Proof of a theorem on discrete groups

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It is shown that every matrix group contains a unique maximal solvable normal subgroup (radical). As a generalisation of a theorem of Bieberbach and Frobenius, it is proved that in a discrete group, all matrices whose components in the representations on irreducible subspaces are arbitrarily close to the identity matrix of the same degree belong to the radical of the group.

§ 1 The radical of a group

Theorem 1 *The product of a solvable normal subgroup $\mathfrak{N}$ with a solvable subgroup $\mathfrak{U}$ is a solvable subgroup.*

H. Fitting[1)] showed that the product of two solvable normal subgroups is again solvable. Theorem 1 is proved by almost identical reasoning.

$\mathfrak{U}\mathfrak{N}$ is a subgroup with solvable normal subgroup $\mathfrak{N}$. The quotient group $\mathfrak{U}\mathfrak{N}/\mathfrak{N}$ is isomorphic to $\mathfrak{U}/(\mathfrak{U} \cap \mathfrak{N})$, hence solvable. $\mathfrak{U}\mathfrak{N}$ is a solvable extension of a solvable group, hence solvable itself.

Definition A solvable subgroup of an arbitrary group is called *maximal solvable normal subgroup*, if any larger normal subgroup is not solvable.

Now it follows from Fitting (already from the lemma proved by him): There is at most one maximal solvable normal subgroup. If it exists, then Fitting calls it the *radical* of the group, and it contains every solvable normal subgroup of the group.

Definition A solvable subgroup of an arbitrary group is called *maximal solvable subgroup* if any larger subgroup is not solvable.

[1)] Fitting, *Beiträge zur Theorie der Gruppen endlicher Ordnung*, Jahresbericht der Deutschen Mathematiker-Vereinigung 48, 1938, 77-141.

It follows from Theorem 1 that every solvable normal subgroup is contained in every maximal solvable subgroup. As the intersection of all solvable subgroups is a solvable normal subgroup, it follows:

Theorem 2 *If a group contains maximal solvable subgroups, then the intersection of all maximal solvable subgroups equals the radical of the group.*

Theorem 3 *If a group $\mathfrak{G}$ and one of its normal subgroups $\mathfrak{N}$ both have radicals $\mathrm{R}\mathfrak{G}$ and $\mathrm{R}\mathfrak{N}$, then*

$$\mathrm{R}\mathfrak{N} = \mathfrak{N} \cap \mathrm{R}\mathfrak{G}.$$

For $\mathfrak{N} \cap \mathrm{R}\mathfrak{G}$ is a solvable normal subgroup of $\mathfrak{N}$, hence contained in $\mathrm{R}\mathfrak{N}$. On the other hand, $\mathrm{R}\mathfrak{N}$ is a solvable characteristic subgroup of $\mathfrak{N}$, hence a solvable normal subgroup of $\mathfrak{G}$ and thus contained in $\mathrm{R}\mathfrak{G}$.

Now we try to embed solvable subgroups of an arbitrary group into maximal solvable subgroups.

Definition A group is called *quasi-solvable* if finitely many of its elements always generate a solvable subgroup.

Every solvable subgroup is quasi-solvable. Every subgroup and every quotient group of a quasi-solvable group are again quasi-solvable.

Definition A subgroup of an arbitrary group is called *maximal quasi-solvable subgroup* if it is quasi-solvable and every larger subgroup is not quasi-solvable.

Theorem 4 *If $\mathfrak{M} = (\dots, \mathfrak{U}, \dots, \mathfrak{B}, \dots)$ is an ascending ordered set of quasi-solvable subgroups, such that $\mathfrak{U} \prec \mathfrak{B}$ implies $\mathfrak{U} \subseteq \mathfrak{B}$, then the union $V(\mathfrak{M})$ of all subgroups from $\mathfrak{M}$ is a quasi-solvable subgroup.*

PROOF: It is clear that $V(\mathfrak{M})$ is a subgroup. Finitely many elements $a_1, \dots, a_r \in V(\mathfrak{M})$ can always be ordered in such a way that a_i is contained in the subgroup $\mathfrak{U}_i$ in $\mathfrak{M}$, and that $\mathfrak{U}_i \subseteq \mathfrak{U}_k$ if $i < k$. But then it follows that $\mathfrak{U}_r$ contains all $\mathfrak{U}_i$, hence all a_i. By assumption, $a_1, \dots, a_r$ generate a solvable subgroup of $\mathfrak{U}_r$, hence of $V(\mathfrak{M})$. Therefore, $V(\mathfrak{M})$ is quasi-solvable. ◇

Theorem 5 *Every quasi-solvable subgroup $\mathfrak{U}$ of an arbitrary group $\mathfrak{G}$ can be embedded into a maximal quasi-solvable subgroup.*

PROOF: Assume the elements of $\mathfrak{G}$ to be well-ordered: $e \prec a_1 \prec a_2 \prec \dots$. For every element $a \in \mathfrak{G}$, define a subgroup $\mathfrak{U}_a$ by transfinite induction. Let $\mathfrak{U}_e = \mathfrak{U}$.

Suppose $\mathfrak{U}_b$ is defined for all $b \prec a$. Let Σ_a be the subgroup generated by all $\mathfrak{U}_b$ for $b \prec a$. If Σ_a together with a generates a quasi-solvable subgroup of $\mathfrak{G}$, then set $\mathfrak{U}_a = \langle \Sigma_a, a \rangle$. Otherwise, set $\mathfrak{U}_a = \Sigma_a$. It is clear that $\mathfrak{U}_b \leq \mathfrak{U}_a$ if $b \preceq a$. Moreover, it follows from Theorem 4 by transfinite induction that $\mathfrak{U}_a$ is quasi-solvable. It also follows from Theorem 4 that the union $\mathfrak{V}$ of all $\mathfrak{U}_a$ is a quasi-solvable subgroup. $\mathfrak{U}$ is contained in $\mathfrak{V}$, and if some element $b \in \mathfrak{G}$ together with $\mathfrak{V}$ generates a quasi-solvable subgroup, then $\langle \Sigma_b, b \rangle$ is quasi-solvable. Therefore $\langle \Sigma_b, b \rangle$ and hence b is contained in $\mathfrak{V}$. So $\mathfrak{V}$ is the required maximal quasi-solvable subgroup. $\diamond$

For the theorems below it is helpful to deduce some simple criteria for solvability.

For the commutator of two elements $a_1, a_2 \in \mathfrak{G}$ we use the following notations

$$a_1 a_2 a_1^{-1} a_2^{-1} = (a_1, a_2) = D^1(a_1, a_2),$$

and we define higher commutators D^i of weight 2^i inductively,

$$D^i(a_1, \ldots, a_{2^i}) = D^1(D^{i-1}(a_1, \ldots, a_{2^{i-1}}), D^{i-1}(a_{2^{i-1}+1}, \ldots, a_{2^i})).$$

According to Hall[2)], it follows that:

Criterion 1 *A group $\mathfrak{G}$ is solvable and at most k-step metabelian if the higher commutator D^k applied to elements of $\mathfrak{G}$ is identically e.*

This implies:

Criterion 2 *A group $\mathfrak{G}$ with generating set $\mathfrak{K}$ is solvable and at most k-step metabelian if finitely many elements of $\mathfrak{K}$ always generate a solvable and at most k-step metabelian subgroup.*

For if $a_1, a_2, \ldots, a_{2^k}$ are 2^k arbitrary elements of $\mathfrak{G}$, every a_i can be written as a product of powers of elements in $\mathfrak{K}$, and in total only finitely many elements $b_1, \ldots, b_r \in \mathfrak{K}$ are used. By assumption, $b_1, \ldots, b_r$ generate a solvable and at most k-step metabelian subgroup. Hence

$$D^k(a_1, \ldots, a_{2^k}) = e.$$

It is interesting and important for later applications that Criteria 1 and 2 can be combined into one, if the condition "solvable" is replaced by "nilpotent".

[2)]See Zassenhaus, *Lehrbuch der Gruppentheorie, Teil I*, Hamburger Mathematische Einzelschriften 21, II §6, Satz 13.

Translator's note: This book is available in English as *The theory of groups*, Dover 1999.

