Universität Karlsruhe (TH)
Forschungsuniversität gegründet 1825

Étale Representations of Reductive Algebraic Groups and Left-Symmetric Algebras

Diploma Thesis

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October 18, 2007

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Preface

The thesis at hand is my diploma thesis on étale representation of reductive algebraic groups, motivated by their relation to left-symmetric algebras. I wrote it independently, using only the sources and resources stated in the thesis.

I would like to thank Prof. Hartmut Prautzsch and HDoz. Oliver Baues for the supervision of my thesis. In particular, I owe gratitude to Oliver Baues for giving advice on the topics of this thesis and for the opportunity to discuss problems which occurred in the making of the thesis.

Karlsruhe, September 2007 *Wolfgang Globke*

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1 Introduction

This first chapter provides a brief summary of the motivation for this thesis, its objectives and the results. This overview is not ordered by the chapters of the thesis, but rather it is structured in a way as to make the background and the motivation for the proceeding in this thesis clear.

1.1 Overview

A *left-symmetric algebra* $(V, *)$ is a vector space V defined over a field $\mathbb{k}$, endowed with a (not necessarily asociative) product $*$, such that

$$x * (y * z) - y * (x * z) = (x * y) * z - (y * x) * z$$

holds. These algebras are introduced in chapter 6.

It follows from this property that every left-symmetic algebra $(V, *)$ becomes a *Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ by taking the commutator

$$[x, y] = x * y - y * x.$$

In this case we say that $\mathfrak{g}$ "admits" a left-symmetric product. Some examples can be found in section 6.2.

Conversely, not every Lie algebra admits a left-symmetric product. For example, it is not possible to define a left-symmetric product on a semisimple Lie algebra. It is still an open problem to determine which Lie algebras admit a left-symmetric product. In particular, the classification of those *reductive Lie algebras* which admit a left-symmetric product is still an open problem. A reductive Lie algebra $\mathfrak{g}$ is a direct sum of Lie algebras

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s},$$

where $\mathfrak{a}$ is an abelian Lie algebra and $\mathfrak{s}$ is a semisimple Lie algebra. A classification of the left-symmetric products on $\mathfrak{gl}_n$ has been given in Baues [2].

In chapter 7, we learn how to exploit the relations between left-symmetric algebras and *affine étale representations* of Lie algebras. An étale representation of a Lie algebra $\mathfrak{g}$ is a representation $\mathrm{d}\varrho$ of $\mathfrak{g}$ by affine transformations of a vector space V,

$$\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{aff}(V) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{gl}(V), v \in V \right\},$$

with the property that there exists a *generic point* $v_0 \in V$, such that the evaluation map

$$\mathrm{ev}_{v_0} : \mathfrak{g} \to V, \quad x \mapsto \mathrm{d}\varrho(x).v_0$$

is an isomorphism of vector spaces. This means starting from v_0, every point in V can be reached by the action of $\mathfrak{g}$ in a unique way.

We find that such an étale representations induces a left-symmetric product - and vice versa: If $\mathrm{d}\varrho$ is an étale representation, we can define a left-symmetric product by

$$x * y = \mathrm{ev}_{v_0}^{-1}\Big(\mathrm{d}\varrho(x)_{\mathrm{lin}}.(\mathrm{ev}_{v_0}(y))\Big),$$

where $\mathrm{d}\varrho(x)_{\mathrm{lin}}$ is the linear part of the affine transformation $\mathrm{d}\varrho(x)$. Conversely, if a left-symmetric product $*$ on $\mathfrak{g}$ and an isomorphism $\varphi : \mathfrak{g} \to V$ of vector spaces are known, then we obtain an étale representation for $\mathfrak{g}$ by

$$\mathfrak{g} \ni x \mapsto \begin{pmatrix} \varphi \circ L_x \circ \varphi^{-1} & \varphi(x) \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(V),$$

where $L_x = (y \mapsto x * y)$ denotes the left-multiplication by x.

If one knows the étale representations, one also knows the left-symmetric algebras.

It is the objective of this thesis to further investigate which reductive Lie algebras $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ admit left-symmetric products. In the case of reductive Lie algebras with $\dim(\mathfrak{a}) = 1$, it is sufficient to consider linear representations $\mathfrak{g} \to \mathfrak{gl}(V) \subset \mathfrak{aff}(V)$. For the case that $\mathfrak{s}$ a simple Lie algebra, this problem has already been dealt with in Baues [2].

This question is strongly linked to the theory of *prehomogeneous vector spaces* (or *prehomogeneous modules*, as we shall call them here). A module (ϱ, V) for a group G is called prehomogeneous if there exists an open (hence Zariski-dense) orbit under the action of G on V, where ϱ is a linear rational representation. In particular, $\dim(G) \geq \dim(V)$ must hold. We introduce this theory in chapter 9, where we study some of the basic properties and get acquainted with the *castling transformation*, which is a potent tool for studying and constructing prehomogeneous modules. In chapter 10, we study how *relative invariants* determine certain useful properties of prehomogeneous module. A rational function f is a relative invariant if its value changes homomorphically under the action of G, i.e. $f(\varrho(g).x) = \chi(g)f(x)$ for some group character χ.

Several classifications of reductive prehomogeneous modules with rational representations have been given by a group of Japanese mathematicians from the 1970s up to the present. Nevertheless, a complete classification of prehomogeneous modules is not available.

The task ahead is to pick those modules out of the available classifications satisfying $\dim(G) = \dim(V)$. If we consider the reductive Lie algebra $\mathfrak{g}$ of G, the representation ϱ of G corresponds to an étale representation $\mathrm{d}\varrho$ of $\mathfrak{g}$ in this case. So, this is a way to find new examples for left-symmetric algebras. This task is approached in chapter 13 and proves to be very easy in some of the cases, i.e. in the cases when group has a simple semisimple part, or when representation is irreducible (see section 13.1). But in other cases, the classification is so general that it is far from obvious which modules satisfy $\dim(G) = \dim(V)$. These are the cases treated in section 13.2. The modules found here are summarised in appendix B.

Aside from picking modules from the classifications, we present a few criteria on when a group admits an étale representation in chapter 12. For example, we prove that groups which do not admit a non-trivial homomorphism $\chi : G \to \mathbb{k}^\times$ also do not admit a linear rational étale representation. Further, we establish that any module for a reductive group with one-dimensional centre and a non-irreducible étale representation must be composed of so-called non-regular irreducible components, even though the module itself must be regular. As an application, these theorems can be used to decide if irreducible prehomogeneous modules can be composed to yield an étale representation. For example, we see that this is not the case for a group $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$ in chapter 14.

Part I provides a brief summary of the mathematical preliminaries for the whole subject. It can be used to look up definitions or some fundamental theorems, but for a reader already acquainted with these topics, it is not essential for the rest of the thesis.

1.2 Description of the Results

We summarise the results on linear rational étale representations of reductive algebraic groups. Here, $\mathbb{k}$ is algebraically closed.

1.2.1 Results known so far

A complete classification of étale representations for groups $\mathrm{GL}_1 \times G$ with one-dimensional centre and simple G is known. These modules are (up to equivalence):

- $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{k}^2)$.
- $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{k}^n)^{\oplus n})$.

For groups $\mathrm{GL}_1 \times G$, where G is semisimple, all irreducible étale representations are given (up to equivalence) by the following modules:

- $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{k}^2)$.
- $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\mathbb{k}^3 \otimes \mathbb{k}^2)$.
- $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{k}^5 \otimes \mathbb{k}^4)$.

If we admit a centre GL_1^k of dimension $k \geq 1$, then all étale representations for groups $\mathrm{GL}_1^k \times G$, where G is simple, are given (up to equivalence) by the following modules:

- $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$.
- $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\Bbbk^n)^{\oplus n+1})$.
- $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*})$.
- $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\Bbbk^2 \otimes \Bbbk^2)$.

If G_1 and G_2 are simple groups, all non-irreducible étale representations of reductive groups $\mathrm{GL}_1^k \times G_1 \times G_2$ of type I (see section 11.3) are (up to equivalence) given by:

- $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus (\Bbbk^4 \otimes \Bbbk^2))$.
- $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \Bbbk^5 \otimes \Bbbk^2) \oplus \Bbbk^{5*} \oplus \Bbbk^{5(*)})$.
- $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.

In particular, there exist étale representations for groups with simple factors other than SL_n:

- $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.

1.2.2 New Results in this Thesis

Let G_1 and G_2 be simple groups and $1 \leq j \leq k$. We construct several examples of non-irreducible étale representations $(\mathrm{GL}_1^j \times G_1 \times G_2, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$ of type II (see section 11.4), which are listed in appendix B and labelled by KII. In particular, this list contains all étale representations with $j = k$, i.e. those where an independent scalar multiplication acts on each irreducible component V_i.

We present a few criteria on when a group admits an étale representation.

- We prove that *groups which do not admit non-trivial characters* $\chi : G \to \Bbbk^\times$ *do not admit linear rational étale representations* (as it is the case for semisimple or unipotent groups).

- Also, we show that *any non-irreducible modules for a reductive algebraic group must be composed of non-regular irreducible components*. From this, we conclude that a group $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$ does not admit any linear rational étale representations. We also conclude that a reductive group with one-dimensional centre admitting a non-irreducible étale representation may contain at most one simple factor $G \neq \mathrm{SL}_n$.

1.2.3 Open Questions

We conjecture that the examples of the étale representations of type II for $\mathrm{GL}_1^j \times G_1 \times G_2$, where G_1 and G_2 are simple and $1 \leq j \leq k$, is a complete list, but this remains to be shown (see remark 13.11 and the cases labelled by KII II and KII III in section 13.2).

Another question yet to be answered is whether there exist special modules for reductive groups whose semisimple part has a simple factor other than SL_n or Sp_2. For the case of a one-dimensional centre, it is even not clear if Sp_2 can appear as a simple factor.

More generally, the classification of non-irreducible étale representations for groups $\mathrm{GL}_1^k \times G$, where G is any semisimple group, is an open problem. A first step should be the classification for the case $k = 1$. Here, the investigation of which non-regular irreducible prehomogeneous modules appear as irreducible components can serve as a starting point.

1.2.4 Primary Sources

Of particular interest are those algebras with one-dimensional centre. The classification of étale representations for $\mathfrak{gl}_n$ has already been done by Baues [2] (cf. chapter 8), see also Burde [5].

For the case $\mathrm{GL}_1 \times G$, where G is semisimple, we can pick all irreducible étale representations from the classification by Sato and Kimura [28].

Kimura et al. [15] studied the prehomogeneity of modules $(\mathrm{GL}_1^k \times G, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$ where G is a simple group, see also theorem 11.2. For each of these modules, the generic isotropy subgroup is known and so we obtain étale representation by picking those modules with finite isotropy subgroup from this classification.

Furthermore, Kimura et al. [16], [17] studied the prehomogeneity of modules $(\mathrm{GL}_1^k \times G_1 \times G_2, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$, where G_1 and G_2 are simple groups, under the assumption that one independent scalar multiplication acts on each irreducible component. This assumption is a non-trivial simplification of the problem, especially for the modules studied in [17], as it is far from obvious if one of these modules could be prehomogeneous with less than k factors GL_1 acting on the module. The étale representations obtained from these articles are considered in proposition 13.5 in section 13.1 and in section 13.2.

1.3 Notation

In this section, we introduce some notation that is used throughout the text.

- We write V^m to denote some abstract vector space of finite dimension m.
- The unit element of a group G is denoted by 1 or 1_G. For matrix groups, we also use I_n to denote the identity matrix.
- When there is no ambiguity about the field $\mathbb{k}$, we will write GL_n instead of $\mathrm{GL}_n(\mathbb{k})$ (resp. SL_n, Sp_n, SO_n, Spin_n).
- The set of $m \times n$-matrices is denoted by $\mathrm{Mat}_{m,n}$. For quadratic matrices we write Mat_n rather than $\mathrm{Mat}_{n,n}$.
- The transpose of a matrix A is denoted by $A^\top$.
- We use the notation $\varrho(g).v$ for the action of a group element g on a vector v via the representation ϱ. When there is no ambiguity about the representation, we sometimes write $g.v$ instead. When $\varrho(g)$ is given by a matrix A, we use the usual notation Av for matrix multiplication.
- For convenience, we will often denote a module (ϱ, V) by the representation ϱ only. In this case, we also write $\dim(\varrho)$ for $\dim(V)$.
- For multiple direct sums or tensor products (of both vector spaces and matrices), we use the notation $V^{\oplus k} = \underbrace{V \oplus \ldots \oplus V}_{k \text{ times}}$ and $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$.
- The dual pairing of an element $v \in V$ with $v^* \in V^*$ is denoted by $\langle v | v^* \rangle$.
- For elements $x_1, \ldots, x_n$ of a vector space (group, algebra or ring, resp.), let $\langle x_1, \ldots, x_n \rangle$ denote the linear span (subgroup, ideal) generated by the x_i. If necessary, we write $\langle x_1, \ldots, x_n \rangle_{\mathbb{k}}$ and $\langle x_1, \ldots, x_n \rangle_{\mathfrak{g}}$ to distinguish between the linear span and the ideal generated by the x_i.

Part I

Preliminaries

2 Multilinear Algebra

At first, we will define tensor products for matrices, providing the coordinate version for some of the abstract definitions in the sections of this chapter.

Definition 2.1 The **direct sum** of matrices $A \in \mathrm{Mat}_m$ and $B \in \mathrm{Mat}_n$ is given by

$$A \oplus B = \begin{pmatrix} A & \\ & B \end{pmatrix} \in \mathrm{Mat}_{m+n}.$$

The **tensor product** of two matrices $A \in \mathrm{Mat}_{m,n}$ and $B \in \mathrm{Mat}_{p,q}$ is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathrm{Mat}_{mp,nq},$$

where the a_{ij} are the coefficients of A.

Remark 2.2 For matrices $A, C \in \mathrm{Mat}_m$ and $B, D \in \mathrm{Mat}_n$, we have:

- $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$.
- $(A \oplus C) \otimes B = (A \otimes B) \oplus (C \otimes B)$.
- $(A \otimes B) = (I_m \otimes B) \cdot (A \otimes I_n) = (A \otimes I_n) \cdot (I_m \otimes B)$.

These identities show that the definition of matrix tensor products is compatible with the abstract definition of tensor products of vector spaces and group representations defined later.

Remark 2.3 For $A \in \mathrm{Mat}_m$, $B \in \mathrm{Mat}_n$ and $v \in \Bbbk^m$, $w \in \Bbbk^n$, it is often convenient to identify $v \otimes w$ with

$$X = v \otimes w^\top = v \cdot w^\top \in \mathrm{Mat}_{m,n},$$

because then the action of $A \otimes B$ on $v \otimes w$ is given by $AXB^\top$.

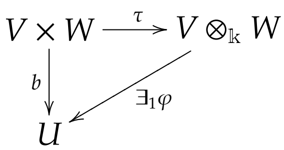
2.1 Tensor Algebra

In this section, we give a definition of the tensor algebra $\bigotimes V$ generated by a vector space V. This is the most general associative algebra over V in the sense that it satisfies the relations for associativity, but no other relations. It is constructed as the direct sum of the vector spaces $V^{\otimes k}$ generated by the products $v_1 \otimes \cdots \otimes v_k$ of k elements of V. Proofs for the propositions in this chapter can be found in appendix A of Knapp [20].

Definition 2.4 Let V and W be vector spaces over $\Bbbk$. The **tensor product** of V and W is a vector space $V \otimes_\Bbbk W$ together with a bilinear map

$$\tau : V \times W \to V \otimes_\Bbbk W, \quad (v, w) \mapsto v \otimes w$$

with the following universal property: For every bilinear map $b : V \times W \to U$, where U is some vector space over $\Bbbk$, there exists a unique surjective linear mapping $\varphi : V \otimes_\Bbbk W \to U$, such that $\varphi \circ \tau = b$, i.e. the diagramm



commutes. When there is no ambiguity about the field $\Bbbk$, we shall write $V \otimes W$ instead of $V \otimes_\Bbbk W$.

Remark 2.5 There exists a unique tensor product for V and W. With definition 2.4 one can define tensor products of more than two vector spaces inductively and show that $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ (see appendix A.1 in Knapp [20]). Further, the tensor product distributes over direct sums, $V \otimes (W \oplus U) = (V \otimes W) \oplus (V \otimes U)$.

Remark 2.6 For finite-dimensional vector spaces V and W with bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ a basis of $V \otimes W$ is given by

$$\{v_1 \otimes w_1, \ldots, v_1 \otimes w_n, \ \ldots, \ v_m \otimes w_1, \ldots, v_m \otimes w_n\},$$

so we have $\dim(V \otimes W) = mn$. For $V = \Bbbk^m$ and $W = \Bbbk^n$, $v_i \otimes w_j$ coincides with the matrix tensor product.

Definition 2.7 Let V_1, V_2, W_1 and W_2 be vector spaces over $\Bbbk$. For linear mappings $\varphi_1 : V_1 \to W_1$ and $\varphi_2 : V_2 \to W_2$, the **tensor product** of φ_1 and φ_2 is the unique map $\varphi_1 \otimes \varphi_2$ given by the universal property of $W_1 \otimes W_2$ such that

$$b(v_1, v_2) = (\varphi_1 \otimes \varphi_2) \circ \tau,$$

with a bilinear map $b : V_1 \times V_2 \to W_1 \otimes W_2, (v_1, v_2) \mapsto \varphi_1(v_1) \otimes \varphi_2(v_2)$.

Remark 2.8 With definition 2.7 one can define tensor products for more than two linear mappings inductively.

