. Perhaps this serves even more to highlight the strength of Riemann's formalism.

Equation (30), which can also be written

$$\int_{m\nwarrow m+1} \frac{e^{-\pi i(x-\eta)^2+2\pi i(x-\eta)(\eta-m)}}{e^{2\pi i x}-1} dx = F(\delta) e^{-\frac{\pi i}{8}-\pi i(\eta-m)^2},$$

allows the following inversion

$$\frac{2}{\sqrt{2\pi}} e^{-\frac{\pi i}{8}-\pi i(\eta-m)^2} \int_{0\swarrow 1} F(u+\delta) e^{iu^2-2\sqrt{2\pi}i(x-\eta)u} du = \frac{e^{-\pi i(x-\eta)^2+2\pi i(x-\eta)(\eta-m)}}{e^{2\pi i x}-1} \tag{47}$$

for $m < \mathrm{Re}(x) < m+1$. This follows either by application of Fourier's theorem or by changing to complex conjugates in (5). From (47) it follows that

$$\frac{e^{-2\pi i m x}}{e^{2\pi i x}-1} = (-1)^m \frac{2}{\sqrt{2\pi}} e^{-\pi i x^2-\frac{\pi i}{8}} \int_{0\swarrow 1} F(u+\delta) e^{i(u+\tau-\sqrt{2\pi}x)^2} du \tag{48}$$

is also valid for $m < \mathrm{Re}(x) < m+1$. It suggests itself to substitute the series

$$F(\delta) + \frac{F'(\delta)}{1!} u + \frac{F''(\delta)}{2!} u^2 + \dots$$

for $F(u+\delta)$ and to determine the contribution of every single term in this series to the integral appearing in (16),

$$\int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} g(x) dx = \int_{\eta+\varepsilon\frac{\eta}{2}}^{\eta-\varepsilon\frac{\eta}{2}} x^{s-1} \frac{e^{-2\pi i m x}}{e^{2\pi i x}-1} dx.$$

In this way, we find the semiconvergent expansion

$$\int_{C_2} g(x) dx \sim (-1)^m \frac{2}{\sqrt{2\pi}} e^{-\frac{\pi i}{8}} \sum_{n=0}^{\infty} \frac{F^{(n)}(\delta)}{n!} \int_{m\nwarrow m+1} x^{s-1} e^{-\pi i x^2} \left(\int_{0\swarrow 1} u^n e^{i(u+\tau-\sqrt{2\pi}x)^2} du \right) dx. \tag{49}$$

On the other hand, by (13) and (44),

$$\int_{C_2} g(x) dx = (-1)^m \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} e^{\vartheta i} R(1-e^{\pi i s}). \tag{50}$$

Since, as we can easily see, the representation of A_n as a homogeneous linear function of the $F^{(k)}(\delta)$ with constant coefficients given in (41) is unique, it follows from (49) and (50) that for $n = 0, 1, 2, \dots$

$$n! D_n(1-e^{\pi i s}) = \frac{2}{\sqrt{2\pi}} \left(\frac{t}{2\pi}\right)^{\frac{1}{4}} e^{-\vartheta i-\frac{\pi i}{8}} \int_{0\nwarrow 1} x^{s-1} e^{-\pi i x^2} \left(\int_{0\swarrow 1} u^n e^{i(u+\tau-\sqrt{2\pi}x)^2} du \right) dx \tag{51}$$

so in particular if we set

$$\frac{1}{\sqrt{2\pi}}\left(\frac{t}{2}\right)^{\frac{1}{4}} e^{\frac{\pi}{4}t}\sqrt{\Gamma\left(\frac{1}{4}+\frac{t\mathrm{i}}{2}\right)\Gamma\left(\frac{1}{4}-\frac{t\mathrm{i}}{2}\right)} = e^{\omega},$$

then

$$\begin{aligned} D_0 &= e^{\omega} \\ D_1 &= -\tau(e^{\omega}-e^{-\omega}) + \frac{\tau e^{\pi \mathrm{i} s-\omega}}{1-e^{\pi \mathrm{i} s}} \sim -\tau(e^{\omega}-e^{-\omega}). \end{aligned} \tag{52}$$

For the remaining D_n we can use (51) to derive a recursion formula via integration by parts. This can also be obtained in the following way without an additional calculation. By (36), (37), (38),

$$S \sim d_0 F(\delta) + \frac{d_1}{1!}F'(\delta) + \frac{d_2}{2!}F''(\delta) + \dots,$$

where by (38) and (39), the d_n satisfy the recursion formula

$$\tau d_n + \frac{1}{2}d_{n+1} + \frac{(n-1)(n-2)}{8}d_{n-3} = 0, \qquad n = 1, 2, 3 \dots$$

Since

$$e^{(\frac{t}{2}\log(\frac{t}{2\pi}-\frac{t}{2}-\frac{uppi}{8}-\vartheta)\mathrm{i}}S = R,$$

we have the following recursion formula for D_n,

$$D_{n+1} = -\frac{2}{n+1}\tau D_n - \frac{1}{4n(n+1)}D_{n-3}, \tag{53}$$

for $n = 1, 2, 3, \dots$, with $D_{-2} = 0$, $D_{-1} = 0$. Using (52), we obtain the values

$$\begin{aligned} D_2 &= -\tau D_1 \sim \tau^2(e^{\omega}-e^{-\omega}) \\ D_3 &= -\frac{2}{3}\tau D_2 \sim -\frac{2}{3}\tau^3(e^{\omega}-e^{-\omega}) \\ D_4 &= -\frac{1}{2}\tau D_3 - \frac{1}{2^4\cdot 3}D_0 \sim \frac{1}{3}\tau^4(e^{\omega}-e^{-\omega}) - \frac{1}{2^4\cdot 3}e^{\omega}. \end{aligned}$$

The semiconvergent expansions of D_0, D_1,... are obtained from (43), namely,

$$\omega \sim \frac{1}{8}\sum_{n=1}^{\infty}\frac{E_n}{n}(2t)^{-2n} = \frac{1}{2^5}t^{-2} + \frac{5}{2^8}t^{-4} + \frac{61}{2^9\cdot 3}t^{-6} + \dots$$

Substituting this into the obtained values for $D_0, \dots, D_4$, it follows that

$$
\begin{aligned}
D_0 &\sim 1 + \frac{1}{2^5}\tau^{-4} + \frac{41}{2^{11}}\tau^{-8} + \dots \\
D_1 &\sim -\frac{1}{2^4}\tau^{-3} - \frac{5}{2^7}\tau^{-7} + \dots \\
D_2 &\sim \frac{1}{2^4}\tau^{-2} + \frac{5}{2^7}\tau^{-6} + \dots \\
D_4 &\sim -\frac{1}{2^3 \cdot 3}\tau^{-1} - \frac{5}{2^6 \cdot 3}\tau^{-5} + \dots \\
D_5 &\sim \frac{19}{2^9 \cdot 3}\tau^{-4} + \dots
\end{aligned}
\tag{54}
$$

From the recursion formula (53) it follows that all exponents of powers appearing in the semiconvergent expansion of the series D_n are congruent n modulo 4. Accordingly, the orders of all derivatives of $F(\delta)$ appearing in C_n are of the form $3n-4k$, as is easily confirmed through the found expressions for C_0, C_1, C_2, C_3, C_4. If we write

$$R \sim \sum b_{kl} F^{3l-4k)}(\delta)\tau^{-l}$$

where the summation index k runs through the values $0, \dots, \lfloor \frac{3l}{4} \rfloor$ and l runs through the values $0, 1, \dots$, then all b_{kl} with $l \leq 4$ are determined by (45), whereas the values $b_{00}, b_{34}, b_{68}, b_{23}, b_{57}, b_{12}, b_{46}, b_{01}, b_{35}, b_{24}$ are know due to (54). We immediately see that the values $b_{00}, b_{34}, b_{23}, b_{12}, b_{01}, b_{24}$, that appear in both (45) and (54), coincide.