Criterion 3 *A group $\mathfrak{G}$ with generating set $\mathfrak{K}$ is nilpotent of class $\leq c$ if and only if all higher commutators $(a_1, \ldots, a_{c+1})$ of weight $c+1$ and of step $c+1$ of elements of $\mathfrak{K}$ are identically e.*

PROOF: It is clear that the condition is necessary. Conversely, if $\mathfrak{G}$ is generated by the complex $\mathfrak{K}$, then let $\mathfrak{K}_i$ denote the complex comprising the higher commutators $(a_1, \ldots, a_i)$ of elements of $\mathfrak{K}$. Clearly, the normal subgroup $\overline{\mathfrak{Z}}_i$ of $\mathfrak{G}$ generated by $\mathfrak{K}_i$ is contained in the ith component $\mathfrak{Z}_i$ of the descending central series of $\mathfrak{G}$. By assumption on $\mathfrak{K}$, $\overline{\mathfrak{Z}}_1 = \mathfrak{G} = \mathfrak{Z}_1$. Suppose we proved that $\overline{\mathfrak{Z}}_n = \mathfrak{Z}_n$, then $(\mathfrak{K}, \mathfrak{K}_n) = \mathfrak{K}_{n+1} \leq \overline{\mathfrak{Z}}_{n+1}$. Since $\overline{\mathfrak{Z}}_{n+1}$ is a normal subgroup, it follows[3)] that

$$(\langle\mathfrak{K}\rangle, \langle\mathfrak{K}_n\rangle) = (\mathfrak{G}, \langle\mathfrak{K}_n\rangle) \leq \overline{\mathfrak{Z}}_{n+1}.$$

Moreover, $(\mathfrak{G}, \overline{\mathfrak{Z}}_{n+1}) \leq \overline{\mathfrak{Z}}_{n+1}$, hence $(\mathfrak{G}, \langle\mathfrak{K}_n\rangle\overline{\mathfrak{Z}}_{n+1}) \leq \overline{\mathfrak{Z}}_{n+1}$. It follows that the subgroup $\langle\mathfrak{K}_n\rangle\overline{\mathfrak{Z}}_{n+1}$ is even a normal subgroup of $\mathfrak{G}$. As the normal subgroup generated by $\mathfrak{K}_n$ equals $\mathfrak{Z}_n$, $\mathfrak{Z}_n$ is contained in $\langle\mathfrak{K}_n\rangle\mathfrak{Z}_{n+1}$. Finally, it follows that

$$\mathfrak{Z}_{n+1} = (\mathfrak{G}, \mathfrak{Z}_n) \leq (\mathfrak{G}, \langle\mathfrak{K}_n\rangle\overline{\mathfrak{Z}}_{n+1}) \leq \overline{\mathfrak{Z}}_{n+1} \leq \mathfrak{Z}_{n+1}, \quad \overline{\mathfrak{Z}}_{n+1} = \mathfrak{Z}_{n+1}.$$

So if $\mathfrak{K}_{c+1} = \{e\}$, then $\mathfrak{Z}_{c+1} = \{e\}$. ◇

Similar to before, we define: A group is called *quasi-nilpotent* if finitely many elements of the group always generate a nilpotent subgroup. The properties derived above for the term "quasi-solvable" also hold for the term "quasi-nilpotent".

As any nilpotent group is solvable, any quasi-nilpotent group is quasi-solvable.

Criterion 4 *A group with generating set $\mathfrak{K}$ is quasi-nilpotent if finitely many elements of $\mathfrak{K}$ always generate a nilpotent subgroup.*

PROOF: If $a_1, \ldots, a_r$ are finitely many elements of the group, then it requires only finitely many elements $b_1, \ldots, b_s \in \mathfrak{K}$ to represent them as powers and products of elements of $\mathfrak{K}$. By assumption, the elements b_i generate a nilpotent subgroup which contains the subgroup generated by the a_i. Hence the elements $a_1, \ldots, a_r$ generate a nilpotent subgroup. ◇

[3)]See the notes in [2)], p. 104; for the proofs and definitions needed for Criterion 3, see II §6 and IV §5.

§ 2 Solvable groups of substitutions

In the following, a matrix is called *n-row* if it is quadratic with n rows and n columns. A *matrix group* or *group of matrices of degree* n is any multiplicative group consisting of non-singular n-row matrices with coefficients in a fixed algebraically closed field. The identity matrix is denoted by E or $E_1, E_2, \ldots$.

Definition For any group $\mathfrak{G}$ of matrices of degree n, the *linear hull* $\mathrm{H}\mathfrak{G}$ is the set of linear combinations

$$\lambda_1 A_1 + \ldots + \lambda_r A_r$$

of finitely many matrices $A_1, \ldots, A_r \in \mathfrak{G}$ with coefficients in the coefficient field.

One readily sees that the linear hull of a group is a hypercomplex system over the coefficient field containing that group, and that every reduction of the group to a semireduced form also effects a reduction of the linear hull to a semireduced form. For later, it is important to remark that a semireduced form of the full group induces a semireduced form for every subgroup whose linear hull coincides with the linear hull of the full group, that is, every irreducible subrepresentation of the full group induces an irreducible subrepresentation of the subgroup.

Theorem 6 *A solvable absolutely irreducible and primitive*[4] *group* $\mathfrak{G}$ *of* n*-row matrices with determinant* 1 *is finite and its order is bounded by a number* $M(n)$ *depending only on* n.

Proof: Every normal subgroup $\mathfrak{N}$ of $\mathfrak{G}$ is fully reducible according to Clifford[5]. As $\mathfrak{G}$ is primitive, the irreducible components of $\mathfrak{N}$ are all equivalent according to Clifford. If $\mathfrak{N}$ is abelian, then the irreducible components of $\mathfrak{G}$ are of degree 1. From what we showed above it follows that an abelian normal subgroup consist of non-zero scalar matrices only. Hence every abelian normal subgroup is contained in the centre $\mathfrak{Z}$ of $\mathfrak{G}$. The centre $\mathfrak{Z}$ consists of scalar matrices of degree n whose determinant is 1. Hence $\mathfrak{Z}$ is cyclic and the order of $\mathfrak{Z}$ divides n. Now let $\mathfrak{N}/\mathfrak{Z}$ be the maximal abelian normal subgroup in the quotient group $\mathfrak{G}/\mathfrak{Z}$.[6] $\mathfrak{G}$ induces a fully reducible representation Δ of $\mathfrak{N}$, which is a multiple of an irreducible representation Γ of $\mathfrak{N}$. The degree m of Γ is thus a divisor of n. By a theorem of Burnside there exist m^2 linearly independent matrices

$$\Gamma(x_1), \ldots, \Gamma(x_{m^2}), \quad x_i \in \mathfrak{N}.$$

[4] *Translator's note:* For the definition of primitive and imprimitive, see [2] II §2.
[5] Clifford, *Representations induced in an invariant subgroup*, Annals of Mathematics 23, 1937.
[6] For the existence, see [2] p. 108.

When transformed by elements in $\mathfrak{N}$, every x_i is changed only by a factor in $\mathfrak{Z}$. Hence the subgroup $\mathfrak{N}_1$ generated by the x_i and $\mathfrak{Z}$ is a normal subgroup of $\mathfrak{N}$, and transformations by elements of $\mathfrak{N}$ yield at most n^{m^2} distinct automorphisms of $\mathfrak{N}_1$. By construction, Γ induces an irreducible representation of $\mathfrak{N}_1$. Hence every element in $\mathfrak{N}$ commuting with the elements of $\mathfrak{N}_1$ is mapped to a non-zero scalar matrix under Γ, so it is contained in $\mathfrak{Z}$.

$\mathfrak{N}$ is finite, and the order of $\mathfrak{N}$ is at most n^{m^2+1}. The elements in $\mathfrak{G}$ commuting with the normal subgroup $\mathfrak{N}$ form a normal subgroup $\mathrm{Z}_{\mathfrak{N}}$ in $\mathfrak{G}$. Since $\mathfrak{G}$ is solvable, $\mathrm{Z}_{\mathfrak{N}}$ is solvable as well. Now suppose that $\mathrm{Z}_{\mathfrak{N}}$ is k-step metabelian. If $k > 1$, then $D^{k-2}\mathfrak{N}$ would be a non-abelian normal subgroup of $\mathfrak{G}$. $D(D^{k-2}\mathrm{Z}_{\mathfrak{N}}) = D^{k-1}\mathrm{Z}_{\mathfrak{N}}$ is an abelian normal subgroup of $\mathfrak{G}$, hence contained in $\mathfrak{Z}$. The centre of $\mathfrak{N}$ is an abelian normal subgroup of $\mathfrak{G}$, hence also contained in $\mathfrak{Z}$. Thus $D^{k-2}\mathrm{Z}_{\mathfrak{N}}$ is no longer contained in $\mathfrak{N}$. But by construction $\mathfrak{N}D^{k-2}\mathrm{Z}_{\mathfrak{N}}/\mathfrak{Z}$ is an abelian normal subgroup of $\mathfrak{G}/\mathfrak{Z}$ which properly contains $\mathfrak{N}/\mathfrak{Z}$, and this contradicts the construction of $\mathfrak{N}$.

$\mathrm{Z}_{\mathfrak{N}}$ is an abelian normal subgroup of $\mathfrak{G}$, hence contained in $\mathfrak{Z}$. The quotient group $\mathfrak{G}/\mathfrak{Z}$ is isomorphic to a group of automorphisms of the finite group $\mathfrak{N}$. As there are at most $n^{m^2+1}!$ distinct automorphisms of $\mathfrak{N}$, it follows that $\mathfrak{G}$ is finite and the order of $\mathfrak{G}$ is at most $M(n) = n^{m^2+1}!n$. $\diamond$

Theorem 7 *Every solvable group of matrices of degree n is at most $k(n)$-step metabelian, where $k(n)$ is a positive number only depending on n, and vice versa.*

PROOF: Every matrix group of degree 1 is abelian, so the theorem holds for $n = 1$ with $k(1) = 1$. Proceed by induction on n. Let $n > 1$ and suppose the theorem holds for matrix groups of degree $< n$. Now let $\mathfrak{G}$ be a solvable group of matrices of degree n.