Remark 2.9 If A is a matrix representation of $\varphi_1 : V_1 \to W_1$, and B is a matrix representation of $\varphi_2 : V_2 \to W_2$, then $A \otimes B$ is a matrix representation of $\varphi_1 \otimes \varphi_2$.

Definition 2.10 Let V be a vector space over $\Bbbk$ and set $V^{\otimes 0} = \Bbbk$. We define the **tensor algebra** generated by V as

$$\bigotimes V = \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

Proposition 2.11 *The tensor algebra $\bigotimes V$ generated by V has the following universal property: Let $\iota : V \to \bigotimes V$ be the embedding of V in $\bigotimes V$. If $\varphi : V \to A$ is a linear map into an associative algebra A with identity, then there exists a unique algebra homomorphism $\Phi : \bigotimes V \to A$ with $\Phi(1) = 1$ and $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \bigotimes V \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle \exists_1 \Phi} & \\ A & & \end{array}$$

commutes.

The tensor algebra is often used to construct associative algebras by taking the quotient over some ideal in $\bigotimes V$ which represents the defining relations of the respective algebra.

2.2 Symmetric Algebra

In this section we construct a symmetric quotient algebra of $\bigotimes V$ by factoring out the ideal of alternating expressions in $\bigotimes V$. To this end, let

$$\mathfrak{A}^k = \langle v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \mid v_1, \ldots, v_k \in V, \sigma \in \mathrm{S}_k \rangle \subset V^{\otimes k}$$

be the ideal of alternating expressions in $V^{\otimes k}$.

Definition 2.12 The k-fold **symmetric product** of a vector space V is

$$\mathrm{Sym}^k V = V^{\otimes k} / \mathfrak{A}^k.$$

We write $v_1 \cdots v_k$ for the image of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Sym}^k V$.

$\mathrm{Sym}^k V$ can be embedded in $V^{\otimes k}$ via the map $v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Proposition 2.13 *The k-fold symmetric product $\mathrm{Sym}^k V$ has the following universal property: Let $\iota : V^{\oplus k} \to \mathrm{Sym}^k V$ be the map $\iota(v_1, \ldots, v_k) = v_1 \cdots v_k$. If*

$\varphi : V^{\oplus k} \to W$ is a symmetric k-linear map into a vector space W, then there exists a unique linear map $\Phi : \mathrm{Sym}^k V \to W$ with $\Phi \circ \iota = \varphi$, i.e. the diagramm

$$\begin{array}{ccc} V^{\oplus k} & \xrightarrow{\iota} & \mathrm{Sym}^k V \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ W & & \end{array}$$

commutes.

Remark 2.14 For $V = \mathbb{k}^n$, the 2-fold symmetric product $\mathrm{Sym}^2\mathbb{k}^n$ can be identified with the symmetric matrices via

$$v_1 v_2 \mapsto \frac{1}{2}(v_1 \cdot v_2^\top + v_2 \cdot v_1^\top).$$

Definition 2.15 Let $\mathfrak{A} = \bigoplus_{k=1}^{\infty} \mathfrak{A}^k$ and $\mathrm{Sym}^0 V = \mathbb{k}$. The **symmetric algebra** over V is defined as

$$\mathrm{Sym} V = (\bigotimes V)/\mathfrak{A} = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k V.$$

Proposition 2.16 *The symmetric algebra $\mathrm{Sym}V$ has the following universal property: Let $\iota : V \to \mathrm{Sym}V$ be the embedding of V in $\mathrm{Sym}V$. If $\varphi : V \to S$ is a linear map into an commutative associative algebra S with identity, then there exists a unique algebra homomorphism $\Phi : \mathrm{Sym}^k V \to S$ with $\Phi(1) = 1$ and $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \mathrm{Sym} V \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ S & & \end{array}$$

commutes.

Remark 2.17 For a finite-dimensional vector space V, the elements of a basis $\{v_1, \ldots, v_n\}$ are algebraically independent in $\mathrm{Sym}V$. It follows that $\mathrm{Sym}V$ can be identified with the polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$. The space $\mathrm{Sym}^k V$ corresponds to the space of homogeneous polynomials of degree k. We have $\dim(\mathrm{Sym}^k V) = \binom{n+k-1}{n-1}$.

Remark 2.18 For the dual V^* of a finite-dimensional vector space V, there is a canonical isomorphism $\mathrm{Sym}V^* \mapsto \mathbb{k}[x_1, \ldots, x_n]$ given by

$$(v_1^* \cdots v_k^*)(w_1, \ldots, w_k) = \sum_{\sigma \in S_k} v_1^*(w_{\sigma(1)}) \cdots v_k^*(w_{\sigma(k)})$$

with $v_i^* \in V^*$ and $w_j \in V$.

2.3 Exterior Algebra

In this section we construct an alternating quotient algebra of $\bigotimes V$ by factoring out the ideal of symmetric expressions in $\bigotimes V$. To this end, let

$$\mathfrak{S}^k = \langle v_1 \otimes \cdots \otimes v_k \mid v_1, \ldots, v_k \in V, \exists\, i \neq j : v_i = v_j \rangle \subset V^{\otimes k}$$

be the ideal of symmetric expressions in $V^{\otimes k}$.

Definition 2.19 The k-fold **exterior product** of a vector space V is

$$\bigwedge^k V = V^{\otimes k} / \mathfrak{S}^k.$$

We write $v_1 \wedge \cdots \wedge v_k$ for the image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge^k V$.

$\bigwedge^k V$ can be embedded in $V^{\otimes k}$ via the map $v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$.

Proposition 2.20 *The k-fold exterior product $\bigwedge^k V$ has the following universal property: Let $\iota : V^{\oplus k} \to \bigwedge^k V$ be the map $\iota(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k$. If $\varphi : V^{\oplus k} \to W$ is an alternating k-linear map into a vector space W, then there exists a unique linear map $\Phi : \bigwedge^k V \to W$ with $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V^{\oplus k} & \xrightarrow{\iota} & \bigwedge^k V \\ {\scriptstyle \varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ W & & \end{array}$$

commutes.

Remark 2.21 For $V = \Bbbk^n$, the 2-fold exterior product $\bigwedge^2 \Bbbk^n$ can be identified with the skew-symmetric matrices via

$$v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \cdot v_2^\top - v_2 \cdot v_1^\top).$$

Definition 2.22 Let $\mathfrak{S} = \bigoplus_{k=1}^{\infty} \mathfrak{S}^k$ and $\bigwedge^0 V = \Bbbk$. The **exterior algebra** over V is defined as

$$\bigwedge V = (\bigotimes V)/\mathfrak{S} = \bigoplus_{k=0}^{\infty} \bigwedge^k V.$$

Proposition 2.23 *The exterior algebra $\bigwedge V$ has the following universal property: Let $\iota : V \to \bigwedge V$ be the embedding of V in $\bigwedge V$. If $\varphi : V \to A$ is a linear map into an associative algebra A with identity, then there exists a unique algebra homomorphism $\Phi : \bigwedge^k V \to A$ with $\Phi(1) = 1$ and $\Phi \circ \iota = \varphi$, i.e. the diagramm*

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \bigwedge V \\ {\scriptstyle \varphi}\downarrow & \swarrow {\scriptstyle \exists_1 \Phi} & \\ A & & \end{array}$$

commutes.

Remark 2.24 For an n-dimensional vector space V, we have $\dim(\bigwedge^k V) = \binom{n}{k}$.

Remark 2.25 Let $n = \dim(V)$. Then $\bigwedge^k V^*$ is canonically isomorphic to $(\bigwedge^k V)^*$ by

$$\langle w_1, \ldots, w_k | v_1^* \wedge \cdots \wedge v_k^* \rangle = \det\Big((\langle v_i^* | w_j \rangle)_{ij} \Big)$$

with $v_i^* \in V^*$ and $w_j \in V$.

Remark 2.26 Let $n = \dim(V)$ and $b_1, \ldots, b_n$ be a basis of V. Then $\bigwedge^{n-k} V$ can be identified with $(\bigwedge^k V)^*$ by

$$\langle v_1 \wedge \cdots \wedge v_k | v_{k+1} \wedge \cdots \wedge v_n \rangle b_1 \wedge \cdots \wedge b_n, = (v_1 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge \cdots \wedge v_n) \in \bigwedge^n V \cong \mathbb{k}$$

with $v_i \in V$.

3 Algebraic Geometry

In this chapter, some basic knowledge about groups, rings and ideals is assumed. A brief introduction to affine algebraic geometry can be found in Kraft's notes [22]. The book by Harris [12] provides a fine introduction to classical algebraic geometry, with emphasis on the geometric aspect. Contrary to this, Hartshorne [13] gives a modern presentation of the subject, emphasising the algebraic concepts.

3.1 Commutative Algebra and Field Extensions

In this section, R is a commutative ring with 1 and $\mathbb{k}$ a field of characteristic 0.

Definition 3.1 R is called **Noetherian** if every Ideal in R is finitely generated.

Proposition 3.2 *If $\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \ldots$ is an ascending chain of ideals in a Noetherian ring R, then this chain is stationary, i.e. for some $n \in \mathbb{N}$ we have $\mathfrak{I}_n = \mathfrak{I}_m$ for $m > n$.*

Theorem 3.3 (Hilbert's Basis Theorem) *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

Definition 3.4 Let $\mathfrak{I} \subseteq R$ be an ideal. Then $\mathfrak{I}$ is called

- **maximal**, if $R/\mathfrak{I}$ is a field (or equivalently: there is no ideal $\mathfrak{J}$ with $\mathfrak{I} \subsetneq \mathfrak{J} \subsetneq R$).
- **prime**, if $R/\mathfrak{I}$ is an integral domain (or equivalently: if $ab \in \mathfrak{I}$, then $a \in \mathfrak{I}$ or $b \in \mathfrak{I}$ holds).
- **primary**, if $ab \in \mathfrak{I}$ and $a \notin \mathfrak{I}$ implies $b^n \in \mathfrak{I}$ for some $n \in \mathbb{N}$.

Definition 3.5 Let $\mathfrak{I} \subseteq R$ an ideal. Then the ideal

$$\sqrt{\mathfrak{I}} = \{a \in R \mid \exists\, n \in \mathbb{N} \text{ with } a^n \in \mathfrak{I}\}$$

is called the **radical** of $\mathfrak{I}$. If $\mathfrak{I} = \sqrt{\mathfrak{I}}$, then $\mathfrak{I}$ is called **radical**.

Lemma 3.6 *If $\mathfrak{I} \subsetneq R$ is a primary ideal, then $\sqrt{\mathfrak{I}}$ is prime.*

Example 3.7 For the ideal $\langle 4\rangle \subset \mathbb{Z}$, the radical is $\sqrt{\langle 4\rangle} = \langle 2\rangle$. For $\langle x^2, y\rangle \subset \mathbb{k}[x, y]$, the radical is $\sqrt{\langle x^2, y\rangle} = \langle x, y\rangle$.

Definition 3.8 Let $S \subset R$ and define the **localisation** $S^{-1}R$ of R at S to be the ring formed by the equivalence classes of fractions $\frac{r}{s}$ with $r \in R$, $s \in S$, where $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ are equivalent if there exists $t \in S$ such that

$$t(s_1 r_2 - s_2 r_1) = 0.$$

Remark 3.9 The localisation of R at $R\backslash\{0\}$ is a field called the **quotient field** of R.

Definition 3.10 Let $\mathbb{K}$ be a field and $\mathbb{k} \subseteq \mathbb{K}$. Then $\mathbb{K}$ is called a **field extension** of $\mathbb{k}$. If M is a subset of $\mathbb{K}$, then $\mathbb{k}(M)$ is the smallest field extension of $\mathbb{k}$ containing M. If M is finite, then the field extension is called **finitely generated**.

Definition 3.11 Let $\mathbb{K}$ be a field extension of $\mathbb{k}$. Then $a \in \mathbb{K}$ is called **algebraic**, if there exists $f \in \mathbb{k}[x]$ with $f(a) = 0$. If a is not algebraic, it is called **transcendent**. If all $a \in \mathbb{K}$ are algebraic, then the field extension is called an **algebraic extension**, otherwise it is called a **transcendent extension**.

Definition 3.12 If $\mathbb{k}$ is field such that every non-constant polynomial $f \in \mathbb{k}[x]$ has a zero in $\mathbb{k}$, then $\mathbb{k}$ is called **algebraically closed**.

Example 3.13 The field of complex numbers $\mathbb{C}$ is algebraically closed.

Definition 3.14 Let $\mathbb{K}$ be a field extension of $\mathbb{k}$ and $a_1, \dots, a_n \in \mathbb{K}$. The $a_1, \dots, a_n$ are called **algebraically independent**, over $\mathbb{k}$ if $f(a_1, \dots, a_n) = 0$ for $f \in \mathbb{k}[x_1, \dots, x_n]$ implies $f = 0$, i.e. algebraically independent elements do not satisfy any algebraic equations over $\mathbb{k}$. A maximal (with respect to the inclusion) subset $T \subset \mathbb{K}$ such that every finite subset of T is algebraically independent is called a **transcendence basis**.

Definition 3.15 Let $\mathbb{K}$ be a field extension of $\mathbb{k}$. Then

$$[\mathbb{K} : \mathbb{k}] = \sup\{n \in \mathbb{N}_0 \mid \exists\, a_1, \dots, a_n \text{ algebraically independent over } \mathbb{k}\}$$

is called the **algebraic degree** of the field extension, and

$$\operatorname{trdeg}_{\mathbb{k}}(\mathbb{K}) = \operatorname{card}(T)$$

for a transcendence basis T of $\mathbb{K}$ is called the **transcendence degree** of the field extension.

Proposition 3.16 *Let $\Bbbk_3$ be a field extension of $\Bbbk_2$, and $\Bbbk_2$ a field extension of $\Bbbk_1$. Then we have*

$$[\Bbbk_3 : \Bbbk_1] = [\Bbbk_3 : \Bbbk_2] \cdot [\Bbbk_2 : \Bbbk_1],$$
$$\operatorname{trdeg}_{\Bbbk_1}(\Bbbk_3) = \operatorname{trdeg}_{\Bbbk_2}(\Bbbk_3) + \operatorname{trdeg}_{\Bbbk_1}(\Bbbk_2).$$

3.2 Algebraic Sets and Varieties

To keeps things simple, we assume the field $\Bbbk$ to be algebraically closed. The space $\Bbbk^n$ is regarded as the n-dimensional affine space.

In algebraic geometry, the geometric object to be studied are zero sets of polynomials in $\Bbbk[x_1, \ldots, x_n]$.

Definition 3.17 An **algebraic set** $X \subseteq \Bbbk^n$ is the common zero set of a collection of polynomials $F \subseteq \Bbbk[x_1, \ldots, x_n]$,

$$X = \mathscr{Z}(F) = \{p \in \Bbbk^n \mid f(p) = 0 \text{ for all } f \in F\}.$$

Obviously, if X is the zero set of the polynomials in F, then it is also the zero set of the polynomials in the ideal generated by F, i.e. $X = \mathscr{Z}(\langle F \rangle)$.

Proposition 3.18 *The union of finitely many algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

This proposition allows us to define a topology on $\Bbbk^n$.

Definition 3.19 The **Zariski topology** on $\Bbbk^n$ is the topology whose closed sets are the algebraic sets.

In the following, all topological terms refer to the Zariski topology.

Definition 3.20 A non-empty subset Y of a topological space X is called **irreducible** if it cannot be express as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y.

Definition 3.21 An **affine algebraic variety** is an irreducible closed subset of $\Bbbk^n$. An open subset of an affine variety is a **quasi-affine variety**.

Proposition 3.22 *Every algebraic set can be expressed uniquely as a union of affine varieties, no one containing another.*

Remark 3.23 Any non-empty open subset of an irreducible subset in $\Bbbk^n$ is dense. Hence, an open subset Y_0 of an affine variety Y is a dense subset of Y. Intuitively, this means that Y_0 contains “almost everything” of Y, except for some “thin” subset.

For any subset Y of $\Bbbk^n$, we can define the ideal

$$\mathfrak{I}(Y) = \{f \in \Bbbk[x_1, \dots, x_n] \mid f(y) = 0 \text{ for all } y \in Y\}$$

Now we have a function $\mathscr{Z}$ mapping subsets of $\Bbbk[x_1, \dots, x_n]$ to algebraic sets, and a function $\mathfrak{I}$ mapping subsets of $\Bbbk^n$ to ideals.

Proposition 3.24

1. *Let $A \subseteq B \subseteq \Bbbk[x_1, \dots, x_n]$. Then $\mathscr{Z}(B) \subseteq \mathscr{Z}(A)$.*
2. *Let $Y \subseteq X \subseteq \Bbbk^n$. Then $\mathfrak{I}(X) \subseteq \mathfrak{I}(Y)$.*
3. *For $X, Y \subseteq \Bbbk^n$, we have $\mathfrak{I}(X \cup Y) = \mathfrak{I}(X) \cap \mathfrak{I}(Y)$.*
4. *For any ideal $\mathfrak{H} \subseteq \Bbbk[x_1, \dots, x_n]$, we have $\mathfrak{I}(\mathscr{Z}(\mathfrak{H})) = \sqrt{\mathfrak{H}}$.*
5. *For any subset $X \subseteq \Bbbk^n$, we have $\mathscr{Z}(\mathfrak{I}(X)) = \mathrm{clos}(X)$, the Zariski closure of X.*

Theorem 3.25 (Hilbert's Nullstellensatz) *Let $\Bbbk$ be an algebraically closed field, $\mathfrak{I}$ an ideal in $\Bbbk[x_1, \dots, x_n]$ and f a polynomial vanishing at all points in $\mathscr{Z}(\mathfrak{I})$. Then $f^k \in \mathfrak{I}$ for some $k \in \mathbb{N}$.*

Corollary 3.26 *There is a bijective inclusion-reversing correspondence between algebraic sets in $\Bbbk^n$ and radical ideals in $\Bbbk[x_1, \dots, x_n]$, given by $X \mapsto \mathfrak{I}(X)$ and $\mathfrak{H} \mapsto \mathscr{Z}(\mathfrak{H})$. Furthermore, an algebraic set is irreducible if and only if its ideal is prime.*

Example 3.27 Examples for irreducible algebraic sets are $\Bbbk^n = \mathscr{Z}(0)$, or the zero set $\mathscr{Z}(f)$ of any irreducible polynomial f. A point $p = (p_1, \dots, p_n) \in \Bbbk^n$ is an irreducible algebraic set, and the corresponding ideal is a maximal ideal $\mathfrak{M}_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle$.