For the numerical computation of the b_{kl} and the practical application of the semiconvergent series, the original form ordered by powers of τ^{-1} is preferable, for computing the D_n via (53) is more laborious than the computation of the C_n treated earlier. Moreover, the orders of the D_n are not monotonously decreasing, but D_{3n-2}, D_{3n-1}, D_{3n} have the exact orders $\tau^{-(n+2)}$, $\tau^{-(n+1)}$, τ^{-n}, so that we would have to additionally compute D_5 to D_{12} to obtain the previous error $\mathrm{O}(\tau^{-5})$.

The transition to the D_n is done via (48). If we try to obtain from (48) an exact expression for $\zeta(s)$ and not just a semiconvergent series, then we are led to to the approach discussed in the following paragraph.

§ 3 The integral representation of the zeta function

The explicit computation of the coefficients of the semiconvergent series for $\zeta(s)$ relies on equation (5) in §1. With the help of this formula, Riemann derived another rather interesting expression for $\zeta(s)$, which apparently escaped the attention of the other mathematicians until 1926.

At first, let $\sigma < 0$, and let u^{-s} assume the principal value on u-plane cut from 0 to $-\infty$. Multiply (5) by u^{-s} and integrate from 0 to $\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\infty$ over u along the bisector of the first quadrant. If we write $\overline{\varepsilon} = \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$ for short, then

$$\begin{aligned}\int_0^{\overline{\varepsilon}\infty} \frac{u^{-s}}{1-\mathrm{e}^{-2\pi\mathrm{i}u}}\mathrm{d}u &= -\int_0^{\overline{\varepsilon}\infty} u^{-s}\sum_{n=1}^{\infty}\mathrm{e}^{2\pi\mathrm{i}nu}\mathrm{d}u = -\sum_{n=1}^{\infty}\int_0^{\overline{\varepsilon}\infty} u^{-s}\mathrm{e}^{2\pi\mathrm{i}nu}\mathrm{d}u \\ &= -\Gamma(1-s)\sum_{n=1}^{\infty}\left(2\pi n\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}\right)^{s-1} \\ &= -(2\pi)^{s-1}\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}\Gamma(1-s)\zeta(1-s)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\overline{\varepsilon}\infty} u^{-s}\left(\int_{0\nwarrow 1}\frac{\mathrm{e}^{-\pi\mathrm{i}x^2+2\pi\mathrm{i}ux}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x\right)\mathrm{d}u &= \int_{0\nwarrow 1}\frac{\mathrm{e}^{-\pi\mathrm{i}x^2}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\left(\int_0^{\overline{\varepsilon}\infty} u^{-s}\mathrm{e}^{2\pi\mathrm{i}ux}\mathrm{d}u\right)\mathrm{d}x \\ &= (2\pi)^{s-1}\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}\Gamma(1-s)\int_{0\nwarrow 1}\frac{\mathrm{e}^{-\pi\mathrm{i}x^2}x^{s-1}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x,\end{aligned}$$

so that by (5),

$$(2\pi)^{s-1}\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(1-s)}\Gamma(1-s)\left(\zeta(1-s)+\int_{0\nwarrow 1}\frac{x^{s-1}\mathrm{e}^{-\pi\mathrm{i}x^2}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x\right)+\int_0^{\overline{\varepsilon}\infty}\frac{u^{-s}\mathrm{e}^{\pi\mathrm{i}u^2}}{\mathrm{e}^{\pi\mathrm{i}u}-\mathrm{e}^{-\pi\mathrm{i}u}} = 0. \tag{55}$$

Here, the second integral can be brought into the form

$$\frac{1}{\mathrm{e}^{\pi\mathrm{i}s}-1}\int_{0\swarrow 1}\frac{u^{-s}\mathrm{e}^{\pi\mathrm{i}u^2}}{\mathrm{e}^{\pi\mathrm{i}u}-\mathrm{e}^{-\pi\mathrm{i}u}}\mathrm{d}u,$$

where $0 \swarrow 1$ indicates that the path of integration is obtained from the original integral by reflection on the real axis. By multiplying (55) with the factor

$$2^{1-s}\pi^{\frac{1-s}{2}}\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(s-1)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}$$

and considering the relation

$$\frac{2^{-s}\pi^{\frac{1-s}{2}}\Gamma(\frac{1-s}{2})}{\sin(\frac{\pi s}{2})\Gamma(1-s)} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$

we obtin the following formula, valid on the whole s-plane,

$$\begin{aligned}\pi^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) &= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx \\ &\quad + \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\int_{0\swarrow 1}\frac{x^{s-1}e^{-\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx.\end{aligned} \tag{56}$$

Riemann does not quite write it in this symmetrical form, but this form seems to be practical for applications. First, it shows the functional identity for $\zeta(s)$. In fact, for $\sigma = \frac{1}{2}$ the two summands on the right hand side are complex conjugates, so $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is real there, and since this function is real for $\sigma > 1$, we obtain from the reflection principle the functional identity

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{57}$$

at $\sigma = \frac{1}{2}$ and thus for arbitrary s.

If in addition we set

$$f(s) = \int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx \tag{58}$$

$$\varphi(s) = 2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)f(s), \tag{59}$$

then by (56) and (57) for $\sigma = \frac{1}{2}$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(1-s) = \mathrm{Re}(\varphi(s)). \tag{60}$$

Thereby the investigation of $\zeta(s)$ on the critical line is reduced to the investigation of its real part.

§ 4 The meaning of Riemann's formulas for the theory of the zeta function

The principle term of the semiconvergent series for $\zeta(s)$, that is, using (32), the expression

$$\sum_{l=1}^{m} l^{-s} + \frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sum_{l=1}^{m} l^{s-1}, \qquad m = \left\lfloor\sqrt{\frac{t}{2\pi}}\right\rfloor,$$

has also been found by Hardy and Littlewood, where is instead of Riemann's expansion for S they only give an upper bound for the absolute value. They also discovered a more general form of the principal term,

$$\sum_{l \leq x} l^{-s} \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \sum_{l \leq y} l^{s-1} \tag{61}$$

with $xy = \frac{t}{2\pi}$. This does not appear in Riemann's work, but we can find without difficulty that the expression (61) can be completed into a semiconvergent series in Riemann's way, namely, here the function $\Phi(\tau, u)$ defined by (7) plays the same role as the special function $\Phi(-1, u)$ for Riemann.