1. Suppose $\mathfrak{G}$ is reducible, so that all matrices in $\mathfrak{G}$ have the form

$$A = \begin{pmatrix} A_1 & \Lambda \\ 0 & A_2 \end{pmatrix},$$

and $A \mapsto A_i$ maps the group homomorphically onto a matrix group $\mathfrak{G}_i$ of degree n_i. For both these homomorphisms let the normal subgroup $\mathfrak{N}_i$ of $\mathfrak{G}$ be mapped to the n_i-row identity matrix. Then it follows from the induction hypothesis: $D^{k(n_i)}\mathfrak{G}_i = \{E_{n_i}\}$ and $D^{k(n_i)}\mathfrak{G} \leq \mathfrak{N}_i$, $i = 1, 2$. The normal subgroup $\mathfrak{N} = \mathfrak{N}_1 \cap \mathfrak{N}_2$ consist of matrices of the form $\begin{pmatrix} E_{n_1} & \Lambda \\ 0 & E_{n_2} \end{pmatrix}$, hence is abelian. It now follows that

$$D^{\max\{k(n_1),k(n_2)\}+1}\mathfrak{G} = \{E\}.$$

Let $\overline{k}(n) = 1 + \max_{1\le n_i<n}\{k(n_i)\}$, then we found that every solvable reducible matrix group of degree n is at most $\overline{k}(n)$-step metabelian.

2. Suppose the matrix group $\mathfrak{G}$ is imprimitive. Then there exists a family of $m > 1$ systems of imprimitivity, so that we may consider $\mathfrak{G}$ to be a group of permutations of these m systems. All elements of $\mathfrak{G}$ that fix every system in this family form a normal subgroup $\mathfrak{N}$ whose quotient group $\mathfrak{G}/\mathfrak{N}$ is isomorphic to a solvable subgroup of the symmetric permutation group on m numbers. It is clear that for every number m there exists an upper bound $\underline{k}(m)$ such that every solvable permutation group on m numbers is at most $\underline{k}(m)$-step metabelian.

   The representation of $\mathfrak{N}$ induced by $\mathfrak{G}$ is fully reducible and decomposes into m irreducible under $\mathfrak{G}$-conjugate components, each of degree $\frac{n}{m}$.[6] Hence $D^{k(\frac{n}{m})}\mathfrak{N} = \{E\}$. Moreover, $D^{\underline{k}(m)}\mathfrak{G}$ is contained in $\mathfrak{N}$ and therefore $D^{\underline{k}(m)+k(\frac{n}{m})}\mathfrak{G} = \{E\}$. We now set

$$\overline{\overline{k}}(n) = \max_{1<m<n}\left\{\underline{k}(m) + k(\tfrac{n}{m})\right\}.$$

   It follows that every irreducible imprimitive solvable group of matrices of degree n is at most $\overline{\overline{k}}(n)$-step metabelian.

3. Suppose $\mathfrak{G}$ is irreducible and primitive and the determinant of every matrix in $\mathfrak{G}$ is 1. By Theorem 6, $\mathfrak{G}$ is finite and the order of $\mathfrak{G}$ is smaller than a certain number $M(n)$. Then there exists an upper bound $k'(n)$ so that $\mathfrak{G}$ is at most $k'(n)$-step metabelian.

4. Suppose $\mathfrak{G}$ is an arbitrary solvable group of matrices of degree n. The matrices in the commutator subgroup of $\mathfrak{G}$ have determinant 1, so one of the three cases above applies to $D\mathfrak{G}$. It follows that if we set

$$k(n) = 1 + \max\{\overline{k}(n), \overline{\overline{k}}(n), k'(n)\},$$

   then every solvable group of matrices of degree n is at most $k(n)$-step metabelian. $\diamond$

[6]See [5], Theorem 1, 2.

This theorem, together with Criterion 2 implies: *Every quasi-solvable group of matrices of degree n is solvable and at most $k(n)$-step metabelian.* From this and Theorem 5 we conclude:

Theorem 8 *Every solvable subgroup of a group of matrices of degree n can be embedded into a maximal solvable subgroup.*

Corollary *Every group of matrices of degree n has a radical.*

We note that the solvability of quasi-solvable matrix groups implies:

Corollary *Every quasi-nilpotent group of matrices of degree n is solvable.*

§ 3 Inequalities

Let $\mathbf{k}$ be a valued field, that is, there is function φ on $\mathbf{k}$ with values in the non-negative reals, such that

1. $\varphi(0) = 0$ and $\varphi(1) = 1$,
2. $\varphi(x + y) \leq \varphi(x) + \varphi(y)$,
3. $\varphi(xy) = \varphi(x)\varphi(y)$.

For a vector $\mathfrak{a} = (a_1, \ldots, a_n)$, define

$$\varphi(\mathfrak{a}) = \left| \sqrt{\sum_{i=1}^{n} \varphi(a_i)^2} \right| . \tag{1}$$

Set $\underline{\mathfrak{a}} = (\varphi(a_1), \ldots, \varphi(a_n))$, so that

$$\varphi(\mathfrak{a}) = |\underline{\mathfrak{a}}|. \tag{2}$$

Let $\mathfrak{b}$ be a second vector with n components. Since $\varphi(a_i + b_i) \leq \varphi(a_i) + \varphi(b_i)$, also $|\underline{\mathfrak{a} + \mathfrak{b}}| \leq |\underline{\mathfrak{a}}| + |\underline{\mathfrak{b}}|$, and

$$\varphi(\mathfrak{a} + \mathfrak{b}) \leq \varphi(\mathfrak{a}) + \varphi(\mathfrak{b}), \tag{3}$$

moreover,

$$\varphi(\mathfrak{a}\mathfrak{b}) = \varphi\left(\sum_{i=1}^{n} a_i b_i\right) \leq \sum_{i=1}^{n} \varphi(a_i)\varphi(b_i) = \underline{\mathfrak{a}}\underline{\mathfrak{b}} \leq |\underline{\mathfrak{a}}||\underline{\mathfrak{b}}|,$$

so

$$\varphi(\mathfrak{a}\mathfrak{b}) \leq \varphi(\mathfrak{a})\varphi(\mathfrak{b}). \tag{4}$$

For matrices $A = (a_{ik})$, where $i = 1, \dots, s$, $k = 1, \dots, r$, we define:

$$\varphi(A) = \left| \sqrt{\sum_{i=1}^{s} \sum_{k=1}^{r} \varphi(a_{ik})^2} \right|. \tag{5}$$

It immediately follows that

$$\varphi(\alpha A) = \varphi(\alpha)\varphi(A) \quad \text{for all } \alpha \in \mathbf{k}. \tag{6}$$

$$\text{If } \varphi(A) = 0, \text{ then } A = 0. \tag{6a}$$

If we consider two matrices A, B as vectors with rs components, then

$$\varphi(A + B) \leq \varphi(A) + \varphi(B). \tag{7}$$

If A is a matrix with r rows and s columns, B is a matrix with s rows and t columns, $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are the row vectors of A, and $\mathfrak{b}_1, \dots, \mathfrak{b}_t$ are the column vectors of B, then

$$\begin{aligned}
\varphi(AB)^2 &= \sum_{i=1}^{r} \sum_{k=1}^{t} \varphi(\mathfrak{a}_i \mathfrak{b}_k)^2 \\
&\leq \sum_{i=1}^{r} \sum_{k=1}^{t} \varphi(\mathfrak{a}_i)^2 \varphi(\mathfrak{b}_k)^2 = \left(\sum_{i=1}^{r} \varphi(\mathfrak{a}_i)^2 \right) \left(\sum_{k=1}^{t} \varphi(\mathfrak{b}_k)^2 \right) = \varphi(A)^2 \varphi(B)^2.
\end{aligned}$$

Hence

$$\varphi(AB) \leq \varphi(A)\varphi(B). \tag{8}$$

Now, let A be an n-row matrix with the property $\varphi(A - E) \leq \kappa$. For any n-row matrix X it follows that

$$\begin{gathered}
\varphi(XA) = \varphi(X + X(A - E)) \leq \varphi(X)(1 + \varphi(A - E)), \\
\varphi(XA) \leq \varphi(X)(1 + \kappa),
\end{gathered}$$

and similarly

$$\varphi(AX) \leq \varphi(X)(1 + \kappa). \tag{9}$$

If in addition $\kappa < 1$, then A is not a left zero divisor, for $AX = 0$ implies:

$$0 \leq \varphi(X) = \varphi((E - A)X) \leq \kappa\varphi(X),$$

hence $\varphi(X) = 0$, so $X = 0$. As A is not a left zero divisor, it follows that A has an inverse matrix.