Definition 3.28 The **affine coordinate ring** of an affine algebraic set $X \subseteq \Bbbk^n$ is

$$\Bbbk[X] = \Bbbk[x_1, \dots, x_n]/\mathfrak{I}(X).$$

Proposition 3.29 *If X is an affine variety, then $\Bbbk[X]$ is an integral domain and a finitely generated $\Bbbk$-algebra. Conversely, any integral domain which is a finitely generated $\Bbbk$-algebra is the coordinate ring of some affine variety.*

3.3 Regular Functions and Morphisms

Definition 3.30 Let X be a (quasi-)affine variety. A function $f : X \to \Bbbk$ is called a **regular function** if for every $x \in X$ there exist polynomials $g, h \in \Bbbk[x_1, \dots, x_n]$ and an open subset $U_x \subset X$ with $x \in U_x$ such that $h(y) \neq 0$ and $f(y) = \frac{g(y)}{h(y)}$ for all $y \in U_x$. The ring of rational functions on X is denoted by $\mathscr{O}(X)$. Its quotient field, the field of **rational functions**, is denoted by $\Bbbk(X)$.

Proposition 3.31 *If X is an affine variety, then the ring of regular functions $\mathscr{O}(X)$ is isomorphic to the coordinate ring $\Bbbk[X]$.*

Due to this proposition and the fact that we consider affine algebraic sets only, we can use $\mathscr{O}(X)$ and $\Bbbk[X]$ interchangably.

Proposition 3.32 *Regular functions are continuous with respect to the Zariski topology.*

Definition 3.33 Let X and Y be algebraic sets. A continuous mapping $\varphi : X \to Y$ is called a **morphism** if for all $f \in \Bbbk[Y]$ the composition $f \circ \varphi$ is a regular function, $f \circ \varphi \in \Bbbk[X]$. If there exists a morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$, then φ is called an **isomorphism**, and X and Y are called **isomorphic**. For a morphism $\varphi : X \to Y$, we call the map φ^* given by

$$\varphi^* : \Bbbk[Y] \to \Bbbk[X], \quad f \mapsto f \circ \varphi$$

the **comorphism** of φ.

Definition 3.34 Two (quasi-)affine varieties are called **birationally equivalent** if there exist open subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that X_0 and Y_0 are isomorphic.

Definition 3.35 A morphism $\varphi : X \to Y$ is **dominant** if $\varphi(X)$ is dense in Y.

Remark 3.36 It can be shown that X and Y are birationally equivalent if and only if $\Bbbk(X) \cong \Bbbk(Y)$.

Remark 3.37 Let $x_1, \dots, x_n$ denote the coordinate functions on $\Bbbk^n$ and let X be any affine variety and Y be an affine variety in $\Bbbk^n$. The fact that a map $\varphi : X \to Y$ is a morphism is equivalent to saying that φ is regular in each coordinate, i.e. $x_i \circ \varphi \in \Bbbk[X]$ for all i.

Proposition 3.38 *Let $X \subseteq \Bbbk^n$ be an affine variety. Then:*

1. *The affine coordinate ring of X is the ring of regular functions on X, i.e. $\Bbbk[X] = \Bbbk[x_1, \dots, x_n]/\mathfrak{I}(X)$.*
2. *For each $x \in X$, there is a maximal ideal $\mathfrak{M}_x \subset \Bbbk[X]$ of functions vanishing at x. Then $x \mapsto \mathfrak{M}_x$ gives a bijective correpondence between points on X and maximal ideals in $\Bbbk[X]$.*

Example 3.39 The ring of regular functions on $\Bbbk^n$ is the polynomial ring $\Bbbk[\Bbbk^n] = \Bbbk[x_1, \dots, x_n]$.

Proposition 3.40 *The correspondence*

$$\varphi \mapsto \varphi^*$$

gives a natural bijection between the morphisms of affine varieties $X \to Y$ and the homomorphisms of rings $\Bbbk[Y] \to \Bbbk[X]$. In particular, X and Y are isomorphic if and only if $\Bbbk[Y]$ and $\Bbbk[X]$ are isomorphic.

3.4 Dimension and Tangent Spaces

Definition 3.41 Let $X \subseteq \mathbb{k}^n$ be an affine variety. The **dimension** of X is defined to be

$$\dim(X) = \operatorname{trdeg}_{\mathbb{k}}(\mathbb{k}(X)).$$

For an algebraic set $Y = Y_1 \cup \ldots \cup Y_k \subseteq \mathbb{k}^n$ with irreducible components $Y_1, \ldots, Y_k$, define

$$\dim(Y) = \max_i \{\dim(Y_i)\},$$

For any $y \in Y$ define the **local dimension** by

$$\dim_y(Y) = \max_{i: y \in Y_i} \{\dim(Y_i)\}.$$

This definition is based on the intuition that the transcendce degree corresponds to the degrees of freedom on a variety.

The **codimension** of a closed subset Y of X is

$$\operatorname{codim}(Y) = \dim(X) - \dim(Y).$$

A **hypersurface** S in $\mathbb{k}^n$ is a Zariski closed subset of $\mathbb{k}^n$ such that S does not contain any irreducible component of codimension > 1.

Proposition 3.42 *Let S be a closed subset of $\mathbb{k}^n$. Then S is a hypersurface if and only if $\mathbb{k}^n \backslash S$ is an affine variety.*

Remark 3.43 The dimension function $x \mapsto \dim_x(X)$ is upper-semicontinuous on X, i.e. for any $r \in \mathbb{R}$, the set $\{x \in X \mid \dim_x(X) < r\}$ is open in X.

Lemma 3.44 *If X is an affine variety and $Y \subsetneq X$ a closed subset, then* $\dim(Y) < \dim(X)$.

Proposition 3.45 *Let X and Y be affine varieties and $\varphi : X \to Y$ a dominant morphism. For $y \in \varphi(X)$, let F be an irreducible component of the fibre $\varphi^{-1}(y)$. Then we have*

$$\dim(F) \geq \dim(X) - \dim(Y).$$

Further, there exists an open subset $U \subseteq \varphi(X)$ such that for any $y \in U$ each irreducible component of $\varphi^{-1}(y)$ has dimension $\dim(X) - \dim(Y)$.

As every point p in an affine variety satisfies certain polynomial equations $f(p) = 0$, one would expect a vector v to be tangent to this variety if the value of f does not change in the direction of v, i.e. $(\operatorname{grad} f)_p(v) = 0$. This motivates the following definition.

Definition 3.46 Let X be an algebraic set, and let $\langle f_1, \dots, f_r\rangle = \mathfrak{I}(X)$. The **tangent space** at $p \in X$ is

$$\mathrm{T}_p X = \left\{ (v_1, \dots, v_n) \in \Bbbk^n \;\middle|\; \sum_{i=1}^{n} v_i \frac{\partial f_k(p)}{\partial x_i} = 0 \text{ for } k = 1, \dots, r \right\}.$$

Remark 3.47 There are some equivalent definitions of tangent spaces. For example, by the above definition, the tangent space at p is the kernel of the Jacobi matrix of $f = (f_1, \dots, f_k)$,

$$\mathrm{Jac}_p(f) = \left(\frac{\partial f_i(p)}{\partial x_j} \right)_{i,j}.$$

Alternatively, a vector v can be identified with the differential operator ∂_v mapping a function f to the directional derivative in the direction of v. Then the tangent space at p is the space of **derivations** of $\mathcal{O}(X)$, i.e. the linear maps $D : \mathcal{O}(X) \to \Bbbk$ satisfying

$$D(fg) = f(p)D(g) + g(p)D(f).$$

See chapter 4 in Kraft [22] for more background on this definition. A third way to define the tangent space is to define its dual space first via

$$\mathrm{T}_p^* X = \mathfrak{M}_p / \mathfrak{M}_p^2,$$

where $\mathfrak{M}_p$ is the maximal ideal of functions vanishing at the point p, and then define $\mathrm{T}_p X = (\mathfrak{M}/\mathfrak{M}_p^2)^*$. See chapter 16 in Tauvel, Yu [30] for more background on this definition.

Remark 3.48 If $\dim(X) = m$, then the rank of the Jacobi matrix of $(f_1, \dots, f_k)$ is at most $n - m$ at any point $p \in X$. We have

$$\dim(\mathrm{T}_p X) \geq \dim_p(X).$$

Definition 3.49 Let X be an algebraic set and $p \in X$. If $\dim(\mathrm{T}_p X) = \dim_p(X)$, then the point p is called **smooth**, and if all points are smooth, then X is called **smooth**. The points $q \in X$ with $\dim(\mathrm{T}_q X) > \dim_q(X)$ are called **singular points**.

Proposition 3.50 *Let X be an affine variety. Then the set of singular points is a proper closed subset of X whose complement is open and dense in X.*

Definition 3.51 Let X and Y be algebraic sets and $\varphi : X \to Y$ a morphism. Then the **differential** $\mathrm{d}\varphi_x$ of φ at x is defined by

$$\mathrm{d}\varphi_x : \mathrm{T}_x X \to \mathrm{T}_{\varphi(x)} Y, \quad D \mapsto D \circ \varphi^*,$$

where we consider the tangent vectors as derivations.

The usual chain rule holds for the differential.

Proposition 3.52 *Let X, Y and Z be algebraic sets and $\varphi : X \to Y$, $\psi : Y \to Z$ morphisms. Then the differential of $\psi \circ \varphi$ at $x \in X$ is given by*

$$\mathrm{d}(\psi \circ \varphi)_x = \mathrm{d}\psi_{\varphi(x)} \circ \mathrm{d}\varphi_x.$$

Example 3.53 Let $\varphi : \Bbbk^n \to \Bbbk^m$ be a morphism. In coordinate representation, the differential $\mathrm{d}\varphi_x$ is given by the Jacobi matrix

$$\mathrm{Jac}_x(\varphi) = \left(\frac{\partial \varphi_i(x)}{\partial x_j}\right)_{i,j},$$

where the φ_i are the coordinate functions of φ.

Proposition 3.54 *Let $\varphi : X \to Y$ be a morphism, $x \in X$ and $F = \varphi^{-1}(\varphi(x))$ the fibre through x. Then $\mathrm{T}_x F \subset \ker(\mathrm{d}\varphi)$. Further, if x is smooth and $\mathrm{d}\varphi_x$ is surjective, then Y is smooth in $\varphi(x)$ and $\dim_x(F) = \dim_x(X) - \dim_{\varphi(x)}(Y)$.*

3.5 Normal Varieties

Definition 3.55 Let $S \subseteq R$ be rings. An element $r \in R$ is **integral** over S if r satisfies a polynomial equation over S, i.e. there exists $f \in S[x]$ such that $f(r) = 0$.

Definition 3.56 Let R be an integral domain and $\mathbb{K} = \mathrm{Quot}(R)$. Then R is called **integrally closed** if every integral element $x \in \mathbb{K}$ over R is already contained in R.

Definition 3.57 Let X be an affine variety. Then X is called **normal** if $\mathscr{O}(X)$ is integrally closed.

Example 3.58 The affine space $\Bbbk^n$ is normal.

Proposition 3.59 *Let X be a normal affine variety. Let S be the subset of singular points in X. Then $\mathrm{codim}(S) \geq 2$.*

Corollary 3.60 *Let X be a normal affine variety and $\dim(X) = 1$. Then X is smooth.*

3.6 Grassmann Varieties

An important structure in algebraic geometry is the **Grassmann variety** $\mathrm{Gr}_k(V)$ of k-dimensional subspaces of an n-dimensional vector space V. Technically speaking, Grassmann varieties are not affine varieties, but *projective varieties*, which we do not treat here.

If U is a k-dimensional subspace of V, it is determined by vectors $v_1, \ldots, v_k$ spanning U. We can associate to U the exterior product

$$v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V.$$

Then $v_1 \wedge \cdots \wedge v_k$ is determined by U up to a scalar factor. Thus, we can identify U with the equivalence class of scalar multiples of $v_1 \wedge \cdots \wedge v_k$, and $\mathrm{Gr}_n(V)$ is defined to be the whole of these equivalence classes.

Grassmann varieties are treated in detail in chapter 6 of Harris [12].

4 Algebraic Groups and Lie Algebras

In this chapter, we summarise the background on affine algebraic groups with a focus on reductive groups. The ground field $\Bbbk$ is of characteristic 0, but not necessarily algebraically closed. For an exhaustive treatment of algebraic groups and all algebraic preliminaries, see Tauvel, Yu [30].

4.1 Algebraic Groups and Homomorphisms

Definition 4.1 An **affine algebraic group** is a group endowed with the structure of an algebraic set, such that the inversion $g \mapsto g^{-1}$ and the multiplication $(g, h) \mapsto gh$ are morphisms of algebraic sets.

Although there exists algebraic groups which are not affine, they will play no part in this thesis, so from now on we use the term "algebraic group" in the sense of "affine algebraic group".

Definition 4.2 A **morphism of algebraic groups** is a group homomorphism which is also a morphism of algebraic varieties.

Isomorphisms and automorphisms of algebraic groups are defined in an obvious way.

Proposition 4.3 *Let G be an algebraic group. Then there exists a unique irreducible component $G^\circ \subseteq G$ which contains the identity 1_G. Further, G° is a subgroup of G.*

The irreducible component G° containing 1_G is called the **connected component** of G. This naming is justified by the following proposition.

Proposition 4.4 *For an algebraic group G, the following are equivalent:*

1. *G is connected.*
2. *G is irreducible (i.e. an affine variety).*
3. *Each closed normal subgroup of finite index in G is G itself.*

Remark 4.5 We shall use the term *connected* algebraic group instead of irreducible algebraic group. As the set of smooth points in G is not empty, every point can be shown to be smooth as it is the image of a smooth point under multiplication with some element of G.

Proposition 4.6 *Let $\varphi : G \to H$ be a morphism of algebraic groups. Then the kernel and the image of φ are closed subgroups of G resp. H. Furthermore, $\varphi(G^\circ) = \varphi(G)^\circ$ and*

$$\dim(G) = \dim(\ker(\varphi)) + \dim(\varphi(G)).$$

4.2 Structure of Algebraic Groups

Definition 4.7 Let G be an algebraic group and H a subgroup of G. Then the subgroup

$$\mathrm{N}_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

is called the **normaliser** of H in G. The subgroup

$$\mathrm{Z}_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}$$

is called the **centraliser** of H in G. The subgroup commuting with all elements of G is $\mathrm{Z}(G) = \mathrm{Z}_G(G)$, the **centre** of G.

Definition 4.8 If G and H are algebraic groups, their Cartesian product $G \times H$ becomes an algebraic group by setting $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. A more general version of this is the **semidirect product** $G \ltimes H$, where the group product is defined by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2,\ h_1(g_1.h_2)),$$

where the action of g_1 on h_2 is defined via some automorphism of H.

In this thesis, semidirect products appear as sets of the form $G \cdot H$, where G and H are matrix groups, so we will also use the notation $G \cdot H$ to denote semidirect products.

Definition 4.9 A connected algebraic group G is called **simple**[1)] if every normal subgroup is finite and $\dim(G) \geq 3$. An algebraic group G is called **semisimple** if its connected component G° is the direct product of simple groups up to a finite subgroup of the centre, i.e.

$$G^\circ = (G_1 \times \cdots \times G_k)/H,$$

where $H \subseteq \mathrm{Z}(G)$ is finite and each G_i is simple.

[1)]Some authors use the term *almost simple* instead, and *simple* only for groups without nontrivial normal subgroups.

Definition 4.10 Let G be an algebraic group. Define the **commutator subgroup** $[G, G]$ as the subgroup generated by the elements

$$[g, h] = ghg^{-1}h^{-1}$$

for $g, h \in G$. Define a sequence $G^{(k)}$ by

$$G^{(0)} = G, \quad G^{(k)} = [G^{(k-1)}, G^{(k-1)}] \text{ for } k > 0.$$

Then G is called **solvable** if there exists an integer n such that $G^{(n)} = \{1_G\}$.

Definition 4.11 Let G be an algebraic group. The **radical** $\mathrm{Rad}(G)$ of G is the largest connected normal solvable subgroup of G.

Definition 4.12 An algebraic group G is called a **torus** if it is isomorphic to the group of diagonal $n \times n$-matrices for some $n \in \mathbb{N}$.

Remark 4.13 For algebraically closed $\mathbb{k}$, a torus is isomorphic to a product of copies of GL_1.