For Hardy and Littlewoods applications of their formula, in particular the estimate of the number $N_0(T)$ of zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$ contained in the interval $0 < t < T$, Riemann's formula apparently does not yield a better result. At the point mentioned in the beginning, Riemann claimed that $N_0(T)$ is asymptotically $\frac{T}{2}\log(\frac{T}{2}) - \frac{T}{2}$, that is, asymptotically equal to the number $N(T)$ of zeros contained in $0 < t < T$ of $\zeta(s)$, and that this could be proven by means of his new expansion. However, from his notes it is not clear how he envisioned this proof. In the following representation, valid on $\sigma = \frac{1}{2}$,

$$\mathrm{e}^{\vartheta \mathrm{i}} \zeta\left(\frac{1}{2} + t\mathrm{i}\right) = 2\sum_{n=1}^{m} \frac{\cos(\vartheta - t\log(n))}{\sqrt{n}} + \mathrm{O}\left(t^{-\frac{1}{4}}\right) \tag{62}$$

$$\vartheta = \frac{t}{2}\log\left(\frac{t}{2\pi}\right) - \frac{t}{2} + \mathrm{O}(1)$$

the first term in the right hand side trigonometric sum, $\cos(\vartheta)$, has indeed asymptotically $\frac{T}{2\pi}\log(\frac{T}{2\pi}) = \frac{T}{2\pi}$ zeros in the interval $0 < t < T$, and the coefficients $\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots$ decrease monotonously. Perhaps Riemann believed that this observation might be helpful for a proof of his claim.

It suggests itself to use the exact Riemann formula to estimate the mean values

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2} + t\mathrm{i}\right)\right|^{2n} \mathrm{d}t \qquad n = 3, 4, \ldots$$

As is well-known, these mean values are in a close relationship to the so-called Lindelöf conjecture. But here we encounter significant difficulties of an arithmetic nature, arising from the number of divisors of the natural numbers.

To establish a numerical table for the zeta function, in particular for the computation of further zeros, the semiconvergent expansion is of great advantage. However, for practical purposes, we need a more precise estimate of the remainder term than the one derived in §2. Riemann made quite extensive computations to determine the positive zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$ using his formula. For the smallest positive zero, he finds the number $a_1 = 14.1386$. Gram later computed the value 14.1347, smaller by less than 3 per mill. A lower bound for a_1 is also given by the equation

$$\sum_{n=1}^{\infty}\left(a_n^2+\frac{1}{4}\right)^{-1} = 1+\frac{1}{2}C-\frac{1}{2}\log(\pi)-\log(2),$$

easily derived from the product representation of $\zeta(s)$, where C is the Euler constant and a_n runs through all solutions of $\zeta(\frac{1}{2}+a\mathrm{i}) = 0$ located in the right half-plane. From this, Riemann obtains

$$\sum_{n=1}^{\infty}\left(a_n^2+\frac{1}{4}\right)^{-1} = 0.02309570896612103381.$$

For a_3, he finds the value 25.31, whereas Gram gives the value 25.01.

The second Riemann formula, namely the integral representation of $\zeta(s)$, may be of greater interest for the theory. One can try to derive from (60) some information on the distribution of zeros on the critical line. Suppose t runs through the interval $t_1 \le t \le t_2$ in the positive direction. If on this path the function $\operatorname{arc}(\varphi(\frac{1}{2}+t\mathrm{i}))$ changes by A, where the change of $\varphi(s)$ in passing a possible zero located on $\sigma = \frac{1}{2}$ is defined to be the multiplicity of the zero multiplied by π, then by (60) the number of zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$ in the interval $t_1 \le t \le t_2$ is greater than $\frac{|A|}{\pi}-1$. But now

$$\operatorname{arc}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right) = \vartheta = \frac{t}{2}\log\left(\frac{t}{2\pi}\right)-\frac{t}{2}+\mathrm{O}(1), \tag{63}$$

and hence by (59) the number of zeros $\zeta(\frac{1}{2}+t\mathrm{i})$ located in the interval $0 < t < T$ would be at least $\frac{T}{2}\log(T)$, that is, asymptotically equal to the number of zeros of $\zeta(s)$ located in the strip $0 < t < T$, if the arcus of the function $f(\frac{1}{2}+t\mathrm{i})$ defined by (58) decreases slower than $-t\log(t)$ for $t \to \infty$. For every half-strip $\sigma_1 \le \sigma \le \sigma_2$, $t > 0$, $f(s)$ can be expanded as a semiconvergent series by the method of §2. However, the principal term is again a sum of $\lfloor\sqrt{\frac{t}{2\pi}}\rfloor$ summands, namely $\sum_{n=1}^{m} n^{-s}$, and studying the arcus of this sum is a task of the same difficulty as studying the zeros of the sum appearing in (62), so that apparently nothing is gained by introducing $f(s)$.

If we now consider the rectangle with sides $\sigma = \frac{1}{2}$, $\sigma = 2$, $t = 0$, $t = T$, where the upper side should not contain any zero of $f(s)$, then the change of $\frac{1}{2\pi}\operatorname{arc}(f(s))$

on a traversal of this rectangle with positive orientation equals the number of zeros of $f(s)$ inside the rectangle. On the lower side, $\operatorname{arc}(f(s))$ changes by $\mathrm{O}(1)$ and on the right side, as follows from the semiconvergent series, the change is also only $\mathrm{O}(1)$. Moreover, with the usual methods in the study of the zeta function we can show that the change on the upper side is at most $\mathrm{O}(\log(T))$. As a consequence, up to an error of order $\log(T)$, the change of $\operatorname{arc}(f(\frac{1}{2}+t\mathrm{i}))$ in the interval $0 < t < T$ equals the number of zeros of $f(t)$ in the rectangle multiplied by -2π. Thus the problem is reduced to the investigation of the zeros of the transcendent $f(s)$.

Riemann tries to obtain a statement on the zeros of $f(s)$ by forming, after (58), the expression

$$|f(\sigma+t\mathrm{i})|^2 = \int_{0\swarrow 1}\int_{0\searrow 1} \frac{x^{-\sigma-t\mathrm{i}}y^{-\sigma+t\mathrm{i}}\mathrm{e}^{\pi\mathrm{i}(x^2-y^2)}}{(\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x})(\mathrm{e}^{\pi\mathrm{i}y}-\mathrm{e}^{-\pi\mathrm{i}y})}\mathrm{d}x\mathrm{d}y$$

and then transforms the complex double integral into a new form by introduction of a new variable, deformation of the domain of integration and application of the residue theorem; but this does not lead to a useful result.