Since

$$\begin{aligned} A^{-1} - E &= -(A - E) - (A - E)(A^{-1} - E), \\ A - E &= -(A^{-1} - E) - (A - E)(A^{-1} - E), \end{aligned}$$

it follows that

$$\begin{aligned} \varphi(A^{-1} - E) &\leq \varphi(A - E) + \varphi(A - E)\varphi(A^{-1} - E), \\ \varphi(A - E) &\leq \varphi(A^{-1} - E) + \varphi(A - E)\varphi(A^{-1} - E), \end{aligned}$$

and

$$\frac{1}{1+\kappa} \leq \frac{\varphi(A - E)}{1 + \varphi(A - E)} \leq \varphi(A^{-1} - E) \leq \frac{\varphi(A - E)}{1 - \varphi(A - E)} \leq \frac{\kappa}{1 - \kappa}. \tag{10}$$

From (9) and (10), it follows for any n-row matrix X that

$$\begin{aligned} \varphi(XA^{-1}) &\leq \varphi(X)\frac{1}{1-\kappa}, \\ \varphi(A^{-1}X) &\leq \varphi(X)\frac{1}{1-\kappa}. \end{aligned} \tag{9a}$$

Let A, B be quadratic and non-singular matrices with

$$\begin{aligned} A &= E + A_1, \quad \varphi(A_1) \leq \kappa < 1, \\ B &= E + B_1, \quad \varphi(B_1) \leq \lambda < 1. \end{aligned}$$

Then

$$AB - BA = A_1B_1 - B_1A_1$$

and

$$\varphi(A_1B_1 - B_1A_1) \leq 2\varphi(A_1)\varphi(B_1) \leq 2\kappa\lambda.$$

Moreover,

$$ABA^{-1}B^{-1} - E = (AB - BA)A^{-1}B^{-1} = (A_1B_1 - B_1A_1)A^{-1}B^{-1},$$

so by (9a):

$$\varphi(ABA^{-1}B^{-1} - E) \leq \frac{2\kappa\lambda}{(1-\kappa)(1-\lambda)}. \tag{11}$$

If A is arbitrary and B non-singular, then

$$\varphi(BAB^{-1}-E)=\varphi(B(A-E)B^{-1})\le\varphi(B)\varphi(B^{-1})\varphi(A-E). \tag{12}$$

If we replace B by B^{-1} and A by $B^{-1}AB$, then:

$$\varphi(BAB^{-1}-E)\ge\frac{\varphi(A-E)}{\varphi(B)\varphi(B^{-1})}. \tag{12a}$$

Now let $A=\left(\begin{smallmatrix}A_1 & P\\ 0 & A_2\end{smallmatrix}\right)$ and $B=\left(\begin{smallmatrix}B_1 & Q\\ 0 & B_2\end{smallmatrix}\right)$ be partially reduced matrices, such that the submatrices A_1, B_1 have n_1 rows, the submatrices A_2, B_2 have n_2 rows, and P, Q are rectangular and moreover

$$\begin{aligned}\varphi(A_i-E_{n_i})&\le\kappa<1,\\ \varphi(B_i-E_{n_i})&\le\lambda<1.\end{aligned}$$

Set $C=ABA^{-1}B^{-1}=\left(\begin{smallmatrix}C_1 & R\\ 0 & C_2\end{smallmatrix}\right)$. We want to estimate R. It is easy to check that

$$\begin{aligned}R&=A_1QA_2^{-1}B_2^{-1}-A_1B_1A_1^{-1}B_1^{-1}QB_2^{-1}+PB_2A_2^{-1}B_2^{-1}-A_1B_1A_1^{-1}PA_2^{-1}B_2^{-1}\\ &=A_1(Q(A_2^{-1}-E_{n_2})-B_1(A_1^{-1}-E_{n_1})B_1^{-1}Q)B_2^{-1}\\ &\quad+(P(B_2-E_{n_2})-A_1(B_1-E_{n_1})A_1^{-1}P)A_2^{-1}B_2^{-1}.\end{aligned}$$

So by (7), (8), (9), (10):

$$\varphi(R)\le(1+\kappa)\varphi(Q)\left(\frac{\kappa}{1-\kappa}+\frac{1+\lambda}{1-\lambda}\frac{\kappa}{1-\kappa}\right)\frac{1}{1-\lambda}+\varphi(P)\left(\lambda+\frac{1+\kappa}{1-\kappa}\lambda\right)\frac{1}{1-\kappa}\frac{1}{1-\lambda},$$

and

$$\varphi(R)\le\frac{2\lambda}{1-\lambda}\frac{1}{(1-\kappa)^2}\varphi(P)+2\kappa\frac{1+\kappa}{1-\kappa}\frac{1}{(1-\lambda)^2}\varphi(Q). \tag{13}$$

§ 4 Proof of the main theorem

Let $\mathfrak{G}$ be a group of partially reduced matrices of degree n of the form

$$A=\begin{pmatrix}A^{(1)} & & & *\\ 0 & A^{(2)} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & A^{(r)}\end{pmatrix},$$

such that the map $A\mapsto A^{(j)}$ is a representation of $\mathfrak{G}$ of degree n_j.

To every complex $\mathfrak{K}$ in $\mathfrak{G}$ we define $\mathfrak{K}_i$ to be the complex of all higher commutators $(A_1, \dots, A_i)$ of weight i and of step 1 with all components from $\mathfrak{K}$. Note that $\mathfrak{K} = \mathfrak{K}_1$. Moreover, let κ_i be the upper bound for all numbers $\varphi(K_i - E)$, where K_i runs through the matrices of $\mathfrak{K}_i$.

Theorem 9 *If $\mathfrak{K}$ contains only finitely many elements, and if the the maximum of all values $\varphi(K^{(\nu)} - E_{n_\nu})$, where $K^{(\nu)}$ runs through the r components of the matrices in $\mathfrak{K}$, is less than $2 - \sqrt{3}$, then the κ_i converge to 0.*[7)]

PROOF: Let $\overline{\kappa}_i$ be the upper bound of all numbers $\varphi(K_i^{(\nu)} - E_{n_\nu})$ whose argument runs through the r components of the matrices in $\mathfrak{K}_i$, minus the identity matrix of the same degree. We have $\overline{\kappa}_1 = \kappa$.

Set $\lambda = \frac{2\kappa}{(1-\kappa)^2}$. Then $\lambda < 1$ since $\kappa < 2 - \sqrt{3}$. Assume it has already been proved that

$$\overline{\kappa}_i \leq \kappa\lambda^{i-1} \leq \kappa. \tag{14}$$

The elements in $\mathfrak{K}_{i+1}$ arise by taking commutators of elements in $\mathfrak{K}$ with elements in $\mathfrak{K}_i$. So it follows from (11), applied to all components of $\mathfrak{G}$, that

$$\overline{\kappa}_{i+1} \leq \frac{2\kappa\overline{\kappa}_i}{(1-\kappa)(1-\kappa_i)} \leq \frac{2\kappa}{(1-\kappa)^2}\overline{\kappa}_i \leq \kappa\lambda^i.$$

So (14) holds for all i, and our claim is proved in the case $r = 1$, for

$$\lim_{i\to\infty} \kappa_i = \lim_{i\to\infty} \overline{\kappa}_i \leq \lim_{i\to\infty} \kappa\lambda^{i-1} = 0.$$

It is clear that we can reduce the proof of the general case via induction to the case $r = 2$. So assume the number of components of $\mathfrak{G}$ is $r = 2$, so that the matrices in $\mathfrak{G}$ are of the form

$$A = \begin{pmatrix} A^{(1)} & A' \\ 0 & A^{(2)} \end{pmatrix}.$$

Let κ_i' be the maximum of all values $\varphi(K_i')$, where the argument runs through the upper right submatrices in K_i. Now (13) implies the following inequality:

$$\kappa_{i+1}' \leq \frac{2\overline{\kappa}_i}{1-\overline{\kappa}_i}\kappa_1' \frac{1}{(1-\kappa)^2} + 2\kappa\frac{1+\kappa}{1-\kappa}\kappa_i' \frac{1}{(1-\overline{\kappa}_i)^2}.$$

[7)]Compare Frobenius, *Über den von L. Bieberbach gefundenen Beweis eines Satzes von C. Jordan*, Berliner Berichte 1911, p. 241-248.

Then

$$2\kappa\frac{1+\kappa}{1-\kappa} \leq 2(2-\sqrt{3})\frac{3-\sqrt{3}}{\sqrt{3}-1} \leq \left(\frac{27}{28}\right)^2.$$

As $\lim_{i\to\infty}\overline{\kappa}_i = 0$, there exists N such that for all $i \geq N$:

$$1-\overline{\kappa}_i \geq \frac{55}{56}$$

and

$$\kappa'_{i+1} \leq \frac{2\overline{\kappa}_i}{1-\overline{\kappa}_i}\kappa'_1\frac{1}{(1-\kappa)^2} + \left(\frac{54}{55}\right)^2\kappa'_i.$$

Since also

$$\overline{\kappa}_i \leq \frac{1}{2}, \quad \frac{\kappa}{\lambda} \leq 1,$$

it follows from (14) that

$$\kappa'_{i+1} \leq 16\kappa'_1\overline{\kappa}_i + \left(\frac{54}{55}\right)^2\kappa'_i \leq 16\kappa'_1\lambda^{i-1} + \left(\frac{54}{55}\right)^2\kappa'_i.$$

Set $\mu = \max\{\lambda, (\frac{54}{55})^2\}$ and it follows that $\mu < 1$, and for all $i \geq N$,

$$\kappa'_{i+1} \leq 16\kappa'_1\mu^{i-1} + \mu\kappa'_i.$$

By induction we prove for $i = 1, 2, \ldots$

$$\kappa'_{N+i} \leq \mu^i(i16\kappa'_1\mu^{N-1} + \kappa'_N).$$

This implies that $\lim_{i\to\infty}\kappa'_i = 0$. By our rules in §3 above,

$$0 \leq \kappa_i \leq 2\overline{\kappa}_i + \kappa'_i,$$

and hence $\lim_{i\to\infty}\kappa_i = 0$. ◇

Definition A group of matrices of degree n is called *discrete*, if the lower bound of distances $\varphi(A-E)$ of all matrices $A \neq E$ in the group is positive.