4.3 Lie Algebras of Algebraic Groups

Lie algebras can be thought of as an infinitesimal version of algebraic groups. They usefulness arises from the fact that many problems for algebraic groups can be reformulated as linear problems for Lie algebras. A fine introduction to Lie algebras (albeit in the context of Lie groups) is given by Hall [11].

Definition 4.14 Let V be a vector space over $\mathbb{k}$. A bilinear product $[\cdot, \cdot] : V \times V \to V$ is called **Lie bracket** if $[x, x] = 0$ for all $x \in V$ and if the **Jacobi identity**

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

is satisfied for all $x, y, z \in V$. In this case, $(V, [\cdot, \cdot])$ is called a **Lie algebra**.

We use small German letters to denote Lie algebras, $\mathfrak{g} = (V, [\cdot, \cdot])$.

Example 4.15 The quadratic matrices with the commutator product

$$[x, y] = xy - yx$$

form a Lie algebra denoted by $\mathfrak{gl}_n$.

Definition 4.16 Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ with $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$ is a **Lie subalgebra**. If $[x, y] \in \mathfrak{h}$ even holds for all $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, then $\mathfrak{h}$ is an **ideal**.

Remark 4.17 For two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, the direct sum of vector spaces becomes a Lie algebra if we define $[x, y] = 0$ for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$.

Definition 4.18 A Lie algebra $\mathfrak{g}$ is called **simple** if the only ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$ itself. It is called **semisimple** if $\mathfrak{g}$ can be written as the direct sum of simple Lie algebras.

Proposition 4.19 *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ be a semisimple Lie algebra with simple components $\mathfrak{g}_i$. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if and only if*

$$\mathfrak{h} = \mathfrak{g}_{i_1} \oplus \ldots \oplus \mathfrak{g}_{i_k}$$

for some $i_1, \ldots, i_k \in \{1, \ldots, m\}$ (where the i_j are pairwise distinct).

Homomorphisms, isomorphisms and automorphisms for Lie algebras are defined in an obvious way, via $\psi([x, y]) = [\psi(x), \psi(y)]$. As for every algebra, the kernel of a homomorphism is an ideal.

We will now define the Lie algebra of an algebraic group. For simplicity, we give the definition for matrix groups only, as more general definition would require excessive preparations. See chapter 23 of Tauvel, Yu [30] or section I.3 in Borel [4] for a general definition.

Proposition 4.20 *Let $G \subseteq \mathbb{k}^{n^2}$ be an algebraic (matrix) group. Let $\mathfrak{Lie}(G)$ denote the tangent space $\mathrm{T}_{1_G}G$ of G at the identity, which can be identified with a subset of $\mathfrak{gl}_n (= \mathrm{Mat}_n)$. Then $\mathfrak{Lie}(G)$ is a Lie algebra, where the Lie bracket is given by the commutator of matrices*

$$[X, Y] = XY - YX$$

for $X, Y \in \mathfrak{gl}_n$.

Theorem 4.21 *Let $\varphi : G \to H$ be a homomorphism of algebraic groups. Then the differential of $\mathrm{d}\varphi_{1_G} : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ at the identity is a homomorphism of Lie algebras.*

Remark 4.22 We speak of a **local homomorphism** φ of algebraic groups G and H of there is an open connected subset $1_G \in U \subseteq G$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in U$. If two groups are locally isomorphic, then their Lie algebras are isomorphic.

Whenever a Lie algebra homomorphism arises as the differential $\mathrm{d}\varphi_{1_G}$ of a group homomorphism φ, we omit the index 1_G and just write $\mathrm{d}\varphi$.

Proposition 4.23 *An algebraic group G is semisimple if and only if its Lie algebra $\mathfrak{Lie}(G)$ is semisimple.*

Definition 4.24 The **commutator subalgebra** $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is the subalgebra spanned by all elements $[X, Y]$ with $X, Y \in \mathfrak{g}$. Define a sequence $\mathfrak{g}^{(k)}$ by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \text{ for } k > 0.$$

Then $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^{(n)} = \{0\}$ for some $n \in \mathbb{N}$.

Remark 4.25 There exists a unique maximal sovable subalgebra of $\mathfrak{g}$. It is called the **radical** $\mathfrak{rad}(\mathfrak{g})$ of $\mathfrak{g}$.

Semisimplicity and solvability can be seen as opposite extremes in the structure of Lie algebras.

Proposition 4.26 *If $\mathfrak{g}$ is a semisimple Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (or $\mathfrak{rad}(\mathfrak{g}) = \{0\}$). Accordingly, if G is a semisimple group, then $[G, G] = G$ (or $\mathrm{Rad}(G) = \{1_G\}$).*

Similarly as in the group case, there are some special subalgebras.

Definition 4.27 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{h}$. Then the subalgebra

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

is called the **normaliser** of $\mathfrak{h}$ in $\mathfrak{g}$. The subalgebra

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid XY = YX \text{ for all } Y \in \mathfrak{h}\}$$

is called the **centraliser** of $\mathfrak{h}$ in $\mathfrak{g}$. The ideal commuting with all elements of $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$, the **centre** of $\mathfrak{g}$.

Remark 4.28 If $\mathfrak{g} = \mathfrak{Lie}(G)$ for some algebraic group G, and H is a subgroup of G with $\mathfrak{h} = \mathfrak{Lie}(H)$, then

$$\mathfrak{Lie}(\mathrm{Z}_G(H)^\circ) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}),$$
$$\mathfrak{Lie}(\mathrm{N}_G(H)^\circ) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}).$$

And if H is a normal subgroup of G, then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

4.4 Dramatis Personae

In this thesis, we will mostly be concerned with subgroups of the **general linear group**,

$$\mathrm{GL}_n(\mathbb{k}) = \{g \in \mathrm{Mat}_n(\mathbb{k}) \mid \det(g) \neq 0\},$$

the group of invertible matrices. As GL_n can be considered as the complement of the closed set of singular matrices in $\mathbb{k}^{n^2}$, it is an affine variety by proposition 3.42, and as such, it is an algebraic group. Its Lie algebra is the set of $n \times n$-matrices, $\mathfrak{gl}_n$.

Definition 4.29 A **linear algebraic group** G is an algebraic group which is a subgroup of GL_n.

Equivalently, linear algebraic groups are the subgroups of GL_n defined by certain polynomial equations.

4.4.1 SL_n - The Hero of the Play

The **special linear group** is the group of unimodular matrices,

$$\mathrm{SL}_n(\mathbb{k}) = \{g \in \mathrm{GL}_n(\mathbb{k}) \mid \det(g) = 1\}$$

with Lie algebra

$$\mathfrak{sl}_n(\mathbb{k}) = \{X \in \mathfrak{gl}_n(\mathbb{k}) \mid \mathrm{tr}(X) = 0\}.$$

Its dimension is

$$\dim(\mathrm{SL}_n) = n^2 - 1.$$

This group is connected and simple (for $n \geq 2$) and its centre is a finite subgroup isomorphic to the group of n-th roots of unity in $\mathbb{k}$.

For $\mathbb{k} = \mathbb{R}$, the elements of SL_n can be interpreted geometrically as those linear transformations preserving volume and orientation.

In the course of this thesis, we will be mostly concerned with SL_n, as most linear étale representations (see chapter 7) arise as representations of SL_n.

4.4.2 Sp_n - The Hero's Sidekick

Define the matrix J by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}.$$

The **symplectic group**[2)] is the group

$$\mathrm{Sp}_n(\mathbb{k}) = \{g \in \mathrm{GL}_{2n}(\mathbb{k}) \mid g^\top J g = J\}$$

with Lie algebra

$$\mathfrak{sp}_n(\mathbb{k}) = \{X \in \mathfrak{gl}_n(\mathbb{k}) \mid X^\top J + JX = 0\}.$$

Its dimension is

$$\dim(\mathrm{Sp}_n) = n(2n + 1).$$

By the condition $gJg^\top$ one easily sees that $\det(g) = \pm 1$, and even $\mathrm{Sp}_n \subset \mathrm{SL}_{2n}$ holds. Further, Sp_n is a simple and connected group.

At least for the cases considered in this thesis, Sp_n is the only simple group aside from SL_n appearing as a simple factor of a group with étale representation.

2) Note that the notation Sp_{2n} instead of Sp_n is also used in the literature.

4.4.3 SO_n and its Alter Ego, Spin_n

The **orthogonal group** is

$$\mathrm{O}_n(\Bbbk) = \{g \in \mathrm{GL}_n(\Bbbk) \mid gg^\top = I_n\},$$

and its intersection with SL_n is the **special orthogonal group**

$$\mathrm{SO}_n(\Bbbk) = \{g \in \mathrm{O}_n(\Bbbk) \mid \det(g) = 1\}.$$

Both groups have the same Lie algebra

$$\mathfrak{o}_n(\Bbbk) = \mathfrak{so}_n(\Bbbk) = \{X \in \mathfrak{gl}_n(\Bbbk) \mid X^\top = -X\}.$$

They are of the same dimension,

$$\dim(\mathrm{O}_n) = \dim(\mathrm{SO}_n) = \frac{1}{2}n(n-1).$$

SO_n is connected, but O_n is not because

$$\mathrm{SO}_n = \mathrm{O}_n/\{\pm I_n\},$$

and $\mathrm{SO}_n = \mathrm{O}_n^\circ$. For $n \geq 3$, both groups are simple. But for $n = 2$, the group O_2 is abelian, hence not simple.

For $\Bbbk = \mathbb{R}$, the elements of O_n can be interpreted geometrically as the linear transformations preserving angles and lengths.

The definition can be generalised by requiring $gQg^\top = Q$ instead of $gg^\top = I_n$, where Q is a matrix defining a symmetric non-degenerate bilinear form.

Closely related to the orthogonal groups is the **spin group** $\mathrm{Spin}_n(\Bbbk)$, of which a detailed introduction can be found in chapter 20 of Fulton, Harris [8]. Here, we will just note that $\mathrm{Spin}_n/\{\pm 1\} \cong \mathrm{SO}_n$. In particular, the spin group has the same Lie algebra as O_n and SO_n (cf. section 4.3), and it is essential in constructing some of the representations of this Lie algebra.

4.4.4 Exceptional Groups

Aside from the simple groups described above, there are five simple **exceptional groups**. These are the groups G_2, F_4, E_6, E_7 and E_8. They are rather complicated to describe in detail, so we will not bother to do this here, but give some references instead.

In the course of § 1 of Sato, Kimura [28], a description of these exceptional groups is given. Chapter 22 of Fulton, Harris [8] is dedicated to the construction of their Lie algebras from the root data. Additionally, a construction of the Lie algebra of G_2 as the derivation algebra of the Cayley numbers is given in chapter 7 of Baues, Globke [3].

4.4.5 Characters

Let G be an algebraic group. The group

$$\mathcal{X}(G) = \{\chi : G \to \mathbb{k}^\times \mid \chi \text{ is a rational group homomorphism}\}$$

is called the **group of characters** of G. It is a free abelian group.

It is well known that the character group of any semisimple group is trival, see for example the proof of proposition 10.21.

The character group of a torus of dimension n is isomorphic to $\mathbb{Z}^n$.

4.4.6 Other Groups

Some other groups which are not simple appear in the course of this thesis.

First, the **additive group** G_m^+ of dimension m which can be considered as the vector space $\mathbb{k}^m$ with its addition as a group operation. In this thesis, this group arises as a semidirect factor of generic isotropy subgroups of prehomogeneous modules, see chapter 11, and in this context it is often written as $\mathrm{G}_{m(n-m)}^+$, which indicates that it appears as a group of matrices of the form

$$\begin{pmatrix} I_{n-m} & 0 \\ A & I_m \end{pmatrix},$$

with $A \in \mathrm{Mat}_{m,n-m}$. Under multiplication, these matrices behave just like the additive group.

Next, there is the n-dimensional **multiplicative group** $(\mathbb{k}^\times)^n$, with componentwise multiplication in $\mathbb{k}^\times$. This group is identical to GL_1^n, and we shall use the latter notation most of the time.

A matrix group is **unipotent** if $(I_n - g)^k = 0$ holds for some $k \in \mathbb{N}$ and any element g. It can be shown that any unipotent group is isomorphic to a closed subgroup of the group of upper triangular matrices with 1 on the diagonal,

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Unipotent groups appear as semidirect factors of generic isotropy subgroups of prehomogeneous modules, see chapter 11. Be warned though that in many cases these subgroups appear in a non-obvious representation, so see the cases in § 5 of Sato, Kimura [28] for the respective appearance of these groups. To be consistent with the notation of Sato, Kimura [28], we let Un_n denote a unipotent group of dimension n, but *not* the group of unipotent $n \times n$-matrices, which would be the more common usage.

4.5 Representations and Group Actions

In representation theory, one studies how a given algebraic group (resp. Lie algebra) can be written as a matrix group (resp. algebra).

Definition 4.30 Let V be a vector space over $\mathbb{k}$. A homomorphism $\varrho : G \to \mathrm{GL}(V)$ of algebraic groups is called a **representation** of G. A Lie algebra homomorphism $\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ is called **representation** of $\mathfrak{g}$. The vector space V together with the representation ϱ (resp. $\mathrm{d}\varrho$) is called a **module** for G (resp. for $\mathfrak{g}$), written (ϱ, V) or (G, ϱ, V) (resp. $(\mathrm{d}\varrho, V)$).

Remark 4.31 The notation $\mathrm{d}\varrho$ for Lie algebra representations is justified as any representation arises as the differential of a group representation ϱ. In fact, many definitions and results for representations and actions of Lie algebras arise as differential expressions of the respective expressions for algebraic groups.

Remark 4.32 If (G, ϱ, V) and (G, σ, W) are modules of G, the direct sum and the tensor product are also modules via

$$(\varrho \oplus \sigma)(g).(v, w) = (\varrho(g).v, \sigma(g).w),$$
$$(\varrho \otimes \sigma)(g).(v \otimes w) = \varrho(g).v \otimes \sigma(g).w.$$

Tensor products representations can also be defined for cartesian products of groups G and H. If (G, ϱ, V) and (H, τ, U) are modules, then the tensor product representation for $G \times H$ is defined via

$$(\varrho \otimes \tau)(g, h).(v \otimes u) = \varrho(g).v \otimes \tau(h).u.$$

The tensor product representations also induce representations for symmetric and exterior products. See chapter 2 to learn how these definitions translate to matrix notation.

Remark 4.33 Let $\mathfrak{g} : \mathrm{d}\varrho \to \mathfrak{gl}(V)$ and $\mathfrak{g} : \mathrm{d}\sigma \to \mathfrak{gl}(W)$ be Lie algebra representations. Representations for direct sums and tensor products are given by

$$(\mathrm{d}\varrho \oplus \mathrm{d}\sigma)(X).(v, w) = (\mathrm{d}\varrho(X).v, \mathrm{d}\sigma(X).w),$$
$$(\mathrm{d}\varrho \otimes \mathrm{d}\sigma)(X).(v \otimes w) = \mathrm{d}\varrho(X).v \otimes w + v \otimes \mathrm{d}\sigma(X).w.$$

Tensor products representations can also be defined for direct sums of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $(\mathfrak{g}, \mathrm{d}\varrho, V)$ and $(\mathfrak{h}, \mathrm{d}\tau, U)$ are modules, then the tensor product representation for $\mathfrak{g} \oplus \mathfrak{h}$ is defined via

$$(\mathrm{d}\varrho \otimes \mathrm{d}\tau)(X, Y).(v \otimes u) = \mathrm{d}\varrho(X).v \otimes u + v \otimes \mathrm{d}\tau(Y).u.$$

Definition 4.34 Let G and H be algebraic groups.

1. Two representations of $\varrho_1 : G \to \mathrm{GL}(V_1)$ and $\varrho_2 : G \to \mathrm{GL}(V_2)$ are called **conjugate** if there exists an isomorphism $\varphi : V_1 \to V_2$ which satisfies

$$\varrho_2(g).\varphi(v) = \varphi(\varrho_1(g).v),$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\varrho_1(g)} & V_1 \\ \varphi\downarrow & & \downarrow\varphi \\ V_2 & \xrightarrow{\varrho_2(g)} & V_2 \end{array}$$

2. Two representations $\varrho : G \to \mathrm{GL}(V)$ and $\sigma : H \to \mathrm{GL}(W)$ are called **equivalent** if there exists an isomorphism $\Psi : \varrho(G) \to \sigma(H)$ and an isomorphism $\varphi : V \to W$ of vector spaces such that

$$\varphi(\varrho(g).v) = \Psi(\varrho(g)).\varphi(v),$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\varrho(g)} & V_1 \\ \varphi\downarrow & & \downarrow\varphi \\ V_2 & \xrightarrow{\Psi(\varrho(g))} & V_2 \end{array}$$

Definition 4.35 If (G, ϱ, V) (resp. $(\mathfrak{g}, \mathrm{d}\varrho, V)$) is a module for G (resp. $\mathfrak{g}$), then the **dual representation** ϱ^* (resp. $\mathrm{d}\varrho^*$) on the dual space V^* of V is defined via $\varrho^*(g) = (\varrho(g)^{-1})^\top$ (resp. $\mathrm{d}\varrho^*(X) = -\mathrm{d}\varrho(X)^\top$).

Definition 4.36 A module (ϱ, V) of an algebraic group G is called **irreducible**[3)] if there exists no non-trival subspace W of V such that W is $\varrho(G)$-invariant, i.e. $(\varrho|_W, W)$ is a representation. It is called **fully reducible** if V decomposes to a direct sum $V = V_1 \oplus \ldots \oplus V_k$ of G-modules $(\varrho|_{V_i}, V_i)$.

This definition is adopted in an obvious way for Lie algebras.