Only very little is known on the location of the zeros of $f(s)$. In Riemann's notes there are no further remarks on this issue. In this historically-mathematically treatise we thus shall keep the following remarks on the theory of $f(s)$ brief. They yield a proof of the inequality

$$N_0(T) > \frac{3}{8\pi}\mathrm{e}^{-\frac{3}{2}T+\mathrm{o}(T)}.$$

For $f(s)$ we can find a semiconvergent series using the method in §2. For the task at hand it is sufficient to know the principal term of this series. First, we show that in the domain $t > 0$, $-\sigma \geq t^{\frac{3}{7}}$, the formula

$$f(s) \sim \mathrm{e}^{\frac{\pi\mathrm{i}}{4}(s-\frac{7}{2})}\pi^{\frac{s-1}{2}}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\frac{\sin(\pi\eta)}{\cos(2\pi\eta)}, \qquad |s|\to\infty, \tag{64}$$

holds, where

$$\eta = \sqrt{\frac{s-1}{2\pi\mathrm{i}}}, \qquad 0 < \operatorname{arc}(\eta) < \frac{\pi}{4}.$$

By (56),

$$f(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\left(\zeta(1-s) - \int_{0\searrow 1}\frac{x^{s-1}\mathrm{e}^{-\pi\mathrm{i}x^2}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}}\mathrm{d}x\right). \tag{65}$$

The saddle point of the function $x^{s-1}\mathrm{e}^{-\pi \mathrm{i} x^2}$ is at $x = \eta$. Set

$$\begin{aligned}
\mathrm{Re}(\eta) &= \eta_1, \\
\mathrm{Im}(\eta) &= \eta_2, \\
m &= \lfloor \eta_1 + \eta_2 \rfloor, \\
z &= x - \eta, \\
w(z) &= \mathrm{e}^{2\pi \mathrm{i}\eta^2 (\log(1+\frac{z}{\eta}) - \frac{z}{\eta} + \frac{1}{2}(\frac{z}{\eta})^2)} - 1.
\end{aligned}$$

By Cauchy's theorem, for every natural number k,

$$\begin{aligned}
&\int_{0\searrow 1} \frac{x^{s-1}\mathrm{e}^{-\pi \mathrm{i} x^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} \mathrm{d}x \\
&= \sum_{n=1}^{m} n^{s-1} + \eta^{s-1}\mathrm{e}^{-\pi \mathrm{i}\eta^2} \left(\int_{k\searrow k+1} \frac{\mathrm{e}^{-2\pi \mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} \mathrm{d}x + \int_{k\searrow k+1} \frac{\mathrm{e}^{-2\pi \mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} w(z) \mathrm{d}x \right).
\end{aligned} \tag{66}$$

If we were to proceed according to the procedure of §2, then we would have to choose $k = m$. But then we would obtain (64) only on the smaller domain $t > 0$, $-\sigma \geq t^{\frac{1}{2}}$, and the extension to the remaining domain $t^{\frac{1}{2}} > -\sigma \geq t^{\frac{3}{7}}$ would require the elimination of additional terms. Thus, we leave k arbitrary for now.

The first integral on the right hand side of (66) can be computed by Riemann's method from §1. We obtain

$$\int_{k\searrow k+1} \frac{\mathrm{e}^{-2\pi \mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} \mathrm{d}x = \frac{\sqrt{2}\mathrm{e}^{\frac{3\pi \mathrm{i}}{8}} \sin(\pi\eta) + (-1)^{k-1}\mathrm{e}^{2\pi \mathrm{i}\eta - 2\pi \mathrm{i}(\eta-k)^2}}{2\cos(2\pi\eta)}. \tag{67}$$

For the second integral choose a path of integration passing through the saddle point $x = \eta$ and parallel to the bisector of the second and fourth quadrant. It meets the real axis in the point $\eta_1 + \eta_2$. But to avoid proximity of the poles $x = m$ and $x = m+1$, further replace those parts of the path of integration passing through the circles $|x - m| = \frac{1}{2}$ and $|x - m - 1| = \frac{1}{2}$ by segments of these circles. If we assume

$$k = m + r \geq m,$$

then

$$\begin{aligned}
\int_{k\searrow k+1} \frac{\mathrm{e}^{-2\pi \mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} w(z) \mathrm{d}x &= \sum_{l=1}^{r} (-1)^{m+l-1}\mathrm{e}^{-2\pi \mathrm{i}(m+l-\eta)^2} w(m+l-\eta) \\
&\quad + \int_{m\searrow m+1} \frac{\mathrm{e}^{-2\pi \mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi \mathrm{i} x} - \mathrm{e}^{-\pi \mathrm{i} x}} \mathrm{d}x.
\end{aligned} \tag{68}$$

For $w(z)$ we require to estimates. The first is with respect to the circle $|z| \leq \frac{1}{2}|\eta|$. In this circle,

$$\left|\log\left(1+\frac{z}{\eta}\right)-\frac{z}{\eta}+\frac{1}{2}\left(\frac{z}{\eta}\right)^2\right| = \left|\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{z}{\eta}\right)^b\right| \leq \frac{1}{3}\left|\frac{z}{\eta}\right|^3\frac{1}{1-|\frac{z}{\eta}|} \leq \frac{2}{3}\left|\frac{z}{\eta}\right|^3$$

and hence

$$|w(z)| \leq \mathrm{e}^{\frac{4\pi}{3}|\frac{z^3}{\eta}|}-1, \qquad |z| \leq \frac{1}{2}|\eta|. \tag{69}$$

The second estimate concerns the part of the path of integration lying outside of this circle. If we further set $u = \mathrm{Re}(z\mathrm{e}^{\frac{\pi i}{4}})$, $v = \mathrm{Im}(z\mathrm{e}^{\frac{\pi i}{4}})$, then on the line of integration we have $-\frac{1}{2} \leq v \leq \frac{1}{2}$, and if $|\eta| > 1$, then outside of the circle $|z| = \frac{1}{2}|\eta|$ the inequality

$$\left|\frac{u}{v}\right| < (|\eta|^2-1)^{-\frac{1}{2}}$$

holds, that is,

$$\mathrm{arc}\left(1+\frac{\mathrm{i}v}{u}\right) \to 0, \qquad |s| \to \infty$$

and

$$\frac{\pi}{4}-\varepsilon < \left|\mathrm{arc}\left(\frac{z}{\eta}\right)\right| < \frac{3\pi}{4}+\varepsilon,$$

with $\varepsilon \to 0$ for $|s| \to \infty$. But then

$$\begin{aligned}\left|2\pi \mathrm{i}\eta^2\left(\log\left(1+\frac{z}{\eta}\right)-\frac{z}{\eta}+\frac{1}{2}\left(\frac{z}{\eta}\right)^2\right)\right| &= 2\pi|\eta|^2\cdot\left|\int_0^{\frac{z}{\eta}}\frac{x^2}{1+x}\mathrm{d}x\right| \\ \leq 2\pi|\eta|^2\int_0^{|\frac{z}{\eta}|}\frac{x}{\sin(\frac{\pi}{4}-\varepsilon)}\mathrm{d}x &= \frac{\pi|z|^2}{\sin(\frac{\pi}{4}-\varepsilon)} \leq \frac{3}{2}\pi|z|^2\end{aligned}$$

and

$$|w(z)| < \mathrm{e}^{\frac{3}{2}\pi|z|^2}.$$

Moreover, on the line of integration,

$$|\mathrm{e}^{-2\pi\mathrm{i}z^2}| = \mathrm{e}^{-2\pi(u^2-v^2)} \leq \mathrm{e}^{\pi-2\pi|z|^2},$$

and thus

$$|\mathrm{e}^{-2\pi\mathrm{i}z^2}w(z)| \leq \begin{cases}\mathrm{e}^{\pi-\frac{\pi}{2}|z|^2}, & |z| > \frac{1}{2}|\eta|, \\ \mathrm{e}^{\pi-2\pi|z|^2}\left(\mathrm{e}^{\frac{4\pi}{3}|\frac{z^3}{\eta}|}-1\right), & |z| \leq \frac{1}{2}|\eta|.\end{cases}$$