By (12a), discrete groups transform again into discrete groups on conjugation with a fixed non-singular matrix. We consider a discrete group $\mathfrak{G}$ of partially reduced matrices of degree n of the form

$$A = \begin{pmatrix} A^{(1)} & & & * \\ 0 & A^{(2)} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & A^{(r)} \end{pmatrix},$$

such that as before the map $A \mapsto A^{(i)}$ is a representation of degree n_i of $\mathfrak{G}$. For every positive number κ we define the complex $\mathfrak{K}^{(\kappa)}$ as the set of all matrices $A \in \mathfrak{G}$ which satisfy

$$\varphi(A^{(i)} - E_{n_i}) \leq \kappa \quad \text{for } i = 1, \ldots, r.$$

$\mathfrak{K}^{(\kappa)}$ generates a subgroup $\mathfrak{G}^{(\kappa)}$ of $\mathfrak{G}$. By Theorem 9, applied to the discrete group $\mathfrak{G}$, if $\kappa < 2 - \sqrt{3}$, for any finite number of matrices $A_1, \ldots, A_r$ in $\mathfrak{K}^{(\kappa)}$ there is an index c, such that all higher commutators formed with these matrices that are of weight $c + 1$ and step 1 equal E. It follows from Criterion 4 that $\mathfrak{G}^{(\kappa)}$ is a quasi-nilpotent group and from the second corollary in §2 it follows that $\mathfrak{G}^{(\kappa)}$ is solvable, if only $\kappa < 2 - \sqrt{3}$ holds both times.

Since $\mathfrak{G}^{(\kappa)} \geq \mathfrak{G}^{(\kappa')}$ for $\kappa \geq \kappa'$, the dimension of the linear hull of $\mathfrak{G}^{(\kappa)}$ is a monotonously growing function of κ with values in the positive integers. Then there exists $\kappa_0 > 0$ such that for all positive numbers κ not greater than κ_0, the linear hulls of the $\mathfrak{G}^{(\kappa)}$ have the same dimension as that of $\mathfrak{G}^{(\kappa_0)}$, and therefore these hulls all coincide. As noted at the beginning of §2, every irreducible subrepresentation $X \mapsto \Gamma(X)$ of degree m of $\mathfrak{G}^{(\kappa_0)}$ induces an irreducible subrepresentation of any group $\mathfrak{G}^{(\kappa)}$ with $0 < \kappa \leq \kappa_0$.

Suppose the representation Γ was imprimitive. Then there is a non-singular matrix T independent of n, such that the matrices in $T\mathfrak{G}^{(\kappa)}T^{-1}$ take the form

$$TAT^{-1} = \begin{pmatrix} * & \cdots & * \\ 0 & \Gamma(X) & \vdots \\ 0 & 0 & * \end{pmatrix}$$

with

$$\overline{\Gamma}(X) = S\Gamma(X)S^{-1} = (\delta_{\pi(i),k}\Gamma_i(X)),$$

where π is a permutation of d numbers, S is a non-singular m-row matrix, and $\Gamma_i(X)$ are $\frac{m}{d}$-row matrices, $i, k = 1, \ldots, d$, $d > 1$. Since for $0 < \kappa \leq \kappa_0$, $S\Gamma(\mathfrak{G}^{(\kappa)})S^{-1}$ is irreducible if $\Gamma(\mathfrak{G}^{(\kappa)})$ is, the generating system $\mathfrak{K}^{(\kappa)}$ of $\mathfrak{G}^{(\kappa)}$ contains least one element $K^{(\kappa)}$ whose imprimitive matrix $\overline{\Gamma}(K^{(\kappa)})$ has a 0 on its diagonal. Then

$$\varphi(TK^{(\kappa)}T^{-1} - E) \geq 1.$$

On the other hand,

$$\varphi(TK^{(\kappa)}T^{-1} - E) = \varphi(T(K^{(\kappa)} - E)T^{-1}) \leq \varphi(T)\varphi(T^{-1})\kappa.$$

If we choose κ sufficiently small, we obtain a contradiction.

The irreducible subrepresentations of the groups $\mathfrak{G}^{(\kappa)}$ are therefore all primitive, if only $0 < \kappa \le \kappa_0$. Now we determine a number κ' satisfying the inequalities $0 < \kappa' \le \kappa_0$ and $\kappa' < 2 - \sqrt{3}$, and we form the sequence

$$\mathfrak{U}_1 = \mathfrak{G}^{(\kappa')},\ \mathfrak{U}_2 = \mathfrak{G}^{(\frac{\kappa'}{2})},\ \ldots,\ \mathfrak{U}_i = \mathfrak{G}^{(\frac{\kappa'}{i})},\ \ldots.$$

The $\mathfrak{U}_i$ form a descending sequence of subgroups with the following four properties:

1. Every element of the sequence is a solvable subgroup.
2. The linear hulls of the groups in the sequence are identical.
3. Every irreducible component of every matrix group in the sequence is primitive.
4. For every index i and every element $X \in \mathfrak{G}$ there is an index $\mu = \mu(i, X)$ such that $X\mathfrak{U}_\mu X^{-1} \le \mathfrak{U}_i$.[8)]

We will call sequences with these four properties *A-sequences*.

Before proving the main theorem on A-sequences, we first show the following lemma: Let $\mathfrak{G}$ be a reducible group of matrices of degree n of the form $A = \begin{pmatrix} A^{(1)} & A \\ 0 & A^{(2)} \end{pmatrix}$, and let $\mathfrak{G}^{(i)}$ be the matrix group formed by all matrices $A^{(i)}$ from the subrepresentation $A \mapsto A^{(i)}$. For short, we call $A^{(i)}$ or $\mathfrak{G}^{(i)}$ the *components* of A and $\mathfrak{G}$, respectively.

Lemma 1

1. *The reducible group $\mathfrak{G}$ is solvable if and only if both of its components are.*
2. *The radical of $\mathfrak{G}$ consists of all elements in $\mathfrak{G}$ whose two components are both contained in the radical of the respective component of $\mathfrak{G}$.*
3. *A subgroup of $\mathfrak{G}$ is contained in the radical of $\mathfrak{G}$ if and only if both of its components are contained in the radical of the respective component of $\mathfrak{G}$.*

PROOF:

[8)] 4. follows from (12).
Translator's note: Then (12) also implies $X\mathfrak{U}_\nu X^{-1} \le \mathfrak{U}_i$ for all $\nu \ge \mu$.

1. If $\mathfrak{G}$ is solvable, the all homomorphic images are solvable, in particular the groups $\mathfrak{G}^{(1)}$, $\mathfrak{G}^{(2)}$. Conversely, if $\mathfrak{G}^{(1)}$ and $\mathfrak{G}^{(2)}$ are solvable, so that $D^k\mathfrak{G}^{(1)} = \{E_{n_1}\}$, $D^k\mathfrak{G}^{(2)} = \{E_{n_2}\}$, then it follows that matrices in $D^k\mathfrak{G}$ are of the form $\left(\begin{smallmatrix} E_{n_1} & A \\ 0 & E_{n_2} \end{smallmatrix}\right)$ and thus form an abelian group. Hence $D^{k+1}\mathfrak{G} = \{E\}$.

2. All elements in $\mathfrak{G}$, both of whose components are contained in the radical of the respective component of $\mathfrak{G}$, form a normal subgroup $\overline{\mathrm{R}}\mathfrak{G}$ of $\mathfrak{G}$. By 1., $\overline{\mathrm{R}}\mathfrak{G}$ is solvable, hence contained in the radical $\mathrm{R}\mathfrak{G}$ of $\mathfrak{G}$. Since every homomorphism maps a solvable normal subgroup to a solvable normal subgroup, each component of $\mathrm{R}\mathfrak{G}$ is contained as a solvable normal subgroup in the radical of the respective component of $\mathfrak{G}$.

3. Follows directly from 2. ◇

Let $\mathfrak{G}$ be an arbitrary group of matrices of degree n, and let

$$\mathfrak{U}_1 \geq \mathfrak{U}_2 \geq \mathfrak{U}_3 \geq \dots$$

be an A-sequence of subgroups.