4.5.1 Adjoint Represenations and Semisimple Lie Algebras

The most natural representation of groups and Lie algebras are those where the group (resp. algebra) acts on itself, as no additional information about the module is required.

Definition 4.37 Let G be an algebraic group and $\mathfrak{g} = \mathfrak{Lie}(G)$. For $g \in G$ and $X \in \mathfrak{g}$, set

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g},\ Y \mapsto gYg^{-1} \quad \text{and} \quad \mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g},\ Y \mapsto [X, Y].$$

The representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}(g)$$

[3)]Note that for representations which are not irreducible, we will use the term *non-irreducible* rather than *reducible*, to avoid further confusion with the terms *reduced*, *reductive* and *fully reducible*.

is called the **adjoint representation** of G. Its differential is

$$\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}(X),$$

the **adjoint representation** of $\mathfrak{g}$.

Theorem 4.38 (Cartan's criterion for semisimplicity) *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the bilinear form*

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}, \quad (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

is non-degenerate.

The bilinear form κ in this theorem is called the **Killing form**, which is a major tool in the study of semisimple Lie algebras.

Definition 4.39 Let $\mathfrak{g}$ be a semisimple Lie algebra. A subalgebra $\mathfrak{c}$ is a **Cartan algebra** of $\mathfrak{g}$ if $\mathfrak{c}$ is a maximal commutative subalgebra and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{c}$ holds. The dimension of $\mathfrak{c}$ is called the **rank** of $\mathfrak{g}$.

Definition 4.40 Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{c}$ a Cartan algebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{c}^*$, define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall\, H \in \mathfrak{c} : \mathrm{ad}(H)X = \alpha(H)X\}.$$

If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, then α is called a **root** and $\mathfrak{g}_\alpha$ its **root space**. The set R of roots of $\mathfrak{g}$ is called the **root system** of $\mathfrak{g}$.

Theorem 4.41 *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{c}$ a Cartan algebra of $\mathfrak{g}$. Let $R \subseteq \mathfrak{c}^*$ be the set of roots. For $\alpha \in R$, we have $-\alpha \in R$ as well. The algebra $\mathfrak{g}$ decomposes into subspaces as follows:*

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Furthermore, for $\alpha, \beta \in R$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (resp. $= \{0\}$ if $\alpha + \beta \notin R$) and $\mathfrak{g}_0 = \mathfrak{c}$.

Definition 4.42 A root system R is called **reduced** if for any $\alpha \in R$ the only roots proportional to α are α and $-\alpha$.

It can be shown that a semisimple Lie algebra $\mathfrak{g}$ is determined uniquely up to isomorphism by its root system R, cf. chapter II.10 in Knapp [20]. Conversely, one can define an abstract notion of a reduced root system independently of a given Lie algebra (see for example chapter 5 in Baues, Globke [3] or chapter II.5 in Knapp [20]) and show that for any reduced root system R, there exists a Lie algebra whose root system is R cf. chapter II.11 in Knapp [20]. After investing quite some effort, one arrives at the following classification result for simple Lie algebras.

Theorem 4.43 (Classification of simple Lie algebras) *Every simple Lie algebra* $\mathfrak{g}$ *over an algebraically closed field* $\mathbb{k}$ *is of one of the following root system types:*

Type	$\mathfrak{g}$		$\dim_{\mathbb{k}}(\mathfrak{g})$
A_n	$\mathfrak{sl}_{n+1}(\mathbb{k})$	$n \geq 1$	$n^2 + 2n$
B_n	$\mathfrak{o}_{2n+1}(\mathbb{k})$	$n \geq 2$	$2n^2 + n$
C_n	$\mathfrak{sp}_n(\mathbb{k})$	$n \geq 3$	$2n^2 + n$
D_n	$\mathfrak{o}_{2n}(\mathbb{k})$	$n \geq 4$	$2n^2 - n$
G_2		-	14
F_4		-	52
E_6		-	72
E_7		-	133
E_8		-	248

Where the index of the respective type is the rank of $\mathfrak{g}$*. Further we have* $A_1 = B_1 = C_1$*,* $B_2 = C_2$ *and* $A_3 = D_3$*.*

4.5.2 Irreducible Representations of Semisimple Lie Algebras

Definition 4.44 Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{c}$ a Cartan algebra of $\mathfrak{g}$. Further, let $d\varrho : \mathfrak{g} \to \mathrm{GL}(V)$ be a representation of $\mathfrak{g}$ on V. For an element $\omega \in \mathfrak{c}^*$, set

$$V_\omega = \{x \in V \mid \forall\, H \in \mathfrak{c} : d\varrho(H).x = \omega(H)x\}.$$

If $V_\omega \neq \{0\}$, it is called the **weight space** for the **weight** ω of $d\varrho$.

Remark 4.45 The roots of $\mathfrak{g}$ are the non-zero weights of the adjoint representation.

Remark 4.46 Now, consider a $\mathfrak{g}$-module $(d\varrho, V)$ of a semisimple Lie algebra $\mathfrak{g}$. Let R be the root system of $\mathfrak{g}$ and consider

$$\mathfrak{c}_0^* = \langle R \rangle_{\mathbb{Q}}.$$

It is possible to introduce a lexicographic order on $\mathfrak{c}_0^*$ such that R can be decomposed into two disjoint subsets, $R = R^+ \cup R^-$. Then, an element $v \in V$ is called a **highest weight vector** of the representation if

$$d\varrho(X).v = 0$$

for all $X \in \mathfrak{g}_\alpha$ with $\alpha \in R^+$. Then v is contained in some weight space,

$$v \in V_\omega \subset V,$$

so $\omega \in \mathfrak{c}_0^*$ is called the **highest weight** of the representation $d\varrho$. The highest weight vector is unique for an irreducible representation. All weight spaces of the representation $d\varrho$ are obtained by successively applying the elements $Y \in \mathfrak{g}_\beta$ with $\beta \in R^-$ to the highest weight vector v. These facts are explained exhaustively in chapter 14 of Fulton, Harris [8], where the important special cases $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$ are treated in chapters 11 to 13.

Remark 4.47 Depending on the choice of a basis of $\mathfrak{c}^*$ and an order in R, there exist certain **fundamental weights** $\omega_1, \ldots, \omega_n \in \mathfrak{c}^*$, where $n = \text{rank}(\mathfrak{g})$, with the property that the highest weight ω can be expressed uniquely as a non-negative integral linear combination of the fundamental weights,

$$\omega = m_1\omega_1 + \ldots + m_n\omega_n$$

for $m_1, \ldots, m_n \in \mathbb{N}_0$.

Theorem 4.48 *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{c}$ a Cartan algebra of $\mathfrak{g}$. Let $\omega \in \mathfrak{c}_0^*$. Then there exists a unique irreducible representation $\mathrm{d}\varrho_\omega$ of $\mathfrak{g}$ with highest weight ω.*

Instead of writing $\mathrm{d}\varrho_\omega$, we will identify the representation with its highest weight and use ω to denote both the representation and the weight throughout the thesis.

Example 4.49 Irreducible representations of simple Lie algebras.

1. For $\mathfrak{g} = \mathfrak{sl}_n$, the representation with highest weight $\omega = \omega_1$ is the standard representation where $\mathfrak{sl}_n$ acts via matrix multiplication on $\mathbb{k}^n$.
2. For $\mathfrak{g} = \mathfrak{sl}_n$ and $1 \leq k \leq n$, the module for the representation $\omega = \omega_k$ is the k-th exterior product $\bigwedge^k \mathbb{k}^n$.
3. For $\mathfrak{g} = \mathfrak{sl}_n$ and $1 \leq k \leq n$, the module for the representation $\omega = k\omega_1$ is the k-th symmetric product $\mathrm{Sym}^k \mathbb{k}^n$.
4. For $\mathfrak{g} = \mathfrak{sl}_2$, an irreducible representation is a symmetric power of the standard representation.
5. For $\mathfrak{g} = \mathfrak{sp}_n$, the representation $\omega = \omega_1$ is the standard representation on $\mathbb{k}^{2n}$.
6. For $\mathfrak{g} = \mathfrak{so}_n$, the representation $\omega = \omega_1$ is the standard representation on $\mathbb{k}^n$.

The book by Fulton and Harris [8] provides a complete classification of the finite-dimensional representations of the simple Lie algebras (for $\mathbb{k} = \mathbb{C}$). In fact, all of them are given by a representation determined by some highest weight, except for some representations of $\mathfrak{so}_n$ which arise as representations of the group Spin_n. These are the spin representation, the (even and odd) half-spin representations and the vector representation, see example 2.15 in Kimura [14].

The dimension of a module of a semisimple Lie algebra can be computed with the help of the Killing form and the knowledge of the highest weight. Each $\alpha \in \mathfrak{c}^*$ is a dual element for some $X_\alpha \in \mathfrak{c}$ with respect to the Killing form. Then the Killing form can be defined on $\mathfrak{c}^*$ via $\kappa(\alpha, \beta) = \kappa(X_\alpha, X_\beta)$.

Theorem 4.50 (Weyl's dimension formula) *Let $\mathfrak{g}$ be a semisimple Lie algebra with root system R. The dimension of the irreducible representation ω is given by*

$$\dim(\omega) = \prod_{\alpha \in R^+} \frac{\kappa(\omega + \beta, \alpha)}{\kappa(\beta, \alpha)},$$

with $\beta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

Now we investigate the correspondence of the irreducible Lie algebra representations to representations of algebraic groups. We assume $\mathbb{k} = \mathbb{C}$.

Definition 4.51 Let G be a connected algebraic group with semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{c}$ be a Cartan algebra of $\mathfrak{g}$ and $C \subset G$ the subgroup corresponding to $\mathfrak{c}$. Then C is called a **Cartan subgroup** of G.

Theorem 4.52 *Let G be a connected algebraic group with semisimple Lie algebra $\mathfrak{g}$, and let C be a Cartan subgroup of G. Then an irreducible representation $\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ corresponds to a rational represenation $\varrho : G \to \mathrm{GL}(V)$ if and only if the highest weight of $\mathrm{d}\varrho$ corresponds to the differential of an element of $\mathcal{X}(C)$.*

We use highest weights to denote both representations for groups and for Lie algebras.

The irreducible representations of non-simple semisimple groups are obtained as tensor products of the irreducible representations of simple groups.

Proposition 4.53 *Let G_1 and G_2 be algebraic groups, and $\varrho : G_1 \times G_2 \to \mathrm{GL}(V)$ an irreducible represenation. Then there exist irreducible representations $\varrho_1 : G \to \mathrm{GL}(V_1)$ and $\varrho_2 : G_2 \to \mathrm{GL}(V_2)$ such that $\varrho = \varrho_1 \otimes \varrho_2$ and $V \cong V_1 \otimes V_2$.*

4.5.3 Reductive Groups and Lie Algebras

Definition 4.54 An algebraic group G is called **reductive** if every G-module is fully reducible.

Theorem 4.55 *An algebraic group G is reductive if and only if its radical is a torus.*

Example 4.56 Reductive groups.

1. Finite groups are reductive.
2. If $G_1, \ldots, G_k$ are simple groups, then

$$\mathrm{GL}_1^j \times G_1 \times \cdots \times G_k$$

is a reductive group with torus GL_1^j.

Remark 4.57 If G is reductive, then $[G, G]$ is semisimple.

Definition 4.58 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$.

1. $\mathfrak{g}$ is called **reductive** if the adjoint representation of $\mathfrak{g}$ is fully reducible.
2. $\mathfrak{h}$ is called **reductive in $\mathfrak{g}$** if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ is fully reducible.

In particular, if $\mathfrak{h}$ is reductive in $\mathfrak{g}$, then $\mathfrak{h}$ is reductive.

Proposition 4.59 *A Lie algebra $\mathfrak{g}$ is reductive if and only if $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is the centre of $\mathfrak{g}$.*

Proposition 4.60 *An algebraic group G is reductive if and only if $\mathfrak{g} = \mathfrak{Lie}(G)$ is reductive and all $X \in \mathfrak{z}(\mathfrak{g})$ are diagonalisable over the algebraic closure of $\mathbb{k}$.*

Remark 4.61 If G is an algebraic group with reductive Lie algebra, then G is not necessarily reductive. For example, this is the case for $G = \mathrm{G}^+$.

Theorem 4.62 (Cartan) *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{k}$, and let $\mathrm{d}\varrho : \mathfrak{g} \to V^n$ be an irreducible representation of $\mathfrak{g}$. Then $\mathrm{d}\varrho(\mathfrak{g})$ is either semisimple or a direct sum of a semisimple Lie algebra and the centre $\mathfrak{z}(\mathrm{d}\varrho(\mathfrak{g})) = \{\lambda I_n \mid \lambda \in \mathbb{k}\} \cong \mathfrak{gl}_1$.*

Remark 4.63 Cartan's Theorem tells us that the torus of an algebraic group acts either trivially or via **scalar multiplication** on an irreducible module (the trivial representation $g \mapsto I_n$ is denoted by 1). We denote the representation where GL_1 acts via scalar multiplication on a module by μ. As a consequence, for a non-irreducible module, at most one scalar multiplication acts on each irreducible component. But this does not necessarily imply that the torus acts via the same scalar multiplication on all irreducible components. For example, if $V = V_1 \oplus V_2 \oplus V_3$ is a module with irreducible components V_i of respective dimensions n_i, then a possible GL_1-action on V is given by

$$v \mapsto \begin{pmatrix} \lambda I_{n_1} & & \\ & \lambda^{-1} I_{n_2} & \\ & & I_{n_3} \end{pmatrix} \cdot v.$$

So GL_1 acts via scalar multiplication on V_1 and V_2, and trivially on V_3. But GL_1 does not act via scalar multiplication on V or even on $V_1 \oplus V_2$.

4.5.4 Actions and Orbits

Definition 4.64 Let G be an algebraic group acting on an algebraic set V. For $v \in V$, the set

$$G.v = \{w \in V \mid \exists\, g \in G : g.v = w\}$$

is called the **orbit** of v under the action of G. The subgroup $G_v \subset G$ fixing v,

$$G_v = \{g \in G \mid g.v = v\},$$

is called the **isotropy subgroup** of v. A subset $W \subseteq V$ is called **G-stable** if $G.W \subseteq W$.

Proposition 4.65 *Let G be an algebraic group acting on an algebraic set V.*

1. *An orbit $G.v$ is open in its closure* $\mathrm{clos}(G.v)$*, and every orbit contains a closed orbit in its closure.*
2. *We have the following dimension formula:*

$$\dim(G) = \dim(G.v) + \dim(G_v).$$

3. *If G is connected, the irreducible components of V are stable under the action of G.*

Proposition 4.66 *Let (G, ϱ, V) be a module for G and $\mathfrak{g} = \mathfrak{Lie}(G)$. The differential (at 1_G) of the orbit map $\beta : G \to V, g \mapsto \varrho(g).v$ is given by*

$$\mathrm{d}\beta : \mathfrak{g} \to V, \quad X \mapsto \mathrm{d}\varrho(X).v.$$

If v is fixed under G, then $\mathrm{d}\varrho(X).v = 0$ for all $X \in \mathfrak{g}$. If $W \subseteq V$ is $\varrho(G)$-stable, then $\mathrm{d}\varrho(\mathfrak{g}).W \subseteq W$.

In particular, we can define the **isotropy algebra** $\mathfrak{g}_v$ of $v \in V$ to be the subalgebra of $\mathfrak{g}$ such that $\mathrm{d}\varrho(X).v = 0$ for $X \in \mathfrak{g}_v$. We have $\mathfrak{g}_v = \mathfrak{Lie}(G_v)$.

Proposition 4.67 *Let G be an algebraic group acting on an algebraic set V. For $v \in V$, the coset space G/G_v is isomorphic to the orbit $G.v$.*

The following theorems are essential for the characterisation of reductive prehomogeneous modules in section 10.3.

Theorem 4.68 (Luna) *Let G be a reductive algebraic group acting on a smooth affine variety V. Assume that for each point $v \in V$ there exists a non-degenerate symmetric bilinear form on the tangent space $\mathrm{T}_v V$ which is invariant under G_v. Then there exists a Zariski dense open subset U of V which is the union of closed G-orbits in V. In particular, if an open dense orbit exists, it is V itself.*

Theorem 4.69 (Matsushima) *Let G be a reductive linear algebraic group and H a closed subgroup of G. Then the coset space G/H is an affine variety if and only if H is a reductive algebraic group.*

A proof of Matsushimas theorem can be found in Richardson [27].

4.6 Lie Groups and Algebraic Groups

We assume $\mathbb{k} = \mathbb{C}$ in this section.

A **complex Lie group** is a group endowed with the structure of a complex manifold, see chapter I.12 in Knapp [20].

Theorem 4.70 *Let G be a semisimple algebraic group. Then G has the structure of a complex Lie group, and this structure is compatible with its structure as an algebraic group. If H is an algebraic group, then every analytic homomorphism $\varphi : G \to H$ is rational.*

This theorem enables us to use facts from the theory of Lie groups for algebraic groups as well. In particular, we need the following fact in section 10.3.

Proposition 4.71 *Let G be a complex Lie group. Then G has a maximal compact subgroup K, and G is the Zariski closure of K.*

In chapter 8, we need a correspondence between representations of algebraic groups and Lie algebras. Part of this is provided by theorem 4.21, and a converse result for certain groups is provided by the following theorem (recall from topology that a set is called simply connected if every loop can be contracted continuously to a point).