This yields

$$\int_{m\searrow m+1} \frac{\mathrm{e}^{-2\pi\mathrm{i}(x-\eta)^2}}{\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x}} w(z)\mathrm{d}x = \mathrm{O}(\mathrm{e}^{-\pi\eta_2}\eta^{-1})$$

and together with (65), (66), (67), (68),

$$\begin{aligned} f(s) = {} & \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \eta^{s-1} \mathrm{e}^{-\pi\mathrm{i}\eta^2} \\ & \cdot \left(\mathrm{e}^{\pi\mathrm{i}\eta^2} \sum_{n=m+r+1}^{\infty} \left(\frac{n}{\eta}\right)^{s-1} - \frac{\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{8}} \sin(\pi\eta) + (-1)^{m+r-1} \mathrm{e}^{2\pi\mathrm{i}\eta - 2\pi\mathrm{i}(\eta-m-r)^2}}{2\cos(2\pi\eta)} \right. \\ & \left. + \sum_{l=1}^{r} (-1)^{m+l} \mathrm{e}^{-2\pi\mathrm{i}(m+l-\eta)^2} w(m+l=\eta) + \mathrm{O}(\mathrm{e}^{-\pi\eta_2}\eta^{-1}) \right). \end{aligned} \tag{70}$$

Now we have to show that on the domain $t > 0$, $-\sigma \geq t^{\frac{3}{7}}$, for a suitable choice of r and for $|s| \to \infty$, the expression

$$-\frac{\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{8}} \sin(\pi\eta)}{2\cos(2\pi\eta)}$$

is of higher order than the remaining terms in the parantheses in (70). Firstly,

$$\begin{aligned} \left| \mathrm{e}^{\pi\mathrm{i}\eta^2} \sum_{n=m+r+1}^{\infty} \left(\frac{n}{\eta}\right)^{s-1} \right| & < \mathrm{e}^{-2\pi\eta_1\eta_2} |\eta^{1-s}| \left(\frac{(m+r+1)^{\sigma}}{-\sigma} + (m+r+1)^{\sigma-1} \right) \\ & < \mathrm{e}^{-2\pi\eta_1\eta_2 + t\,\mathrm{arc}(\eta)} \left(\frac{m+r+1}{|\eta|} \right)^{\sigma-1} \left(\frac{m+r+1}{-\sigma} + 1 \right). \end{aligned} \tag{71}$$

Since

$$2\pi\eta_1\eta_2 + \mathrm{arc}(\eta) < -2\pi\eta_1\eta_2 + t\frac{\eta_2}{\eta_1} = -2\pi\frac{\eta_2^3}{\eta_1} < 0 \tag{72}$$

and

$$\left(\frac{m+1}{|\eta|}\right)^{\sigma-1} < \left(\frac{\eta_1^2+\eta_2^2+2\eta_1\eta_2}{\eta_1^2+\eta_2^2}\right)^{\frac{\sigma-1}{2}} < \mathrm{e}^{\frac{\eta_1\eta_2}{\eta_1^2+\eta_2^2}\frac{\sigma-1}{2}} < \mathrm{e}^{\frac{\eta_2}{2\eta_1}\frac{\sigma-1}{2}} = \mathrm{e}^{-\pi\eta_2^2},$$

we have

$$\left| \mathrm{e}^{\pi\mathrm{i}\eta^2} \sum_{n=m+1}^{\infty} \left(\frac{n}{\eta}\right)^{s-1} \right| = \mathrm{O}\left(\mathrm{e}^{-\pi\eta_2^2} \left(1 + \frac{\eta_1}{-\sigma} \right) \right). \tag{73}$$

Moreover,

$$|(-1)^{m-1}\mathrm{e}^{2\pi\mathrm{i}\eta-2\pi\mathrm{i}(\eta-m)^2}| = \mathrm{e}^{-2\pi\eta_2-4\pi(m-\eta_1)\eta_2} < \mathrm{e}^{-4\pi(\eta_2-\frac{1}{2})\eta_2}. \qquad (74)$$

Now, since on the subdomain $t > 0, -\sigma \geq t^{\frac{5}{8}}$, the inquality

$$\eta_2 = \frac{1-\sigma}{4\pi\eta} > \frac{1-\sigma}{2}(2\pi(t+1-\sigma))^{-\frac{1}{2}} > \frac{1}{2}t^{\frac{5}{8}}\left(2\pi\left(t+t^{\frac{5}{8}}\right)\right)^{-\frac{1}{2}}$$

holds, and the right hand side goes to infity with t, it follows in light of (73) and (74) that the expression in the paranthese of (70) takes the value

$$-\frac{\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{8}}\sin(\pi\eta)}{2\cos(2\pi\eta)}(1+\mathrm{o}(1)), \qquad |s| \to \infty, \qquad (75)$$

for $r = 0$ on this subdomain. In the remaining subdomain $t > 0, t^{\frac{5}{8}} > -\sigma > t^{\frac{3}{7}}$, choose

$$r = \left\lfloor |\sigma|^{\frac{1}{5}} \right\rfloor.$$

Then for sufficiently large t,

$$\left(\frac{m+r+1}{|\eta|}\right)^{\sigma-1} < \left(\frac{|\eta|+r}{|\eta|}\right)^{\sigma-1} < \mathrm{e}^{\frac{\sigma-1}{2}\frac{r}{2\eta_1}} = \mathrm{e}^{-\pi r\eta_2},$$

so by (71) and (72),

$$\mathrm{e}^{\pi\mathrm{i}\eta^2}\sum_{n=m+r+1}^{\infty}\left(\frac{n}{\eta}\right)^{s-1} = \mathrm{O}\left(\mathrm{e}^{-\pi r\eta_2}\left(1+\frac{|\eta|}{|\sigma|}\right)\right) = \mathrm{O}\left(\mathrm{e}^{-\frac{1}{2}t^{\frac{1}{70}}}\right). \qquad (76)$$