Theorem 10 *From a certain index on, the subgroups in an A-sequence are contained in the radical of the whole group $\mathfrak{G}$.*

Proof: It is sufficient to show that a certain subgroup $\mathfrak{U}_\nu$ is contained in the radical of $\mathfrak{G}$. To this end, note that any subsequence of an A-sequence is again an A-sequence, so we may replace any A-sequence in the proof by a subsequence. Since for any $X \in \mathfrak{G}$ there is an index $\mu = \mu(1, X)$ such that $X\mathfrak{U}_\mu X^{-1} \leq \mathfrak{U}_1$, it follows from property 2 of A-sequences that $\mathrm{H}\mathfrak{U}_\mu = \mathrm{H}\mathfrak{U}_1 = H$, and further

$$X(\mathrm{H}\mathfrak{U}_\mu)X^{-1} \leq \mathrm{H}\mathfrak{U}_1, \quad XHX^{-1} \leq H = \mathrm{H}\mathfrak{U}_1.$$

All elements of $\mathfrak{G}$ contained in H form a normal subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ which contains $\mathfrak{U}_1$ and whose linear hull is H. If we have shown that $\mathfrak{U}_\nu$ is contained in the radical of $\mathfrak{G}_1$, then it follows from Theorem 3 that $\mathfrak{U}_\nu$ is also contained in the radical of $\mathfrak{G}$. By replacing $\mathfrak{G}$ by $\mathfrak{G}_1$, we may assume that the linear hull of $\mathfrak{G}$ coincides with the linear hull of every subgroup in the A-sequence. By Lemma 1, part 3, we may further assume that $\mathfrak{G}$ is irreducible. So now we need to prove Theorem 9 under the additional assumption that all matrix groups $\mathfrak{G}, \mathfrak{U}_1, \mathfrak{U}_2, \dots$

are irreducible. It follows from property 3 of A-sequences that in our case all $\mathfrak{U}_i$ are primitive. Now we want to ensure that all matrices in $\mathfrak{U}_i$ have determinant 1. Let $\mathfrak{Z}$ be the group of non-zero scalar matrices. Let $\overline{\mathfrak{U}}_i$ be the normal subgroup of all matrices with determinant 1 in $\mathfrak{U}_i\mathfrak{Z}$. Clearly $\overline{\mathfrak{U}_i}\mathfrak{Z} = \mathfrak{U}_i\mathfrak{Z}$, as for every matrix $A \in \mathfrak{U}_i$, $\det(A) = \alpha^n$ is solvable and $\alpha^{-1}A$ is contained in $\overline{\mathfrak{U}}_i$. If we already proved that $\overline{\mathfrak{U}}_\nu$ is contained in the radical of $\mathfrak{G}\mathfrak{Z}$, then it follows that $\overline{\mathfrak{U}}_\nu\mathfrak{Z}$ is also contained in the radical. As $\overline{\mathfrak{U}}_\nu\mathfrak{Z} = \mathfrak{U}_\nu\mathfrak{Z}$, by Theorem 3, $\mathfrak{U}_\nu$ is also contained in the radical of $\mathfrak{G}$.

Now we can and will assume that all matrices in $\mathfrak{U}_1$ have determinant 1. By Theorem 6, all subgroups $\mathfrak{U}_i$ are finite and there exists an index N from which on all subgroups in the A-sequence are identical. From property 4 of A-sequences it follows now that $\mathfrak{U}_N$ is a solvable normal subgroup in the whole group $\mathfrak{G}$, hence $\mathfrak{U}_N$ is contained in the radical of $\mathfrak{G}$. $\diamond$

We apply the above theorem to the A-sequence $\mathfrak{U}_1 \leq \mathfrak{U}_2 \leq \mathfrak{U}_3 \leq \ldots$ constructed on page 14 for a discrete subgroup $\mathfrak{G}$. Then we find:

Theorem 11 *For every discrete group $\mathfrak{G}$ of matrices of degree n of the form*

$$A = \begin{pmatrix} A^{(1)} & & & * \\ 0 & A^{(2)} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & A^{(r)} \end{pmatrix},$$

there exists a positive number ε such that all matrices $A \in \mathfrak{G}$ that satisfy the inequality $\varphi(A^{(i)} - E) \leq \varepsilon$ are contained in the radical of $\mathfrak{G}$.

§ 5 Refinements

Now we want to show that for sufficiently small κ all irreducible components of $\mathfrak{G}^{(\kappa)}$ have degree 1.

Let $\mathfrak{U}_1$ be a group of matrices of the form

$$A = \begin{pmatrix} A^{(1)} & & & * \\ 0 & A^{(2)} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & A^{(\nu)} \end{pmatrix}.$$

An A-sequence $\mathfrak{U}_1 \geq \mathfrak{U}_2 \geq \mathfrak{U}_3 \geq \ldots$ is called a *B-sequence*, if there exists a sequence of positive numbers κ_i converging to 0, such that the $\mathfrak{U}_i$ are generated by matrices A_i with the property $\varphi(A_i^{(\nu)} - E_{n_\nu}) \leq \kappa_i$, $\nu = 1, \ldots, r$. We show:

The irreducible components of the subgroups in a B-sequence have degree 1.

PROOF: Considering (12), (12a), it follows by the same reasoning as in the proof of Theorem 10 that we may restrict to the case in which all groups in the B-sequence are irreducible. Moreover, by changing to a subsequence, we can assume that $\kappa_i < 1$.

Again, we want to change the $\mathfrak{U}_1$ so that all matrices in $\mathfrak{U}_1$ have determinant 1.

All matrices[9)] A_i in $\mathfrak{U}_1$ with $\varphi(A_i - E) \leq \kappa_i$ that generate $\mathfrak{U}_i$ form a complex $\mathfrak{B}_i$. For the determinant α_i of A_i, an estimate of the form[10)]

$$\varphi(\alpha_i - 1) \leq C\kappa_i$$

holds, where C is a positive constant which is independent of the κ_i. Let ζ denote a primitive nth root of unity contained in the algebraically closed coefficient field of $\mathfrak{U}_1$. Then:

$$1 - \alpha_i = \pm \prod_{\nu=1}^{n} (1 - \zeta^\nu \sqrt[n]{\alpha_i}),$$
$$\varphi(1 - \alpha_i) = \prod_{\nu=1}^{n} \varphi(1 - \zeta^\nu \sqrt[n]{\alpha_i}),$$

and there is at least one ν for which

$$\varphi(1 - \zeta^\nu \sqrt[n]{\alpha_i}) \leq \sqrt[n]{C\kappa_i},$$

and hence

$$\varphi(\zeta^\nu \sqrt[n]{\alpha_i}) > 1 - \sqrt[n]{C\kappa_i}.$$

Let $\overline{\mathfrak{B}}_i$ be the complex consisting of the matrices $\frac{1}{\zeta^\nu \sqrt[n]{\alpha_i}} A_i$. The matrices in the group $\overline{\mathfrak{U}}_i$ generated by $\overline{\mathfrak{B}}_i$ have determinant 1, and as in the proof of Theorem 11, $\overline{\mathfrak{U}}_i \mathfrak{Z} = \mathfrak{U}_i \mathfrak{Z}$, where $\mathfrak{Z}$ is the group of non-zero scalar matrices. After changing

9) *Translator's note:* In the original it is somewhat unclear whether Zassenhaus wants $\mathfrak{U}_i$ to contain all matrices A_i with $\varphi(A_i - E) \leq \kappa_i$ or just be generated by some of them. However, this is not relevant for the following argument.

10) *Translator's note:* This inequality holds if we assume $\kappa_i < 2 - \sqrt{3} < 1$. Otherwise replace it by $\varphi(\alpha_i - 1) \leq C\kappa_i^n$, which does not affect the validity of the ensuing argument.

to a subsequence we may assume that $C\kappa_i \leq \frac{1}{2}$, and then we have the following estimate for $A_i - E$:

$$\varphi(A_i - E) \leq \frac{\varphi(A_i - E) + \varphi((1-\zeta^\nu \sqrt[n]{\alpha_i})E)}{\varphi(\zeta^\nu \sqrt[n]{\alpha_i})} \leq \frac{\kappa_i + \sqrt{n}\sqrt[n]{C\kappa_i}}{1 - \sqrt[n]{C\kappa_i}} \leq D\sqrt[n]{\kappa_i}$$

for a suitable constant D.

Hence $\overline{\mathfrak{U}}_1 \geq \overline{\mathfrak{U}}_2 \geq \ldots$ is a B-sequence. As in the proof of Theorem 11 it follows that the groups $\overline{\mathfrak{U}}_i$ are finite and coincide from a certain index on. For the distance from E of the matrices in $\mathfrak{U}_1 \backslash \{E\}$ we have a positive lower bound.[11)] Hence, for sufficiently large index N, we have $\overline{\mathfrak{U}}_N = \{E\}$, $\mathfrak{U}_N \leq \mathfrak{Z}$. Since $\mathfrak{U}_N$ is irreducible, the degree of $\mathfrak{U}_N$ is 1. ◇

We know that in a discrete group $\mathfrak{G}$ the subgroups $\mathfrak{G}^{(\kappa)}$ are quasi-nilpotent if $0 \leq \kappa < 2 - \sqrt{3}$. Now we want to show that they are even nilpotent. This follows from the above together with the following lemma:

Lemma 2 *Let $\mathfrak{U}$ be a semireducible group of matrices of degree n of the form*

$$A = \begin{pmatrix} A^{(1)} & A' \\ 0 & A^{(2)} \end{pmatrix},$$

such that the maps $A \mapsto A^{(i)}$ map the group $\mathfrak{U}$ homomorphically to the matrix group $\mathfrak{U}^{(i)}$ of degree n_i, for $i = 1, 2$. If the whole group $\mathfrak{U}$ is quasi-nilpotent and each of the two components is nilpotent, then $\mathfrak{U}$ is nilpotent.