Theorem 4.72 *Let $\mathfrak{g}, \mathfrak{h}$ be complex Lie algebras and $\psi : \mathfrak{g} \to \mathfrak{h}$ a homomorphism of Lie algebras. There exist complex Lie groups G and H, where G is simply connected, and a homomorphism $\varphi : G \to H$ of Lie groups such that $\psi = \mathrm{d}\varphi_{1_G}$.*

Remark 4.73 Even though the homomorphism φ in the previous theorem is constructed using the exponential map (see chapter 10 in Knapp [20]), theorem 4.70 tells us that the homomorphism φ is rational, hence a homomorphism of algebraic groups.

5 Invariant Theory

In this chapter we study polynomial functions which are invariant under the action of some algebraic group. For an introduction to this subject, see Kraft [23], [24]. A more abstract approach to the subject is taken by Schmitt [29].

We assume $\mathbb{k}$ to be algebraically closed.

Definition 5.1 Let G be an algebraic group and $\varrho : G \to \mathrm{GL}(V)$ a representation. A function $f \in \mathbb{k}[V]$ is called **invariant** under G, if

$$f(v) = f(\varrho(g).v)$$

for all $g \in G$ and $v \in V$. The polynomial invariants form a subring of $\mathbb{k}[V]$, the **invariant ring**, denoted by $\mathbb{k}[V]^G$.

Example 5.2 Invariants.

1. All constant functions are invariants.

2. For $G = \mathrm{SL}_n$ acting on $\Bbbk^n$ via matrix multiplication, an invariant is given by the determinant. In fact, the invariant ring $\Bbbk[\Bbbk^n]^{\mathrm{SL}_n}$ is generated by the determinant.

3. For $G = \mathrm{GL}_n$ acting via $\omega_1 \oplus \omega_1^*$ on $\Bbbk^n \oplus \Bbbk^{n*}$. The invariant ring $\Bbbk[\Bbbk^n \oplus \Bbbk^{n*}]^{\mathrm{GL}_n}$ is generated by the dual pairing, $\langle x|y\rangle = \langle \varrho(g).x|\varrho^*(g).y\rangle$.

4. The action of $G = \mathrm{SL}_{2n}$ on the space $\bigwedge^2 \Bbbk^{2n}$ of skew-symmetric matrices via $\varrho(g).X = gXg^\top$. The determinant of a skew-symmetric matrix X can always be written as the square of a polynomial in the matrix entries. This polynomial is the **Pfaffian**

$$\mathrm{Pf}(X) = \sqrt{\det(X)}.$$

The Pfaffian is an irreducible polynomial and it generates the ring of invariants $\Bbbk[\bigwedge^2 \Bbbk^{2n}]^{\mathrm{SL}_{2n}}$.

5. The action of $G = \mathrm{SL}_2$ on the space of binary cubic forms $\mathrm{Sym}^3\Bbbk^2$ is given by $\varrho(g).f = f(\varrho(g).\cdot)$. In general, the **discriminant** of a polynomial f is a function which is equal to 0 if and only if f has multiple roots. For the binary cubic forms, the discriminant is given by

$$\mathrm{dis}(f) = a_2^2a_3^2 + 18a_1a_2a_3a_4 - 4a_1a_3^3 - 4a_2^3a_4 - 27a_1^2a_4^2,$$

where $f(x,y) = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3$. The ring of invariants $\Bbbk[\mathrm{Sym}^3\Bbbk^2]^{\mathrm{SL}_2}$ is generated by the discriminant.

5.1 Algebraic Quotients

Theorem 5.3 (Hilbert, Nagata) *If G is a reductive algebraic group and V an algebraic set such that G acts in V. Then the invariant ring $\Bbbk[V]^G$ is a finitely generated $\Bbbk$-algebra.*

Now it follows from proposition 3.29, that there exists an affine variety with coordinate ring $\Bbbk[V]^G$.

Definition 5.4 Let V be an algebraic set and G an algebraic group acting on V. Then let $V/\!\!/G$ denote the variety with coordinate ring $\Bbbk[V]^G$, and $\pi : V \to V/\!\!/G$ be the comorphism of the embedding $\Bbbk[V]^G \to \Bbbk[V]$. Then the pair $(V/\!\!/G, \pi)$ is called the **algebraic quotient** of V by G.

Proposition 5.5 *Let V be an algebraic set and G an algebraic group acting on V, and let $(V/\!\!/G, \pi)$ be its algebraic quotient.*

1. *The map π is surjective and constant on the G-orbits.*

2. *The algebraic quotients satisfies the following universal property: If $\varphi : V \to W$ is a morphism which is constant on the G-orbits, then there exists a unique morphism $\psi : V/\!\!/G \to W$ such that $\varphi = \psi \circ \pi$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle\pi}\downarrow & \nearrow_{\exists_1 \psi} & \\ V/\!\!/G & & \end{array}$$

3. *If V is irreducible, then $V/\!\!/G$ is irreducible, and if V is normal, then $V/\!\!/G$ is normal.*

4. *If Y is a closed G-stable subset of V, then $\pi(Y)$ is a closed subset of $V/\!\!/G$. Furthermore, $(\pi(Y), \pi|_Y)$ is the algebraic quotient of Y by G.*

5. *If $(Y_i)_i$ is a family of closed G-stable subsets of V, then*

$$\pi\Big(\bigcap_i Y_i\Big) = \bigcap_i \pi(Y_i).$$

6. *For $v \in V$, the fibre $\pi^{-1}(\pi(v))$ contains a unique closed G-orbit X and we have*

$$\pi^{-1}(\pi(v)) = \{w \in V \mid X \subset \operatorname{clos}(G.w)\}.$$

Remark 5.6 By the last part of proposition 5.5, we can interprete $V/\!\!/G$ as the set of *closed* G-orbits in V, or as the set of closures of orbits.

Proposition 5.7 *Let V be a finite-dimensional module for an algebraic group G. If the character group $\mathcal{X}(G)$ is trivial, i.e. $\mathcal{X}(G) = \{1\}$, then*

$$\Bbbk(V/\!\!/G) = \Bbbk(V)^G.$$

Example 5.8 The assumptions of proposition 5.7 are satisfied by any module of a semisimple group.

Proposition 5.9 *Let V be a finite-dimensional module for an algebraic group G. Then*

$$\operatorname{trdeg}_{\Bbbk}(\Bbbk(V)^G) = \dim(V) - \max\{\dim(G.v) \mid v \in V\}.$$

Theorem 5.10 (Rosenlicht) *Let V be a variety and G an algebraic group acting on V. Then there exists an open dense G-invariant subset $U \subseteq V$, an algebraic set W, and a morphism $\varphi : U \to W$ such that*

1. *Every fibre $\varphi^{-1}(w)$ for $w \in W$ is precisely a single G-orbit.*

2. *U and W are smooth.*

3. $\varphi^* : \Bbbk(W) \to \Bbbk(U)^G \cong \Bbbk(V)^G$ *is an isomorphism.*

4. $\varphi^* : \Bbbk[W] \to \Bbbk[U]^G$ *is an isomorphism.*

Proposition 5.11 *Let V be a module of an algebraic group G. If there exists an orbit of codimension ≤ 1, then $V /\!\!/ G$ is either a point or isomorphic to the affine line $\Bbbk$.*

We shall learn some facts about the fibres of the quotient map π.

Definition 5.12 Let $\varphi : V \to W$ be a morphism and consider the fibre $F = \varphi^{-1}(w)$ for some $w \in \varphi(V) \subseteq W$. Then F is called **reduced** if $\varphi^*(\mathfrak{M}_w)$ generates a radical ideal in $\Bbbk[V]$, i.e. $\sqrt{\varphi^*(\mathfrak{M}_w)\Bbbk[V]} = \varphi^*(\mathfrak{M}_w)\Bbbk[V]$.

In the following, let V be a module for an algebraic group G, and let

$$F_0 = \pi^{-1}(\pi(0)) = \{v \in V \mid 0 \in \mathrm{clos}(G.v)\}$$

denote the **zero fibre** of π.

Proposition 5.13 *Some facts about the zero fibre.*

1. *If the zero fibre F_0 contains a dense orbit, so does any other fibre of π. Then all fibres are of the same dimension.*

2. *We have $\dim(F_0) \geq \dim(F)$ for any fibre F of π. In particular, all fibres are of the same dimension if and only if*

$$\dim(F_0) = \dim(V) - \dim(V /\!\!/ G),$$

   i.e. $\dim(F_0)$ is minimal.

3. *If F_0 is reduced and irreducible of dimension $\dim(F_0) = \dim(V) - \dim(V /\!\!/ G)$, the all fibres are reduced and irreducible. If additionally F_0 is normal, then all fibres are normal.*

Theorem 5.14 *Let G be an algebraic group acting on an algebraic set V. If*

$$W = \{v \in F_0 \mid \mathrm{d}\pi_v : V \to \mathrm{T}_{\pi(0)}(V /\!\!/ G) \text{ is surjective}\}$$

is not empty and $\mathrm{codim}_{F_0}(\mathrm{clos}(F_0 \backslash W)) \geq 2$, then all fibres of π are normal and $V /\!\!/ G$ is an affine space.

Proposition 5.15 *If $\pi(0) \in V /\!\!/ G$ is a smooth point, then $V /\!\!/ G \cong \Bbbk^n$ for some $n \in \mathbb{N}$. In particular, $\Bbbk[V]^G = \Bbbk[V /\!\!/ G]$ is generated by algebraically independent homogeneous elements $f_1, \ldots, f_n \in \mathfrak{M}_{\pi(0)}$.*

Theorem 5.16 *If $\dim(F_0) = \dim(V) - \dim(V /\!\!/ G)$ and if F_0 is reduced at some point $v \in F_0$, then the quotient $V /\!\!/ G$ is an affine space.*

Theorem 5.17 *If G is semisimple and $\dim(V /\!\!/ G) = 2$, then $V /\!\!/ G \cong \Bbbk^2$.*

5.2 The Fundamental Theorems

The Fundamental Theorems of invariant theory describe the ring of invariants for certain modules in terms of generators and relations. These two are usually seperated in "First" and "Second" Fundamental Theorems, but here we shall state them together. By the fundamental theorems we obtain some information about the quotient, e.g. if the generators are algebraically independent, then the quotient is isomorphic to some affine space.

Theorem 5.18 (Fundamental Theorem for GL_n) *Consider the module*

$$\left(\mathrm{GL}_n,\ (\omega_1)^{\oplus p} \oplus (\omega_1^*)^{\oplus q},\ (\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}\right).$$

The ring of invariants $\Bbbk[(\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}]^{\mathrm{GL}_n}$ *is generated by the dual pairings*

$$\langle x_i | \lambda_j \rangle = \lambda_j(x_i) \quad \text{for } i = 1, \ldots, p,\ j = 1, \ldots, q,$$

where $(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) \in (\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}$. *The relations are generated by*

$$0 = \det \begin{pmatrix} \langle x_{i_1} | \lambda_{j_1} \rangle & \cdots & \langle x_{i_1} | \lambda_{j_{n+1}} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_{n+1}} | \lambda_{j_1} \rangle & \cdots & \langle x_{i_n} | \lambda_{j_{n+1}} \rangle \end{pmatrix}$$

for $1 \leq i_1 < \ldots < i_{n+1} \leq p$ *and* $1 \leq j_1 < \ldots < j_{n+1} \leq q$.

Theorem 5.19 (Fundamental Theorem for SL_n) *Consider the module*

$$\left(\mathrm{SL}_n,\ (\omega_1)^{\oplus p} \oplus (\omega_1^*)^{\oplus q},\ (\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}\right).$$

The ring of invariants $\Bbbk[(\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}]^{\mathrm{SL}_n}$ *is generated by the dual pairings*

$$\langle x_i | \lambda_j \rangle = \lambda_j(x_i) \quad \text{for } i = 1, \ldots, p,\ j = 1, \ldots, q,$$

and, if $p \geq n$ *(resp.* $q \geq n$*), the determinants*

$$\det(x_{i_1}, \ldots, x_{i_n}), \quad \det(\lambda_{j_1}, \ldots, \lambda_{j_n})$$

where $(x_1, \ldots, x_p, \lambda_1, \ldots, \lambda_q) \in (\Bbbk^n)^{\oplus p} \oplus (\Bbbk^{n*})^{\oplus q}$. *The relations are generated by*

$$\det(x_{i_1}, \ldots, x_{i_n}) \det(\lambda_{j_1}, \ldots, \lambda_{j_n}) = \det \begin{pmatrix} \langle x_{i_1} | \lambda_{j_1} \rangle & \cdots & \langle x_{i_1} | \lambda_{j_n} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_n} | \lambda_{j_1} \rangle & \cdots & \langle x_{i_n} | \lambda_{j_n} \rangle \end{pmatrix},$$

$$\sum_{k=1}^{n} (-1)^k \det(x_{i_1}, \ldots, \cancel{x_{i_k}}, \ldots, x_{i_{n+1}}) \langle x_{i_k} | \lambda \rangle = 0,$$

$$\sum_{k=1}^{n} (-1)^k \det(\lambda_{i_1}, \ldots, \cancel{\lambda_{i_k}}, \ldots, \lambda_{i_{n+1}}) \langle x | \lambda_{i_k} \rangle = 0,$$

$$\sum_{k=1}^{n} (-1)^k \det(x_{i_1}, \ldots, \cancel{x_{i_k}}, \ldots, x_{i_{n+1}}) \det(x_{i_k}, y_{j_1}, \ldots, y_{j_{n-1}}) = 0,$$

$$\sum_{k=1}^{n} (-1)^k \det(\lambda_{i_1}, \ldots, \cancel{\lambda_{i_k}}, \ldots, \lambda_{i_{n+1}}) \det(\lambda_{i_k}, \xi_{j_1}, \ldots, \xi_{j_{n-1}}) = 0,$$

for $1 \leq i_1 < \ldots < i_{n(+1)} \leq p$ *and* $1 \leq j_1 < \ldots < j_{n(+1)} \leq q$.

Theorem 5.20 (Fundamental Theorem for SO_n) *Consider the module*

$$\left(\mathrm{SO}_n,\ (\omega_1)^{\oplus p},\ (\Bbbk^n)^{\oplus p}\right).$$

The ring of invariants $\Bbbk[(\Bbbk^n)^{\oplus p}]^{\mathrm{SO}_n}$ *is generated by the inner products*

$$\langle x_i | x_j \rangle \quad \text{for } i, j = 1, \ldots, p,$$

where $(x_1, \ldots, x_p) \in (\Bbbk^n)^{\oplus p}$, *and, for* $n \leq p$, *the determinants*

$$\det(x_{i_1}, \ldots, x_{i_n}),$$

where $1 \leq i_1 < \ldots < i_n \leq p$. *The relations are generated by*

$$0 = \det \begin{pmatrix} \langle x_{i_1} | y_{j_1} \rangle & \cdots & \langle x_{i_1} | y_{j_{n+1}} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_{n+1}} | y_{j_1} \rangle & \cdots & \langle x_{i_{n+1}} | y_{j_{n+1}} \rangle \end{pmatrix},$$

$$\det(x_{i_1}, \ldots, x_{i_n}) \det(y_{j_1}, \ldots, y_{j_n}) = \det \begin{pmatrix} \langle x_{i_1} | y_{j_1} \rangle & \cdots & \langle x_{i_1} | y_{j_n} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_n} | y_{j_1} \rangle & \cdots & \langle x_{i_n} | y_{j_n} \rangle \end{pmatrix},$$

where $1 \leq i_1 < \ldots < i_{n(+1)} \leq p$ *and* $1 \leq j_1 < \ldots < j_{n(+1)} \leq p$.

Theorem 5.21 (Fundamental Theorem for Sp_n) *Consider the module*

$$\left(\mathrm{Sp}_n,\ (\omega_1)^{\oplus p},\ (\Bbbk^{2n})^{\oplus p}\right).$$

The ring of invariants $\Bbbk[(\Bbbk^{2n})^{\oplus p}]^{\mathrm{Sp}_n}$ *is generated by the inner products*

$$\langle x_i | x_j \rangle \quad \text{for } i, j = 1, \ldots, p,$$

where $(x_1, \ldots, x_p) \in (\Bbbk^{2n})^{\oplus p}$. *The relations are generated by*

$$0 = \mathrm{Pf}\begin{pmatrix} \langle x_{i_1} | y_{j_1} \rangle & \cdots & \langle x_{i_1} | y_{j_{n+2}} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{i_{n+2}} | y_{j_1} \rangle & \cdots & \langle x_{i_{n+2}} | y_{j_{n+2}} \rangle \end{pmatrix}.$$

Part II

Left-Symmetric Algebras

6 Introduction

Left-symmetric algebras generalise associative algebras in a way that they still induce a Lie algebra structure when considering the commutator product. They arise in many different areas of mathematics, of which a survey can be found in an article by Burde [6].

In this and the following chapters, we shall follow Baues [2], [1] in the introduction of left-symmetric algebras and some of their properties leading to a first classification result. The exposition is further supplied by some examples taken from Burde [6].

Definition 6.1 Let V be a vector space over the field $\mathbb{k}$ endowed with a $\mathbb{k}$-bilinear product $*$ satisfying

$$x * (y * z) - y * (x * z) = (x * y) * z - (y * x) * z.$$

Then the algebra $(V, *)$ is called a **left-symmetric algebra** or **pre-Lie algebra**.

Introducing the **associator product**

$$(x, y, z) = x * (y * z) - (x * y) * z,$$

we can rewrite the defining condition for left-symmetric algebras as

$$(x, y, z) = (y, x, z).$$

6.1 Left-Symmetric Algebras and Lie Algebras

Proposition 6.2 *Let $(V, *)$ be a left-symmetric algebra. The commutator product*

$$[x, y] = x * y - y * x$$

for $x, y \in V$ satisfies the Jacobi identity, i.e. $(V, [\cdot, \cdot])$ is a Lie algebra.