Moreover, for $l = 1, \ldots, r$,

$$|m+l-\eta|^2 \leq (r+\eta_2)+\eta_2^2 = \mathrm{O}(|\sigma|^{\frac{2}{5}}) = \mathrm{O}(t^{\frac{1}{4}}), \qquad (77)$$

that is, for $m+l-\eta$ lies in the circle $|z| \leq \frac{1}{2}|\eta|$ for sufficiently large t, and (69) is applicable for $z = m+l-\eta$. By (77),

$$w(m+l-\eta) = \mathrm{O}(|\sigma|^{\frac{3}{5}}|\eta|^{-1}). \qquad (78)$$

Finally, for $l = 1, \ldots, r$,

$$|\mathrm{e}^{-2\pi\mathrm{i}(m+l-\eta)^2}| < \mathrm{e}^{-4\pi(\eta_2+l-1)\eta_2} \leq \mathrm{e}^{-4\pi\eta_2^2} \qquad (79)$$

and for sufficiently large t,

$$|\mathrm{e}^{2\pi\mathrm{i}\eta-2\pi\mathrm{i}(m+r-\eta)^2}| < \mathrm{e}^{-3\pi r\eta_2} = \mathrm{O}(\mathrm{e}^{-t^{\frac{1}{70}}}), \qquad (80)$$

so that by (78) and (79),

$$\sum_{l=1}^{r}(-1)^{m+l}\mathrm{e}^{-2\pi\mathrm{i}(m+l-\eta)^2}w(m+l-\eta) = r\mathrm{O}(\mathrm{e}^{-4\pi\eta_2^2}|\sigma|^{\frac{3}{5}}|\eta|^{-1}) \\ = \mathrm{O}(\mathrm{e}^{-4\pi\eta_2^2}|\sigma|^{\frac{4}{5}}|\eta|^{-1}). \tag{81}$$

If we consider the inequalities

$$|\sin(\pi\eta)| \geq \sinh(\pi\eta_2) > \pi\eta_2 > \frac{|\sigma|}{4|\eta|},$$
$$|\cos(2\pi\eta)| \leq \cosh(2\pi\eta_2) < 2\mathrm{e}^{2\pi\eta_2},$$

the the estimates (76), (80), (81) show that on the domain $t > 0$, $t^{\frac{5}{8}} < -\sigma < t^{\frac{3}{7}}$ the value of the expression in parantheses in (70) is given by (75).

The claim in (64) now follows by application of Stirling's formula.

It is even possible to prove (64) for the larger domain $t > 0$, $-\sigma \geq t^{\varepsilon}$, where ε is any fixed positive number. But for the following, any value of ε below $\frac{1}{2}$, for example $\varepsilon = \frac{3}{7}$, is sufficient.

Aside from (64) we need a rough estimate of the order of $f(s)$ for fixed σ and $t \to \infty$. This is obtained from the semiconvergent expansion of $f(s)$ for in the domain $t > 0$, $-\sigma \leq t^{\frac{3}{7}}$. A look at the proof of (64) shows that up to equation (70), the assumption $-\sigma \geq t^{\frac{3}{7}}$ was only used in the weaker form $\sigma < \sigma_0$, where σ_0 is any real number. Thus, in analogy to (70) with $r = 0$,

$$f(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\cdot\Bigg(\zeta(1-s) - \sum_{n=1}^{m} n^{s-1} \\ - \eta^{s-1}\mathrm{e}^{-\pi\mathrm{i}\eta^2}\Bigg(\frac{\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{8}}\sin(\pi\eta) + (-1)^{m-1}\mathrm{e}^{2\pi\mathrm{i}\eta-2\pi\mathrm{i}(\eta-m)^2}}{2\cos(2\pi\eta)} + \mathrm{O}(\eta^{-1})\Bigg)\Bigg) \tag{82}$$

with $\eta = \sqrt{\frac{s-1}{2\pi\mathrm{i}}}$, $|\mathrm{arc}(\eta)| < \frac{\pi}{4}$, $m = \lfloor \mathrm{Re}(\eta) + \mathrm{Im}(\eta) \rfloor$ in the quarter plane $\sigma < \sigma_0$, $t > 0$. Additional terms of the semiconvergent series can be obtained by the method from §2, but are not needed for our present purposes.

A second semiconvergent expansion of $f(s)$, for which the quarter plane $\sigma > \sigma_0$, $t > 0$, is convenient, can be obtained by applying the saddle point method to (58) rather than the representation of $f(s)$ obtained from (65). There is no need to repeat the calculation, for the integral in (58) is obtained from the one in (65) if we pass to complex conjugate quantities and replace σ by $1-\sigma$. Hence

$$f(s)=\sum_{n=1}^{m_1} n^{-s}+\eta_1^{-s}\mathrm{e}^{\pi\mathrm{i}\eta_1^2}\left(\frac{\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{8}}\sin(\pi\eta_1)+(-1)^{m_1-1}\mathrm{e}^{2\pi\mathrm{i}\eta_1-2\pi\mathrm{i}(\eta_1-m_1)^2}}{2\cos(2\pi\eta_1)}+\mathrm{O}(\eta_1^{-1})\right) \tag{83}$$

with $\eta_1=\sqrt{\frac{s}{2\pi\mathrm{i}}}$, $|\mathrm{arc}(\eta_1)|<\frac{\pi}{4}$, $m_1=\lfloor\mathrm{Re}(\eta_1)-\mathrm{Im}(\eta_1)\rfloor$ in the quarter plane $\sigma>\sigma_0$, $t>0$. Comparison of (82) and (83) yields the semiconvergent series for $\zeta(s)$ on the half strip $\sigma_1<\sigma<\sigma_2$, $t>0$. This derivation is possibly easier with respect to the necessary estimates than the one in §2, but the individual terms of the series appear at first in a more complicated form.

From (83) it follows that

$$\begin{cases} f(s)=\sum_{n=1}^{m_1} n^{-s}+\mathrm{O}\left(\left(\frac{|s|}{2\pi\mathrm{e}}\right)^{-\frac{\sigma}{2}}\right), & \sigma\geq 0, t>0,\\ f(s)=\mathrm{O}(t^{\frac{1}{4}}), & \sigma\geq\frac{1}{2},\\ |f(s)-1|<\frac{3}{4}, & \sigma\geq 2, t>t_0, \end{cases} \tag{84}$$

and from (82) it follows that

$$\begin{cases} f(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\left(\zeta(1-s)-\sum_{n=1}^{m} n^{s-1}+\mathrm{O}(1)\right), & \sigma\leq 1, t>0,\\ f(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\mathrm{O}\left(\left(\frac{t}{2\pi}\right)^{\frac{\sigma}{2}}|\sigma|^{-1}\right), & 0<-\sigma\leq t^{\frac{3}{7}}, t>0,\\ f(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\mathrm{O}(\log(t)), & 0\leq-\sigma\leq t^{\frac{3}{7}}, t>0 \end{cases} \tag{85}$$

For the following it is convenient to consider the function

$$g(s)=\pi^{-\frac{s+1}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}s}{4}}\Gamma\left(\frac{s+1}{2}\right)f(s)$$

instead of $f(s)$. By (64), for $t>0$, $-\sigma\geq t^{\frac{3}{7}}$ with $\eta=\sqrt{\frac{s-1}{2\pi\mathrm{i}}}$,

$$g(s)\sim\mathrm{e}^{-\frac{7\pi\mathrm{i}}{8}}\tan\left(\frac{\pi s}{2}\right)\frac{\sin(\pi\eta)}{\cos(2\pi\eta)},\quad |s|\to\infty. \tag{86}$$