PROOF: All matrices $A \in \mathfrak{U}$ with $A^{(i)} = E_{n_i}$ form a normal subgroup $\mathfrak{N}_i$ of $\mathfrak{U}$, so that $\mathfrak{U}/\mathfrak{N}_i \cong \mathfrak{U}^{(i)}$. We assume that $\mathfrak{U}^{(1)}$ and $\mathfrak{U}^{(2)}$ are nilpotent. Hence there is a number c such that

$$\mathfrak{Z}_{c+1}(\mathfrak{U}^{(i)}) = \{E_{n_i}\}, \quad i = 1, 2,$$
$$\mathfrak{Z}_{c+1}(\mathfrak{G}) = \mathfrak{N} \leq \mathfrak{N}_1 \cap \mathfrak{N}_2.$$

The matrices in the normal subgroup $\mathfrak{N}$ have the form $\begin{pmatrix} E_{n_1} & A' \\ 0 & E_{n_2} \end{pmatrix}$.

Let $\mathfrak{K}$ be a complex of matrices of the form

$$K = \begin{pmatrix} \lambda E_{n_1} & K' \\ 0 & \lambda E_{n_2} \end{pmatrix}$$

11) *Translator's note:* We are still assuming that $\mathfrak{G}$ is discrete.

that contains the identity matrix, and let $\mathfrak{V}$ be a group of non-singular matrices of the form
$$V = \begin{pmatrix} V^{(1)} & V' \\ 0 & V^{(2)} \end{pmatrix}.$$
We set
$$(V, K) = \begin{pmatrix} E_{n_1} & K' - V^{(1)} K' (V^{(2)})^{-1} \\ 0 & E_{n_2} \end{pmatrix}.$$
If $\lambda \neq 0$, then $(V, K) = VKV^{-1}K^{-1}$.

Let $(\mathfrak{V}, \mathfrak{K})$ be the complex of all matrices (V, K). For the linear hulls we can easily derive the following formula from the above:

$$\mathrm{H}(\mathfrak{V}, \mathrm{H}\mathfrak{K}) = \mathrm{H}(\mathfrak{V}, \mathfrak{K}). \tag{15}$$

Now let $\mathfrak{V}$ be a subgroup generated by finitely many elements of $\mathfrak{U}$. By our assumptions, all subgroups of this kind are nilpotent. Set

$$\mathfrak{K}_1 = \mathfrak{V} \cap \mathfrak{N},\ \mathfrak{K}_2 = (\mathfrak{V}, \mathfrak{K}_1),\ \ldots,\ \mathfrak{K}_{i+1} = (\mathfrak{V}, \mathfrak{K}_i), \ldots$$

Since $\mathfrak{V}$ is nilpotent, there exists an index $m = m(\mathfrak{V})$ such that $\mathfrak{K}_{m+1} = \{E\}$. From (15):
$$\mathrm{H}(\mathfrak{V}, \mathrm{H}\mathfrak{K}_i) = \mathrm{H}\mathfrak{K}_{i+1}.$$
It follows that
$$\mathrm{H}\mathfrak{K}_1 > \mathrm{H}\mathfrak{K}_2 > \ldots > \mathrm{H}\mathfrak{K}_{m+1} = \mathbf{k}E.$$
As $\mathrm{H}\mathfrak{K}_1$ contains at most $n_1 n_2 + 1$ linearly independent matrices, we have in any case
$$\mathrm{H}\mathfrak{K}_{n_1 n_2 + 1} = \mathbf{k}E, \quad \mathfrak{K}_{n_1 n_2 + 1} = \{E\}.$$
Now let $X_1, \ldots, X_{n_1 n_2}$ be $n_1 n_2$ arbitrary elements in $\mathfrak{U}$ and N an arbitrary element in $\mathfrak{N}$. Set
$$\mathfrak{V} = \langle X_1, \ldots, X_{n_1 n_2}, N \rangle$$
and obtain
$$(X_1, \ldots, X_{n_1 n_2}, N) = E.$$
By Hall's substitution principle[12)] this implies

$$(\underbrace{\mathfrak{U}, \ldots, \mathfrak{U}}_{n_1 n_2 \text{ times}}, \mathfrak{N}) = \{E\},$$

[12)]See [2)], II, §6, Satz 13.

and together with $\mathfrak{N} = \mathfrak{Z}_{c+1}(\mathfrak{U})$,

$$\mathfrak{Z}_{c+1+n_1 n_2}(\mathfrak{U}) = \{E\},$$

that is, $\mathfrak{U}$ is nilpotent. ◇

We have obtained the following result:

In a discrete matrix group of degree n whose matrices are all of the form

$$A = \begin{pmatrix} A^{(1)} & & & * \\ 0 & A^{(2)} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & A^{(r)} \end{pmatrix},$$

all matrices A in the group that satisfy

$$\varphi(A^{(\nu)} - E_{n_\nu}) \leq \varepsilon \qquad (\nu = 1, \ldots, r)$$

generate a nilpotent subgroup with irreducible components of degree 1*, where ε is a sufficiently small number.*

§ 6 Groups of affinities

We apply Theorem 11 to groups of affinities. By an *affinity* in n dimensions we mean any $n+1$-row matrix of the form

$$A = \begin{pmatrix} \underline{A} & \mathfrak{p} \\ 0 & 1 \end{pmatrix},$$

where the *homogeneous part* $\underline{A}$ is an n-row matrix, and the *translation component* is a vector with n components. The affinity is called *non-degenerate* if the homogeneous part is non-singular. The totality of non-degenerate affinities forms a group, the *affine group* in n dimensions. Its subgroups will be called *groups of affinities* for short.

A group of affinities is called *decomposable* after Bieberbach, if all of its matrices are of the form

$$\begin{pmatrix} A^{(1)} & 0 & 0 \\ 0 & A^{(2)} & \mathfrak{p}_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

An affinity is called a *translation* if its homogeneous part is the n-row identity matrix. All translations in a group $\mathfrak{G}$ of affinities form an abelian normal subgroup $\mathfrak{T}$ of the group, such that the quotient group $\mathfrak{G}/\mathfrak{T}$ is isomorphic to the group consisting of the homogeneous parts of $\mathfrak{G}$. A finite number of translations in $\mathfrak{G}$ is called *independent* if their translation parts are linearly independent. This property is preserved by transformations with non-degenerate affinities. A group of affinities in n dimensions is called *isotropic* if it contains n independent translations. An isotropic group is indecomposable.

Definition A matrix all of whose eigenvalues are 1 is called a *semi-translation*. Translations are always semi-translations.

Theorem 12 *Every abelian normal subgroup $\mathfrak{A}$ in an indecomposable group $\mathfrak{G}$ of affinities in n dimensions consists of semi-translations.*

PROOF: The matrices in $\mathfrak{G}$ have the form

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{M} = \operatorname{span}\{u_1, \ldots, u_{n+1}\}$ be a representation module for $\mathfrak{G}$ with operation

$$u_i A = \sum_{k=1}^{n+1} a_{ik} u_k, \quad (i = 1, \ldots, n), \qquad u_{n+1} A = u_{n+1}.$$

The vectors u in $\mathfrak{M}$ that satisfy $u(A - E)^{n+1} = 0$ for all $A \in \mathfrak{A}$ form an $\mathfrak{A}$-invariant submodule, which is the submodule $\mathfrak{M}_1$ corresponding to the eigenvalue 1. It follows from the representation theory of abelian groups that $\mathfrak{M}$ is the direct sum of $\mathfrak{M}_1$ and a second $\mathfrak{A}$-invariant submodule $\mathfrak{M}_2$, and that this $\mathfrak{M}_2$ is uniquely determined. As $\mathfrak{A}$ is a normal subgroup of $\mathfrak{G}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are also invariant under $\mathfrak{G}$. As $\mathfrak{G}$ is indecomposable and $\mathfrak{M}_1$ contains the vector u_{n+1}, it follows that $\mathfrak{M}_2 = \mathbf{0}$, hence $\mathfrak{M} = \mathfrak{M}_1$. Thus $\mathfrak{A}$ consists of semi-translations. ◇

§ 7 Groups of motions

Mow suppose the field of coefficients is the field of complex numbers, and the valuation φ is the absolute value. Let $\overline{A}$ denote the complex conjugate matrix to

A. An n-row matrix is called *unitary* if $A\overline{A}^\top = E$. If A is unitary, so is $\overline{A}^\top$. The unitary matrices form the *unitary group*. The unitary matrices A are characterised by the relation $\varphi(A\mathfrak{x}) = \varphi(\mathfrak{x})$ for all vectors $\mathfrak{x}$. This implies: $\varphi(AB) = \varphi(B)$ for all n-row unitary matrices A and arbitrary n-row matrices B. If C is a matrix with n columns, then

$$\varphi(CA) = \varphi((A^\top C^\top)^\top) = \varphi(A^\top C^\top) = \varphi(C^\top) = \varphi(C).$$

Hence

$$\varphi(ABA^{-1}) = \varphi(B) \tag{16}$$

for all unitary n-row matrices A and arbitray n-row matrices B.