Proof: We have

$$\begin{aligned}[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = {} & x * (y * z - z * y) - (y * z - z * y) * x \\ & + y * (z * x - x * z) - (z * x - x * z) * y \\ & + z * (x * y - y * x) - (x * y - y * x) * z,\end{aligned}$$

and reordering terms yields

$$\underbrace{(x,y,z)-(y,x,z)}_{=0}+\underbrace{(z,x,y)-(x,z,y)}_{=0}+\underbrace{(y,z,x)-(z,y,x)}_{=0}=0,$$

so the Jacobi identity holds. ■

If $\mathfrak{g}$ is the Lie algebra arising from taking the commutator product of a left-symmetric algebra $(V,*)$, we call $\mathfrak{g}$ the Lie algebra **associated** to $(V,*)$. Conversely, if $\mathfrak{g}$ is a Lie algebra such that a left-symmetric product $*$ can be defined on $\mathfrak{g}$ (as a vector space), then we say that $\mathfrak{g}$ **admits** a left-symmetric product.

Remark 6.3 Although every left-symmetric algebra has a Lie algebra associated to it, not every Lie algebra admits a left-symmetric product. For example, semisimple Lie algebras do not admit left-symmetric products, which follows from corollary 8.7.

Definition 6.4 For a left-symmetric algebra $(\mathfrak{g},*)$, let

$$L_x:\mathfrak{g}\to\mathfrak{g},\quad y\mapsto x*y$$

denote the **left-multiplication** on $\mathfrak{g}$. The map

$$L:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}),\quad x\mapsto L_x$$

is called the **left-regular representation** of $(\mathfrak{g},*)$.

Proposition 6.5 *For a left-symmetric algebra $(\mathfrak{g},*)$, the left-regular representation*

$$L:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}),\quad x\mapsto L_x$$

is a Lie algebra homomorphism.

Proof: We have to show $L([x,y])=[L_x,L_y]$ holds for all $x,y\in\mathfrak{g}$. The map $L([x,y])=L_{[x,y]}$ is defined to be

$$\Big(z\mapsto[x,y]*z\Big)=\Big(z\mapsto(x*y)*z-(y*x)*z\Big).$$

On the other hand, $[L(x),L(y)]=[L_x,L_y]$ is defined to be

$$\Big(z\mapsto(L_xL_y-L_yL_x)(z)\Big)=\Big(z\mapsto x*(y*z)-y*(x*z)\Big).$$

By the left-symmetry of $*$, both expressions are equal. ■

Given a left-symmetric product on $\mathfrak{g}$, we can construct new left-symmetric products by the following transformation.

Proposition 6.6 *Let $(\mathfrak{g},*)$ be a left-symmetric algebra and $\psi:\mathfrak{g}\to\mathfrak{g}$ a Lie algebra endomorphism of $\mathfrak{g}$ such that $\mathrm{id}_{\mathfrak{g}}-\psi$ is bijective. Let $\varphi=(\mathrm{id}_{\mathfrak{g}}-\psi)^{-1}$. Then*

$$x\bullet y=\varphi\Big(x*\varphi^{-1}(y)-\varphi^{-1}(y)*\psi(x)\Big)$$

defines a left-symmetric product $\bullet$ on $\mathfrak{g}$.

Proof: Direct computation. ■

The product $\bullet$ from proposition 6.6 is called the **Helmstetter transform** of the product $*$.

Two left symmetric products $\star$ and $*$ on $\mathfrak{g}$ are called **isomorphic** if there exists an automorphism φ of $\mathfrak{g}$ such that $\varphi(x * y) = \varphi(x) \star \varphi(y)$ holds for every $x, y \in \mathfrak{g}$.

Next, we meet the most gregarious elements of a left-symmetric algebra, those who associate with everyone.[4)]

Definition 6.7 Let $(\mathfrak{g}, *)$ be a left-symmetric algebra. The **nucleus** of $\mathfrak{g}$ is

$$\mathfrak{nuc}(\mathfrak{g}, *) = \{z \in \mathfrak{g} \mid (x, y, z) = 0 \text{ for all } x, y \in \mathfrak{g}\}.$$

The nucleus is an associative subalgebra of $(\mathfrak{g}, *)$.

6.2 Examples

Example 6.8 The most obvious example for a left-symmetic algebra is an associative algebra $(V, \cdot)$. As $(x, y, z) = 0$ for all $x, y, z \in V$, the condition defining a left-symmetric product are trivially satisfied.

Example 6.9 Let $\mathfrak{vect}(M)$ be the Lie algebra of smooth vector fields on a manifold M with an affine connection ∇. The connection ∇ is **torsion free** if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

holds for any $X, Y \in \mathfrak{vect}(M)$. Further, ∇ is **flat** if

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

holds for all $X, Y, Z \in \mathfrak{vect}(M)$. Such a flat, torsion free connection determines a covariant derivative

$$\nabla_X : \mathfrak{vect}(M) \to \mathfrak{vect}(M), \quad Y \mapsto \nabla_X Y.$$

If we define a product $*$ on $\mathfrak{vect}(M)$ by

$$X * Y = \nabla_X Y,$$

we see that this is a bilinear product over the field $\mathbb{k} = \mathbb{R}$, and the conditions that ∇ is flat and torsion free makes $\mathfrak{vect}(M)$ into a left-symmetric algebra over $\mathbb{R}$. This example is further explained in section 2.8 in Burde [6].

[4)] This pun must be credited to Kevin McCrimmon.

Example 6.10 A **convex homogeneous cone** C is a non-empty open subset of $\mathbb{R}^n$ such that $x \in C$ implies $\lambda x \in C$ for $\lambda > 0$, that $x, y \in C$ implies $x + y \in C$ and that the closure of C does not contain any subspace of $\mathbb{R}^n$ of dimension > 0. Without loss of generality, we may assume that the vertex of the cone C is 0, so the group G_C of affine transformations leaving C invariant is contained in $\mathrm{GL}_n(\mathbb{R})$. We can factor G_C into

$$G_C = HT,$$

where H is the isotropy subgroup of some $v_0 \in C$ and T is the maximal connected triangular subgroup of G_C. Let $\mathfrak{g}_C = \mathfrak{Lie}(G_C)$ and $\mathfrak{t} = \mathfrak{Lie}(T)$. Then

$$\mathrm{eval} : \mathfrak{t} \to \mathbb{R}^n, \quad X \mapsto X \cdot v_0$$

is an isomorphism of vector spaces. Let $X_v = \mathrm{eval}^{-1}(v)$. Define a product $*$ on $\mathbb{R}^n$ by

$$v * w = X_v \cdot w.$$

By the commutation rules in $\mathfrak{g}_C$, we have

$$\begin{aligned} [X_v, X_w](v_0) &= X_v \cdot w - X_w \cdot v = v * w - w * v, \\ [X_v, X_w] &= X_{v*w-w*v}. \end{aligned}$$

This implies $(u, v, w) = (v, u, w)$ for all $u, v, w \in \mathbb{R}^n$, so $(\mathbb{R}^n, *)$ is a left-symmetric algebra over $\mathbb{k} = \mathbb{R}$. It even shows that there exists a two-sided unit element with respect to $*$. See section 2.7 in Burde [6] for a more general exposition of this example.

7 Étale Representations

In this section we explain how certain representations of an affine Lie algebra $\mathfrak{g}$ are related to left-symmetric products on $\mathfrak{g}$.

7.1 Affine Representations

Definition 7.1 Let V be vector space over $\mathbb{k}$. We can identify V with the hyperplane $V \times \{1\}$ in $V \oplus \mathbb{k}$. The **affine group** of V is

$$\mathrm{Aff}(V) = \left\{ \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \,\middle|\, g \in \mathrm{GL}(V), v \in V \right\} \subset \mathrm{GL}(V \oplus \mathbb{k}).$$

Its Lie algebra is

$$\mathfrak{aff}(V) = \left\{ \begin{pmatrix} X & w \\ 0 & 0 \end{pmatrix} \,\middle|\, X \in \mathfrak{gl}(V), w \in V \right\} \subset \mathfrak{gl}(V \oplus \mathbb{k}).$$

The affine group provides us with a matrix representation for the affine transformations on V. If we identify $x \in V$ with $\binom{x}{1} \in V \oplus \mathbb{k}$, then the affine transformation given by multiplication with a matrix $g \in \mathrm{GL}(V)$ and translation by a vector v,

$$y = g \cdot x + v,$$

can be rewritten in matrix form:

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Hence, for $A = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(V)$ (resp. $A = \begin{pmatrix} X & w \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(V)$) we call g (resp. X) the **linear part** A_{lin} of the affine transformation A, and v (resp. w) the **translational part** A_{tra} of A.

For every $v \in V$ we have a map

$$\mathrm{eval}_v : \mathfrak{aff}(V) \to V, \quad A \mapsto A.v = A_{\mathrm{lin}} v + A_{\mathrm{tra}},$$

which we call the **evaluation map** at v.

Remark 7.2 Let $A \in \mathrm{Aff}(V)$. Then eval_{Ax} and eval_x are related by

$$\mathrm{eval}_{Ax} \circ \mathrm{Ad}(A) = A_{\mathrm{lin}} \mathrm{eval}_x.$$

To see this, let $g = A_{\mathrm{lin}}$, $v = A_{\mathrm{tra}}$. For any $B = \begin{pmatrix} X & w \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(V)$ we have

$$\begin{aligned} \big(\mathrm{eval}_{Ax} \circ \mathrm{Ad}(A)\big)(B) &= \mathrm{eval}_{Ax}(AXA^{-1}) = (AXA^{-1}).Ax = (AX).x \\ &= \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & w \\ 0 & 0 \end{pmatrix} .x = \begin{pmatrix} gX & gw \\ 0 & 0 \end{pmatrix} .x \end{aligned}$$

and the last term equals $g(Xx + w) = A_{\mathrm{lin}}(B.x)$.

Remark 7.3 For $A = \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} Y & w \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(V)$, define a composition by the matrix multiplication AB. Then

$$\mathrm{eval}_x(AB) = X \cdot \mathrm{eval}_x(B),$$

because

$$\begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} Y & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} XY & Xw \\ 0 & 0 \end{pmatrix},$$

which implies $\mathrm{eval}_x(AB) = X(Yx + w) = X(B.x)$.

Let $\mathfrak{g}$ be a Lie algebra and $d\varrho : \mathfrak{g} \to \mathfrak{aff}(V)$ an affine representation of $\mathfrak{g}$. The **evaluation map** at $x \in V$ is defined by

$$\mathrm{ev}_x = \mathrm{eval}_x \circ d\varrho.$$

Remark 7.4 Let $\mathrm{d}\tilde{\varrho} = \mathrm{Ad}(A) \circ \mathrm{d}\varrho$ be a representation conjugate to $\mathrm{d}\varrho$. Then the evaluation map $\tilde{\mathrm{ev}}_{Ax}$ with respect to $\mathrm{d}\tilde{\varrho}$ satisfies

$$\tilde{\mathrm{ev}}_{Ax} = A_{\mathrm{lin}}\mathrm{ev}_x.$$

Definition 7.5 A representation $\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{aff}(V)$ is called an **étale representation** if there exists a point $x_0 \in V$ such that the evaluation map

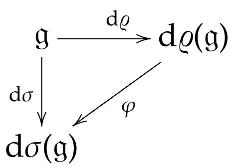
$$\mathrm{ev}_{x_0} : \mathfrak{g} \to V, \quad X \mapsto \mathrm{d}\varrho(X).x_0$$

is an isomorphism of vector spaces. In this case, x_0 is called a **generic point**. Equivalently, a representation $\varrho : G \to \mathrm{Aff}(V)$ of an algebraic group G is called an **étale representation** if there exists a point $x_0 \in V$ such that the orbit $\varrho(G).x_0$ is Zariski-open in V and the isotropy subgroup G_{x_0} at x_0 is a finite group.

We write $(\mathrm{d}\varrho, x_0)$ to indicate that x_0 is the generic point of $\mathrm{d}\varrho$.

Remark 7.6 Let $(\mathrm{d}\varrho, x)$ and $(\mathrm{d}\sigma, y)$ be étale representations of $\mathfrak{g}$ on V.

1. If $(\mathrm{d}\varrho, x)$ and $(\mathrm{d}\sigma, y)$ are conjugate with $\mathrm{Ad}(g) \circ \mathrm{d}\varrho = \mathrm{d}\sigma$, $g \in \mathrm{Aff}(V)$, then we have $y = gx$.
2. As étale representations must be faithful, $(\mathrm{d}\varrho, x)$ and $(\mathrm{d}\sigma, y)$ are equivalent if and only if $(\mathrm{d}\varrho \circ \psi, x)$ and $(\mathrm{d}\sigma, y)$ are conjugate for some automorphisms ψ of $\mathfrak{g}$.[5)] This is because equivalence of representations requires the existence of a Lie algebra isomorphism $\varphi : \mathrm{d}\varrho(\mathfrak{g}) \to \mathrm{d}\sigma(\mathfrak{g})$:



As all maps in this diagram are bijective, we have $\mathrm{d}\sigma(\mathfrak{g}) = (\varphi \circ \mathrm{d}\varrho)(\mathfrak{g})$ and we can set $\psi = \mathrm{d}\sigma^{-1} \circ \varphi \circ \mathrm{d}\varrho$.

Definition 7.7 An affine étale representation $(\mathrm{d}\sigma, x)$ is called **linear** if it is equivalent to an étale representation $\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$, or equivalently, if it has a fixed point in V.

7.2 The Correspondence with Left-Symmetric Algebras

Assume $\dim(V) = \dim(\mathfrak{g})$ throughout this section. We will now show that étale representations induce left-symmetric products and vice versa.

5) It is in this sense that equivalence is defined in Baues [2] (where conjugate representations are called *isomorphic*) and we will use it in this sense for the rest of the chapter.

Proposition 7.8 *Let $\varphi : \mathfrak{g} \to V$ a vector space isomorphism and $*$ are left-symmetric product on $\mathfrak{g}$ with left-regular representation L. Then the map*

$$\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{aff}(V), \quad X \mapsto \begin{pmatrix} \varphi \circ L_X \circ \varphi^{-1} & \varphi(X) \\ 0 & 0 \end{pmatrix}$$

defines an étale representation of $\mathfrak{g}$ with generic point 0.

Proof: We have

$$\mathrm{ev}_0(X) = (\varphi(L_X(\varphi^{-1}(0))) + \varphi(X) = 0 + \varphi(X),$$

and as φ is an isomorphism, so is ev_0. For $[X, Y] \in \mathfrak{g}$ we have

$$\begin{aligned} [\mathrm{d}\varrho(X), \mathrm{d}\varrho(Y)] &= \begin{pmatrix} \varphi \circ L_X \circ \varphi^{-1} & \varphi(X) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \circ L_Y \circ \varphi^{-1} & \varphi(Y) \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} \varphi \circ L_Y \circ \varphi^{-1} & \varphi(Y) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \circ L_X \circ \varphi^{-1} & \varphi(X) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi \circ L_X L_Y \circ \varphi^{-1} & (\varphi \circ L_X)(Y) \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \varphi \circ L_Y L_X \circ \varphi^{-1} & (\varphi \circ L_Y)(X) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi \circ [L_X, L_Y] \circ \varphi^{-1} & \varphi([X, Y]) \\ 0 & 0 \end{pmatrix} \\ &= \mathrm{d}\varrho([X, Y]), \end{aligned}$$

so $\mathrm{d}\varrho$ is a Lie algebra representation. ■

Proposition 7.9 *Let $\mathrm{d}\varrho : \mathfrak{g} \to \mathfrak{aff}(V)$ be an affine étale representation of $\mathfrak{g}$ with generic point 0. Then*

$$X * Y = \mathrm{ev}_0^{-1}\Big(\mathrm{d}\varrho(X)_{\mathrm{lin}}.(\mathrm{ev}_0(Y))\Big),$$

defines a left-symmetric product on $\mathfrak{g}$.