Now we wish to estimate the mean value of $|g(s)|^2$ on the half line $\sigma=\sigma_0<\frac{1}{2}$, $t\geq 0$, that is, the expression

$$T^{-1}\int_0^T|g(\sigma+t\mathrm{i})|^2\mathrm{d}t,\quad \sigma<\frac{1}{2}.$$

This could be done using the asymptotic expansion (82), but the most elegant derivation uses (58), for according to this, for $\varepsilon > 0$,

$$\int_0^\infty |f(\sigma+t\mathrm{i})|^2 \mathrm{e}^{-\varepsilon t}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\varepsilon t}\left(\int_{0\swarrow 1}\int_{0\searrow 1}\frac{x^{-\sigma-t\mathrm{i}}y^{-\sigma+t\mathrm{i}}\mathrm{e}^{\pi\mathrm{i}(x^2-y^2)}}{(\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x})(\mathrm{e}^{\pi\mathrm{i}y}-\mathrm{e}^{-\pi\mathrm{i}y})}\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t,$$

and here the right hand side can be transformed by deformation of the path of integration, interchange of the order of integration and application of the residue theorem. This computation yields

$$\int_0^\infty |f(\sigma+t\mathrm{i})|^2 \mathrm{e}^{-\varepsilon t}\mathrm{d}t \sim \frac{1}{2\varepsilon}(2\pi\varepsilon)^{\sigma-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\sigma\right),$$

valid for $\sigma < \frac{1}{2}$ and $\varepsilon \to \infty$, and it further follows that

$$\int_1^\infty |f(\sigma+t\mathrm{i})|^2 \left(\frac{t}{2\pi}\right)^\sigma \mathrm{e}^{-\varepsilon t}\mathrm{d}t \sim \frac{(2\varepsilon)^{-\frac{3}{2}}}{1-2\sigma}.$$

Hence, for any fixed $\sigma < \frac{1}{2}$,

$$\int_1^T |f(\sigma+t\mathrm{i})|^2 \left(\frac{t}{2\pi}\right)^\sigma \mathrm{d}t \sim \frac{1}{3\sqrt{2\pi}}\frac{T^{\frac{3}{2}}}{\frac{1}{2}-\sigma}.$$

On the other hand, by Stirling's formula,

$$|g(s)| \sim \sqrt{2}\pi^{-\frac{\sigma}{2}}\left(\frac{t}{2}\right)^{\frac{\sigma}{2}}|f(s)|, \tag{87}$$

and thus the desired formula

$$T^{-1}\int_1^T |g(\sigma+t\mathrm{i})|^2\mathrm{d}t \sim \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{T^{\frac{1}{2}}}{\frac{1}{2}-\sigma}$$

is obtained for fixed $\sigma < \frac{1}{2}$. It follows further that

$$\int_0^T \log|g(\sigma+t\mathrm{i})|\mathrm{d}t < \frac{T}{2}\log\left(\frac{\sqrt{2}T^{\frac{1}{2}}}{3\sqrt{\pi}(\frac{1}{2}-\sigma)}\right) + \mathrm{o}(T), \qquad \sigma < \frac{1}{2}, T \to \infty. \tag{88}$$

For $\sigma = \frac{1}{2}$ we can state a lower bound for $\int_0^T \log|g(\sigma+t\mathrm{i})|\mathrm{d}t$. In fact, by (60), on the critical line we have

$$\left|\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\right| \le \left|2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)f(s)\right|,$$

that is, by (87),

$$|g(s)| \geq (8\pi)^{-\frac{1}{4}} t^{\frac{1}{4}} |\zeta(s)|(1 + o(1)), \qquad \sigma = \frac{1}{2},$$

$$\int_0^T \log \left| g\left(\frac{1}{2} + t\mathrm{i}\right) \right| \mathrm{d}t > \frac{T}{4} \log(T) - (\log(8\pi) + 1)\frac{T}{4} + \int_0^T \log \left| \zeta\left(\frac{1}{2} + t\mathrm{i}\right) \right| \mathrm{d}t + o(T). \tag{89}$$

Finally, for $\sigma \geq 2$, by (87) and (84),

$$\int_0^T \log |(g\sigma + t\mathrm{i})| \mathrm{d}t = \sigma \left(\frac{T}{2} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2} \right) + \frac{T}{2} \log(2) + o(T). \tag{90}$$

Now let $t_0 > 0$, $T > t_0$ and the lines $t = t_0$, $t = T$ are assumed to contain no zeros of $g(s)$. Moreover, let $\sigma_0 > -T^{\frac{3}{7}} = \sigma_1$. Consider the rectangle with the sides $\sigma = \sigma_0$, $t = T$, $\sigma = \sigma_1$, $t = t_0$. On the left side $\sigma = \sigma_1$, $t_0 \leq t \leq T$, there is no zero of $g(s)$ for sufficiently large T according to (64). The zeros contained in the inside of the rectangle are to be connected to the right side $\sigma = \sigma_0$ by sections that are parallel to the real axis. In the cut up rectangle $\log(g(s))$ is unique. A branch of this function is determined by the requirement $0 \leq \mathrm{arc}(g(\sigma_1 + T\mathrm{i})) < 2\pi$. As is well-known, then

$$2\pi \sum_{\alpha < \sigma_0} (\sigma_0 - \alpha) = \int_{t_0}^T \log |g(\sigma_0 + t\mathrm{i})| \mathrm{d}t - \int_{\sigma_1}^{\sigma_0} \mathrm{arc}(g(\sigma + T\mathrm{i})) \mathrm{d}\sigma - \int_{t_0}^T \log |g(\sigma_1 + t\mathrm{i})| \mathrm{d}t + \int_{\sigma_1}^{\sigma_0} \mathrm{arc}(g(\sigma + t_0\mathrm{i})) \mathrm{d}\sigma, \tag{91}$$

where α runs through the real parts of all zeros of $g(s)$ contained in the rectangle. The first integral can be estimated from above for $\sigma_0 < \frac{1}{2}$, from below for $\sigma_0 = \frac{1}{2}$, and given precisely for $\sigma_0 \geq 2$. The third and fourth integral contribute only a term of order $T^{\frac{13}{14}}$, as can be easily seen from (86). The second integral can be estimated as $O(T^{\frac{6}{7}} \log(T))$ in the usual way using (84) and (85). Therefore, by (88),

$$\sum_{\alpha < \sigma} (\sigma - \alpha) < \frac{T}{8\pi} \log(T) - \frac{T}{4\pi} \log\left(3\sqrt{\frac{\pi}{2}}\left(\frac{1}{2} - \sigma\right)\right) + o(T), \qquad \sigma < \frac{1}{2}, \tag{92}$$

by (89)

$$\sum_{\alpha < \frac{1}{2}} \left(\frac{1}{2} - \alpha\right) > \frac{T}{8\pi} \log(T) - (1 + \log(8\pi))\frac{T}{8\pi} + \frac{1}{2\pi} \int_0^T \log \left| \zeta\left(\frac{1}{2} + t\mathrm{i}\right) \right| \mathrm{d}t + o(T), \tag{93}$$

and by (90),

$$\sum_{\alpha}(\sigma-\alpha)=\sigma\left(\frac{T}{4\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{4\pi}\right)+\frac{T}{4\pi}\log(2)+\mathrm{o}(T),\qquad \sigma\geq 2. \quad (94)$$