By a *motion* we mean any affinity whose homogeneous component is unitary. The motions in n dimensions form a group. In a given discrete group $\mathfrak{G}$ of motions, let as before $\mathfrak{G}^{(\kappa)}$ denote the subgroup generated by all matrices A whose homogeneous component $\underline{A}$ satisfies the inequality $\varphi(\underline{A} - \underline{E}) \leq \kappa$. Then $\mathfrak{G}^{(\kappa)}$ is solvable if $\kappa < 2 - \sqrt{3}$.[10)] Moreover, by (16), $\mathfrak{G}^{(\kappa)}$ is a normal subgroup. So in this case, Theorem 10 is not required to prove Theorem 11. As the identity matrix is an accumulation point in an infinite unitary group, Theorem 11 yields:[11)]

Every infinite discrete group of motions has a radical different from $\{E\}$.

More generally:

Theorem 13 *Every infinite discrete group of matrices with complex coefficients whose irreducible components are bounded has a radical different from $\{E\}$.*

According to Maschke, a finite group of motions $\mathfrak{G}$ is always decomposable, that is, any representation module $\mathfrak{M}$ of $\mathfrak{G}$ decomposes uniquely into invariant submodules $\mathfrak{M}_1$ and $\mathfrak{M}_2$, such that $\mathfrak{M}_1$ consists of all the vectors that are fixed by every transformation in $\mathfrak{G}$. *An indecomposable group of motions does not contain any finite normal subgroup with more than one element.*

[10)] It is sufficient to require $\kappa < \frac{1}{2}$, as the inequalities (11), (13) can be sharpened for unitary matrices A, B:

$$\varphi(ABA^{-1}B^{-1} - E) \leq 2\kappa\lambda,$$
$$\varphi(R) \leq 2\lambda\varphi(P) + 2\kappa\varphi(Q),$$

and this implies a sharpened version of Theorem 8.

[11)] *Translator's note:* The reasoning here is erroneous, as the identity is only an accumulation point in a continous unitary group, but not in a discrete one. This also affects Theorem 13.

If we note further that every motion that is a semi-translation is even a translation, then Theorems 12 and 13 imply, that *every infinite discrete group of motions contains translations other than the identity.*

Moreover, Frobenius and Bieberbach proved: *In an isotropic discrete group of motions, the index of the normal subgroup of translations is finite in the full group.*

§ 8 Decompositions by translations

Let $\mathfrak{G}$ be a group of affinities in n dimensions. The translations in $\mathfrak{G}$ form an abelian normal subgroup $\mathfrak{T}$.

We choose a representation module $\mathfrak{M} = \operatorname{span}\{u_1, \ldots, u_{n+1}\}$ such that

$$Au_k = \sum_{i=1}^{n+1} a_{ik} u_k$$

for all matrices A in $\mathfrak{G}$,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The submodule $\mathfrak{M}' = \operatorname{span}\{u_1, \ldots, u_n\}$ is $\mathfrak{G}$-invariant. All vectors $(T - \underline{E})u_{n+1}$ with $T \in \mathfrak{T}$ generate a $\mathfrak{G}$-invariant submodule $\mathfrak{m}$ of $\mathfrak{M}'$, whose dimension p is the maximal number of linearly independent translations. The invariance follows, because

$$(T - \underline{E})u_i = 0, \quad \text{if } i \leq n,$$

hence

$$\begin{aligned} (T - \underline{E})\mathfrak{M} &\subseteq \mathfrak{m}, \\ A(T - \underline{E})\mathfrak{M} &= A(T - \underline{E})A^{-1}\mathfrak{M} \\ &= (ATA^{-1} - \underline{E})\mathfrak{M} \subseteq \mathfrak{m}. \end{aligned}$$

Choose a basis $v_1, \ldots, v_p$ of $\mathfrak{m}$ and extend it to a basis $v_1, \ldots, v_n$ of $\mathfrak{M}'$. Define a non-degenerate affinity S via $Su_i = v_i$ (for $1 \leq i \leq n$), $Su_{n+1} = u_{n+1}$, so that

$$SAS^{-1} = \begin{pmatrix} A^{(1)} & 0 & \mathfrak{p}_1 \\ \Lambda & A^{(2)} & \mathfrak{p}_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $A^{(1)}$ p-row, $A^{(2)}$ $(n-p)$-row, for all $A \in \mathfrak{G}$. The map $A \mapsto A^{(i)}$ maps $\mathfrak{G}$ homomorphically onto a matrix group $\mathfrak{G}^{(i)}$. All matrices in $\mathfrak{G}$ that are mapped to the identity by the map for $i = 1$ or $i = 2$, form a normal subgroup $\mathfrak{N}_1$ or $\mathfrak{N}_2$, respectively. The map

$$A \mapsto \begin{pmatrix} A^{(1)} & \mathfrak{p}_1 \\ 0 & 1 \end{pmatrix}$$

also gives rise to a group $\mathfrak{G}^{(11)}$ of affinities. $\mathfrak{N}_{11}$ is the normal subgroup containing the matrices mapped to the identity. Then

$$\mathfrak{G}/\mathfrak{N}_i \cong \mathfrak{G}^{(i)} \qquad (i = 1, 2), \tag{17}$$

$$\mathfrak{G}/\mathfrak{N}_{11} \cong \mathfrak{G}^{(11)}. \tag{18}$$

$\mathfrak{G}^{(11)}$ is isotropic. In fact, $\mathfrak{N}_1^{(11)}$ is precisely the normal subgroup of all translations in $\mathfrak{G}^{(11)}$.

$$\mathfrak{T}\mathfrak{N}_{11} \leq \mathfrak{N}_1, \tag{19}$$

$$\mathfrak{T} \cap \mathfrak{N}_{11} = \{E\}, \tag{20}$$

$$D\mathfrak{N}_1 \leq \mathfrak{N}_{11}. \tag{21}$$

Now let $\mathfrak{G}$ be a group of motions. The module $\mathfrak{M}$ has a non-degenerate unitary orthogonal metric and the orthogonal space $\mathfrak{m}^\perp$ to $\mathfrak{m}$ is mapped to itself by $\mathfrak{G}$. As the decomposition $\mathfrak{M} = \mathfrak{m} + \mathfrak{m}^\perp$ is direct, let the preceding construction provide that

$$\mathfrak{m}^\perp = \operatorname{span}\{v_{p+1}, \dots, v_n\}.$$

Moreover, we can assume that $\mathfrak{G}^{(1)}$ and $\mathfrak{G}^{(2)}$ are unitary. Then the matrices in $\mathfrak{G}$ are all of the form

$$SAS^{-1} = \begin{pmatrix} A^{(1)} & 0 & \mathfrak{p}_1 \\ 0 & A^{(2)} & \mathfrak{p}_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map

$$A \mapsto \begin{pmatrix} A^{(2)} & \mathfrak{p}_2 \\ 0 & 1 \end{pmatrix}$$

maps $\mathfrak{G}$ homomorphically onto a group $\mathfrak{G}^{(22)}$ of motions. Let $\mathfrak{N}_{22}$ be the normal subgroup of matrices mapped to E. Then:

$$\mathfrak{G}/\mathfrak{N}_{22} \cong \mathfrak{G}^{(22)}, \tag{22}$$

$$\mathfrak{T}^{(22)} = \{E\}. \tag{23}$$

If in addition $\mathfrak{G}$ in indecomposable, then $\mathfrak{G}^{(22)}$ is indecomposable. Now assume further that $\mathfrak{G}$ is discrete. If $\mathfrak{N}_{11}$ contained a non-trivial solvable normal subgroup, then $\mathfrak{N}_{11}$ also contained a non-trivial abelian normal subgroup of $\mathfrak{G}$. But, as shown above[12)], $\mathfrak{N}_{11}$ would then contain non-trivial translations. This is not possible, so the radical of $\mathfrak{N}_{11}$ is the identity, hence $\mathfrak{N}_{11}$ is a finite group of motions. As the full group $\mathfrak{G}$ is indecomposable, this normal subgroup $\mathfrak{N}_{11}$ is trivial. By (21), $\mathfrak{N}_1$ is an abelian normal subgroup, hence contained in $\mathfrak{T}$ by Theorem 12, and hence $\mathfrak{N}_1 = \mathfrak{T}$ by (19). As $\mathfrak{G}^{(11)}$ is isotropic and $\mathfrak{T}^{(11)}$ is the largest group of translations in $\mathfrak{G}^{(11)}$, it follows that $\mathfrak{G}^{(11)}/\mathfrak{T}^{(11)}$ and hence $\mathfrak{G}/\mathfrak{T}$ is finite. Now $\mathfrak{T}^{(22)}$ is trivial, hence $\mathfrak{G}^{(22)}$ is finite. As $\mathfrak{G}^{(22)}$ is indecomposable, $\mathfrak{G}^{(22)}$ is necessarily trivial. Hence $p = n$. This proves Bieberbach's fundamental theorem:

Every indecomposable discrete group of motions contains n linearly independent translations.

[12)]*Translator's note:* Here, Zassenhaus refers to Theorem 12 and the remark immediately following Theorem 13.

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Original: *Beweis eines Satzes über diskrete Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 12, 1937, no. 1, 289-312

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