In the last equation, α runs through the real parts of all zeros of $g(s)$ contained in the strip $0<t<T$. If their number is denoted by $N_1(T)$, then due to (94),

$$N_1(T)=\frac{T}{4\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{4\pi}+\mathrm{o}(T). \quad (95)$$

In the upper half plane the zeros of $g(s)$ coincide with those of $f(s)$. If all except $\mathrm{o}(T)$ many zeros of $N_1(T)$ were located to the right of $\sigma=\frac{1}{2}$, then the change of $\mathrm{arc}(f(\frac{1}{2}+t\mathrm{i}))$ in the interval $0<t<T$ would equal $-(\frac{T}{2}\log(\frac{T}{2\pi})-\frac{T}{2})+\mathrm{o}(T)$, and we would not obtain any statement on the zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$. However, firstly it follows from (94) that

$$\sum_{\alpha}\alpha=-\frac{T}{4\pi}\log(2)+\mathrm{o}(T),$$

there are certainly infinitely many zeros of $f(s)$ located to the left of $\sigma=0$. Moreover, (92) and (93) yield, independently of (94), by subtraction a lower bound for the number of zeros of $f(s)$ located in the domain $\sigma<\frac{1}{2}$, $0<t<T$. Namely, if $N_2(T)$ denotes this number, then it follows for every $\sigma<\frac{1}{2}$

$$\left(\frac{1}{2}-\sigma\right)N_2(T)>\frac{T}{4\pi}\log\left(\frac{3}{4}\mathrm{e}^{-\frac{1}{2}}\left(\frac{1}{2}-\sigma\right)\right)+\frac{1}{2\pi}\int_0^T\log\left|\zeta\left(\frac{1}{2}+t\mathrm{i}\right)\right|\mathrm{d}t+\mathrm{o}(T).$$

This estimate is the best for

$$\sigma=\frac{1}{2}-\frac{4}{3}\mathrm{e}^{\frac{3}{2}}$$

and yields

$$N_2(T)>\frac{3}{16\pi}\mathrm{e}^{-\frac{3}{2}}T+\frac{3}{8\pi}\mathrm{e}^{-\frac{3}{2}}\int_0^T\log\left|\zeta\left(\frac{1}{2}+t\mathrm{i}\right)\right|\mathrm{d}t+\mathrm{o}(T).$$

As is well-known, and follows from an ansatz (91) with $\zeta(s)$ instead of $g(s)$,

$$\frac{1}{2\pi}\int_0^T\log\left|\zeta\left(\frac{1}{2}+t\mathrm{i}\right)\right|\mathrm{d}t=\sum_{\alpha_\zeta>\frac{1}{2}}\left(\alpha_\zeta-\frac{1}{2}\right)+\mathrm{O}(\log(T)),$$

where α_ζ runs through the real parts of the zeros of the zeta function located in the strip $0 < t < T$ to the right of the critical line. This implies

$$N_2(T) > \frac{3}{16\pi}\mathrm{e}^{-\frac{3}{2}}T + \frac{3}{4}\mathrm{e}^{-\frac{3}{2}}\sum_{\alpha_\zeta > \frac{1}{2}}\left(\alpha_\zeta - \frac{1}{2}\right) + \mathrm{o}(T). \tag{96}$$

Within the strip $0 < t < T$, at most $N_1(T) - N_2(T)$ zeros of $f(s)$ are located to the right of $\sigma = \frac{1}{2}$, and therefore $\mathrm{arc}(f(\frac{1}{2}+t\mathrm{i}))$ decreases by at most $2\pi(N_1(T) - N_2(T)) + \mathrm{O}(\log(T))$ in the interval $0 < t < T$. As a consequence, $\mathrm{arc}(\varphi(\frac{1}{2}+t\mathrm{i}))$ increases by at most

$$\vartheta(T) - 2\pi N_1(T) + 2\pi N_2(T) + \mathrm{O}(\log(T))$$

in this interval, and this number is at least $2\pi N_2(T) + \mathrm{o}(T)$ by (63), (95), (96). Hence $N_0(T)$, the number of zeros of $\zeta(\frac{1}{2}+t\mathrm{i})$ in the interval $0 < t < T$, satisfies the inequality

$$N_0(T) > \frac{3}{8\pi}\mathrm{e}^{-\frac{3}{2}}T + \frac{3}{2}\mathrm{e}^{-\frac{3}{2}}\sum_{\alpha_\zeta > \frac{1}{2}}\left(\alpha_\zeta - \frac{1}{2}\right) + \mathrm{o}(T). \tag{97}$$

The density of the zeros of $\zeta(s)$ located on the critical line, that is, the lower bound of the ratio $N_0(T) : T$ for $T \to \infty$, is therefore positive, namely it is at least $\frac{3}{8\pi}\mathrm{e}^{-\frac{3}{2}}$, and thus greater than $\frac{1}{38}$. Aside from this numerical value, this insight is not at all new, but was already proved by Hardy and Littlewood in 1920 in a much easier way. Despite of this rather negligable result, the current proof may have some value in its own right due to the insight into the properties of $f(s)$.

A further remark on equation (97). In a way, the sum $\sum(\alpha_\zeta - \frac{1}{2})$ measures the "wrongness" of Riemann's conjecture. It is known from Littlewood's work that this sum is at most $\mathrm{O}(T\log(\log(T)))$, but we do not know a better estimate. If Riemann's conjecture is wrong, then the sum might grow faster than T. But then $N_0(T)$ would grow faster than T by (97), and Riemann's conjecture couldn't bee "all too wrong". If $\psi(t)$ is any positive function of t that diverges more slowly than $\log(t)$, then (97) implies further that in the narrow domain $0 \le \sigma - \frac{1}{2} \le \frac{\psi(t)}{\log(t)}$, $2 \le t \le T$, at least $\frac{3}{4\pi}\mathrm{e}^{-\frac{3}{2}}T\psi(T)(1+\mathrm{o}(1))$ zeros of $\zeta(s)$ are contained. This is a new result in the situation that $\psi(t)$ also grows more slowly than $\log(\log(t))$. For example, the domain $0 \le \sigma - \frac{1}{2} \le \frac{19}{\log(t)}$, $2 \le t \le T$, contains more than $T + \mathrm{o}(T)$ zeros.

The question remains whether the lower bound for $N_2(T)$ given in (96) can be improved. For the proof of Riemann's claim that $N_0(T)$ is asymptotically $\frac{T}{2\pi}\log(\frac{T}{2\pi})-$

$\frac{T}{2\pi}$, it is sufficient to show the corresponding statement for $N_2(T)$. It seems this can hardly be achieved by the analytical methods employed so far in the theory of the zeta function without any significant new idea. In particular, this is so for any attempt to prove Riemann's conjecture.

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