Proof: For $X, Y, Z \in \mathfrak{g}$, let

$$\mathrm{d}\varrho(X) = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \ \mathrm{d}\varrho(Y) = \begin{pmatrix} B & b \\ 0 & 0 \end{pmatrix}, \ \mathrm{d}\varrho(Z) = \begin{pmatrix} C & c \\ 0 & 0 \end{pmatrix}.$$

As we evaluate at 0, we have

$$\mathrm{ev}_0(\mathrm{d}\varrho(X)) = \mathrm{d}\varrho(X)_{\mathrm{tra}} = a, \ \mathrm{ev}_0(\mathrm{d}\varrho(Y)) = b, \ \mathrm{ev}_0(\mathrm{d}\varrho(Z)) = c.$$

To prove the left-symmetry of $*$, we must show

$$(X, Y, Z) - (Y, X, Z) = 0.$$

Because ev_0 is an isomorphism, it is sufficient to show

$$\begin{aligned} \mathrm{ev}_0((X, Y, Z) - (Y, X, Z)) &= \mathrm{ev}_0(X * (Y * Z) - Y * (X * Z)) - \mathrm{ev}_0((X * Y) * Z - (Y * X) * Z) \\ &= 0. \end{aligned} \tag{$*$}$$

First, we have

$$\begin{aligned}\mathrm{ev}_0(X*(Y*Z)) &= \mathrm{d}\varrho(X)_{\mathrm{lin}}.\mathrm{ev}_0(Y*Z)\\ &= A.(\mathrm{d}\varrho(Y)_{\mathrm{lin}}.\mathrm{ev}_0(Z))\\ &= A.(B.c)\end{aligned}$$

and accordingly $\mathrm{ev}_0(Y*(X*Z)) = B.(A.c)$. Then it follows that

$$\begin{aligned}\mathrm{ev}_0(X*(Y*Z)-Y*(X*Z)) &= A.(B.c)-B.(A.c)\\ &= [A,B].c\\ &= [\mathrm{d}\varrho(X),\mathrm{d}\varrho(Y)]_{\mathrm{lin}}.c. \qquad (**)\end{aligned}$$

Next, we have

$$\begin{aligned}\mathrm{ev}_0((X*Y)*Z) &= \mathrm{d}\varrho(X*Y)_{\mathrm{lin}}.c\\ &= \mathrm{d}\varrho(\mathrm{ev}_0^{-1}(\mathrm{d}\varrho(X)_{\mathrm{lin}}.\mathrm{ev}_0(Y)))_{\mathrm{lin}}.c\\ &= \mathrm{d}\varrho(\mathrm{ev}_0^{-1}(A.b))_{\mathrm{lin}}.c\end{aligned}$$

and accordingly $\mathrm{ev}_0((Y*X)*Z) = \mathrm{d}\varrho(\mathrm{ev}^{-1}(B.a))_{\mathrm{lin}}.c$. This implies

$$\begin{aligned}\mathrm{ev}_0((X*Y)*Z-(Y*X)*Z) &= \mathrm{d}\varrho(\mathrm{ev}_0^{-1}(A.b))_{\mathrm{lin}}.c-\mathrm{d}\varrho(\mathrm{ev}_0^{-1}(B.a))_{\mathrm{lin}}.c\\ &= \mathrm{d}\varrho(\mathrm{ev}_0^{-1}(A.b)-\mathrm{ev}_0^{-1}(B.a))_{\mathrm{lin}}.c\\ &= \mathrm{d}\varrho((\mathrm{ev}_0^{-1}(A.b-B.a))_{\mathrm{lin}}.c \qquad (***)\end{aligned}$$

Now, $\mathrm{ev}_0^{-1}(A.b-B.a)$ must be some element of $\mathfrak{g}$ which gets mapped to a matrix with translational part $A.b-B.a$ under $\mathrm{d}\varrho$. As ev_0 is bijective, there is only one such matrix in $\mathrm{d}\varrho(\mathfrak{g})$. On the other hand, such a matrix is given by

$$[\mathrm{d}\varrho(X),\mathrm{d}\varrho(Y)] = \begin{pmatrix}[A,B] & A.b-B.a\\ 0 & 0\end{pmatrix}.$$

Now, by comparing (**) and (***), we get

$$[\mathrm{d}\varrho(X),\mathrm{d}\varrho(Y)] = \mathrm{d}\varrho((\mathrm{ev}_0^{-1}(A.b-B.a)),$$

and so (*) holds. ■

Note that the constructions of proposition 7.9 can be extended to étale representations with generic point $v \neq 0$ by conjugation with a translation $g \in \mathrm{Aff}(V)$ moving the origin 0 to v.

Remark 7.10 For $X, Y \in \mathfrak{g}$ we have

$$\mathrm{ev}_v(\mathrm{d}\varrho(X*Y)) = \mathrm{ev}_v(\mathrm{d}\varrho(X)\cdot\mathrm{d}\varrho(Y)),$$

where $\cdot$ is the associative matrix product.

For a fixed Lie algebra $\mathfrak{g}$, let $\mathcal{E}_0$ (resp. $\mathcal{E}$) be the set of étale representations with generic point 0 (resp. with any generic point, i.e. $\mathcal{E}_0 \subset \mathcal{E}$), and let $\mathcal{L}$ be the set of left-symmetric products. By proposition 7.8 and 7.9, we obtain two maps

$$\Phi : \mathcal{L} \to \mathcal{E}_0 \quad \text{and} \quad \Psi : \mathcal{E} \to \mathcal{L}.$$

Proposition 7.11 *The maps Φ and Ψ induce a bijection between $\mathcal{L}$ and $\mathcal{E}_0$, where isomorphic left-symmetric products are mapped to equivalent étale representations and vice versa.*

Sketch of proof: Let $(\mathrm{d}\varrho, x)$ be an étale representation with generic point x. For $g \in \mathrm{Aff}(V)$, the representation $(\mathrm{d}\sigma = \mathrm{Ad}(g) \circ \mathrm{d}\varrho, gx)$ is conjugate to $\mathrm{d}\varrho$. Using that the evaluation map ev_{gx} for $\mathrm{d}\sigma$ is given by $\mathrm{ev}_{gx} = g_{\mathrm{lin}}.\mathrm{ev}_x$, we see that the left-symmetric products defined for $\mathrm{d}\varrho$ and $\mathrm{d}\sigma$ coincide. This means that Ψ can be defined on the set of conjugacy classes of étale affine representations.

From the definition of Φ and Ψ we have $\Psi \circ \Phi = \mathrm{id}_{\mathcal{L}}$. To prove that $\Phi \circ \Psi(\mathrm{d}\varrho, x)$ is an étale representation with generic point 0 conjugate to $(\mathrm{d}\varrho, x)$, conjugate the étale representation with some suitable element $g \in \mathrm{Aff}(V)$.

Similarly, one shows that isomorphic left-symmetric products are mapped to equivalent étale representations, and vice versa.

For a full proof, see proposition 2.1 in Baues [2]. ∎

The next result gives us a criterion of linearity for an affine étale representation.

Proposition 7.12 *An étale representation is isomorphic to a linear representation if and only if the corresponding left-symmetric algebra has a right-identity.*

Proof: Use the formula

$$X * Y = \mathrm{ev}_{x_0}^{-1}\Big(\mathrm{d}\varrho(X)_{\mathrm{lin}}.(\mathrm{ev}_{x_0}(Y))\Big).$$

Then $y \in V$ is a fixed point if and only if $\mathrm{ev}_{x_0}^{-1}(x_0 - y)$ is a right identity. ∎

8 Left-Symmetric Algebras for $\mathfrak{gl}_n$

In this chapter we assume $\mathbb{k} = \mathbb{C}$.

To prepare the classification of left-symmetric products for $\mathfrak{gl}_n$ in section 8.3, we have to study some properties of reductive and semisimple algebraic groups and their relations to étale representations. In definition 7.5, we also gave a definition of étale representations for groups.

Definition 8.1 Let G be an algebraic group and $\varrho : G \to \mathrm{Aff}(V)$ a rational étale representation of G. Then the module (G, ϱ, V) is called a **special module**.[6)]

In this chapter, let $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{s}$ be a reductive Lie algebra with centre $\mathfrak{gl}_1$ and semisimple part $\mathfrak{s}$. For $\mathfrak{s}$, there exists a connected, simply-connected algebraic group S such that $\mathfrak{s} = \mathfrak{Lie}(S)$. If $(\mathrm{d}\varrho, x)$ is an étale representation of $\mathfrak{g}$, then $\mathrm{d}\sigma = \mathrm{d}\varrho|_{\mathfrak{s}}$ arises as the differential of a rational representation $\sigma : S \to \mathrm{Aff}(V)$ (see theorem 4.72) and the module $(\mathrm{GL}_1 \times S, \mu \otimes \sigma, V)$ is a special module with Lie algebra $\mathfrak{g}$. Thus we can view V as an algebraic variety for S. The restriction of the evaluation map ev_x on $\mathfrak{s}$ is the differential of the orbit map of S,

$$g \mapsto \sigma(g).x.$$

As $\mathrm{d}\varrho$ is an étale representation, the orbit $\sigma(S).x$ must be of maximal dimension $\dim(S)(= \dim(G) - 1 = \dim(V) - 1)$, and this is the case for all orbits on a Zariski-open subset of V (this follows from proposition 3.45).

We may even assume σ to be a linear representation, as any semisimple group has a fixed point in V.

8.1 Algebraic Quotients for Semisimple Groups

Recall from section 5.1 that the algebraic quotient $V /\!\!/ G$ of the action of a reductive algebraic group on an module V is the variety with coordinate ring $\mathbb{C}[V]^G$. It can be interpreted as the set of closed orbits under the action of G.

We study the algebraic quotient for the action of a semisimple group S such that S has orbits of dimension $\dim(S) = \dim(V) - 1$ on a Zariski-open subset of V. In particular, this is the case if the action of S arises by restricting an étale representation of a reductive group with centre GL_1 to its semisimple part. We will see in this section that the converse holds.

Proposition 8.2 *The quotient variety $V /\!\!/ S$ is isomorphic to the affine line $\mathbb{C}$. The ring of invariants $\mathbb{C}[V]^S$ is generated by an irreducible, homogeneous $f \in \mathbb{C}[V]^S$ with* $\deg(f) > 0$.

Proof: As S is semisimple, we have $\mathbb{C}(V)^S = \mathrm{Quot}(\mathbb{C}[V]^S)$. By proposition 5.9, we have

$$\begin{aligned}\dim(V /\!\!/ S) &= \mathrm{trdeg}_{\mathbb{C}}(\mathrm{Quot}(\mathbb{C}[V]^S)) \\ &= \mathrm{trdeg}_{\mathbb{C}}(\mathbb{C}(V)^S) = \dim(V) - \max\{\dim(\sigma(S).v) \mid v \in V\} \\ &= 1.\end{aligned}$$

[6)]This definition of special modules differs slightly from the definition in Baues [2], where the definition is made for semisimple algebraic groups.

Now the proposition follows from proposition 5.11, where f is the generator of the homogeneous ideal of invariants vanishing at 0. ■

As a consequence of this proposition, we can identify the quotient mapping $\pi : V \to V/\!\!/S$ with $f : V \to \mathbb{C}$.

Remark 8.3 The zero fiber $F_0 = f^{-1}(0)$ is invariant under the action of GL_1, and $\{0\}$ is the only closed S-orbit in F_0, since the elements of $V/\!\!/S$ correspond bijectively to the closed S-orbits. For $c \in \mathbb{C}^\times$, let $F_c = f^{-1}(c)$. Then the action of GL_1 permutes the fibers, as $a \in \mathrm{GL}_1$ maps F_c to

$$aF_c = f^{-1}(f(aF_c)) = f^{-1}(a^d f(F_c)) = f^{-1}(a^d c) = F_{a^d c},$$

where $d = \deg(f)$.

Proposition 8.4 *Every fiber $F_c = f^{-1}(c)$ for $c \in \mathbb{C}^\times$ consists of a single closed S-orbit of codimension 1.*

Sketch of proof: S has an orbit of codimension 1 in every fiber F_c where $c \neq 0$. Since the isotropy subgroup of such an orbit is finite, the orbit is an affine variety by theorem 4.69. One can show that every such orbit is closed.

Since every fiber F_c contains exactly one closed orbit in its closure, F_c must consist of a single closed orbit.

For a full proof, see theorem 3.4 in Baues [1]. ■

Corollary 8.5 *The only fixed point for the action of S on V is 0.*

Combining remark 8.3 and proposition 8.4, we arrive at the following theorem.

Theorem 8.6 *Let S be a semisimple algebraic group acting on a vector space V, such that $\dim(S) = \dim(V) - 1$ and S has orbits of codimension 1 on a Zariski-open subset of V. This is the case if and only if V is a special module for the action of $\mathrm{GL}_1 \times S$ on V induced by the action of S.*

Corollary 8.7 *The semisimple group S has no étale representations.*

Proof: See corollary 3.7 in Baues [1]. ■

8.2 Classification of Étale Representations

We consider an étale representation $\mathrm{d}\varrho$ of the reductive Lie algebra $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{s}$. We may assume that the representation $\mathrm{d}\sigma = \mathrm{d}\varrho|_{\mathfrak{s}}$ is linear and that $\mathrm{d}\sigma$ is the differential of a representation $\sigma : S \to \mathrm{GL}(V)$ for semisimple S such that S has orbits of dimension $\dim(S) = \dim(V) - 1$ on a Zarsiki-open subset of V.

Remark 8.8 Consider the normaliser $\mathrm{N}_{\mathrm{Aff}(V)}(\sigma(S))$. Then the fixed point 0 is a fixed point for $\mathrm{N}_{\mathrm{Aff}(V)}(\sigma(S))$ as well, so the normaliser is contained in $\mathrm{GL}(V)$. As a consequence, we may assume $\mathrm{d}\varrho(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$.

Lemma 8.9 *The generic point x of $\mathrm{d}\varrho$ is not contained in the zero fiber $F_0 = f^{-1}(0)$ of the quotient $V /\!\!/ S$, where f is the invariant polynomial from proposition 8.2.*

Proof: See lemma 3.10 in Baues [2]. ■

Lemma 8.10 *The representation $\mathrm{d}\varrho$ of $\mathfrak{g}$ is étale if and only if $\mathrm{d}f_x(\mathrm{d}\varrho(\mathfrak{gl}_1).x) \neq \{0\}$.*

Proof: The map ev_x restricted to $\mathfrak{s}$ is the differential of the orbit mapping $g \mapsto \sigma(g).x$ of S, so it is injective. As ev_x is surjective, it is an isomorphism if and only if it is injective on $\mathfrak{gl}_1$, and this is the case if and only if $\mathrm{d}f_x(\mathrm{d}\varrho(\mathfrak{gl}_1).x) \neq \{0\}$. ■

Remark 8.11 Since $\mathrm{d}\varrho$ is linear, $\mathrm{d}\varrho(\mathfrak{gl}_1)$ is contained in the centraliser $\mathfrak{z}$ of $\mathrm{d}\varrho(\mathfrak{s})$ in $\mathfrak{gl}(V)$. The normaliser $\mathrm{N}_{\mathrm{GL}(V)}(\sigma(S))$ acts on $\mathfrak{z}$ via conjugation. Let

$$N_x = \{g \in \mathrm{N}_{\mathrm{GL}(V)}(\sigma(S)) \mid g.x = x\}.$$

Then we have

$$\mathrm{N}_{\mathrm{GL}(V)}(\sigma(S)) = \mathrm{GL}_1 \cdot \sigma(S) \cdot N_x,$$

and $\mathrm{GL}_1 \cdot \sigma(S)$ acts trivially on $\mathfrak{z}$. Hence, the functional $Y \mapsto \mathrm{d}f_x(Y.x)$ is invariant under the action of $\mathrm{N}_{\mathrm{GL}(V)}(\sigma(S))$.

Theorem 8.12 *Let S, x and f be as above. The set of equivalence classes of étale affine representations of $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{s}$ which induce (S, σ, V) (up to equivalence) by restricting the representation to $\mathfrak{s}$ is parametrised by the conjugacy classes under $\mathrm{N}_{\mathrm{GL}(V)}(\sigma(S))$ of elements Y which satisfy $\mathrm{d}f_x(Y.x) = 1$.*

Sketch of proof: Let (ϱ_1, x) and (ϱ_2, y) be étale representations which induce (S, σ, V). As the generic point is not cointained in the zero fiber, one can find a $g \in \mathrm{GL}_1 \times \sigma(S)$ that moves x to y, so ϱ_1 and ϱ_2 are equivalent.

Let $a \in \mathfrak{gl}_1$. Then $\mathrm{d}\varrho(a) \in \mathfrak{z}$ and $\mathrm{d}f_x(\mathrm{d}\varrho(a)) \neq 0$ by lemma 8.10. Conjugation with an element $g \in N_x$ induces an automorphism φ of $\mathfrak{s}$,

$$\mathrm{Ad}(g)(\mathrm{d}\varrho(X)) = \mathrm{d}\varrho(\varphi(X))$$

for $X \in \mathfrak{s}$. Extend φ to an automorphism ψ of $\mathfrak{g}$ such that $\psi|_{\mathfrak{s}} = \varphi$ and $\psi(a) = \lambda a$ for some $\lambda \in \mathbb{C}^\times$.

Now, $\mathrm{d}\tilde{\varrho} = \mathrm{Ad}(g)(\mathrm{d}\varrho \circ \psi^{-1})$ is a representation of $\mathfrak{g}$ which induces the same representation σ of S, and we have $\mathrm{d}\tilde{\varrho}(a) = \frac{1}{\lambda}\mathrm{Ad}(g)(\mathrm{d}\varrho(a))$. Thus, $\mathrm{d}\varrho(a)$ is defined up to scalar multiplication and up to conjugation with $g \in \mathrm{N}_{\mathrm{GL}(V)}(\sigma(S))$. Now the claimed correspondence follows easily. ■

8.3 The Classification for $\mathfrak{gl}_n$

In this section we apply the results on modules for semisimple groups to left-symmetric algebras. As before, we consider a reductive algebra $\mathfrak{g} = \mathfrak{gl}_1 \oplus \mathfrak{s}$, and we assume that there exists a left-symmetric product $*$ on $\mathfrak{g}$.

Proposition 8.13 *The left-symmetric algebra* $(\mathfrak{g}, *)$ *has a unique right-identity.*

Proof: This follows immediately from remark 8.8 and proposition 7.12. ∎

Let $(\mathfrak{g}, *)$ and $(\mathfrak{g}, \star)$ be left-symmetric algebras with respective left-regular representations $L^{[*]}$ and $L^{[\star]}$. These representations give the vector space $\mathfrak{g}$ the structure of a module for the subalgebra $\mathfrak{s}$.

Definition 8.14 We say that $(\mathfrak{g}, *)$ and $(\mathfrak{g}, \star)$ belong to the same **family**, if the representations $L^{[*]}$ and $L^{[\star]}$ for $\mathfrak{s}$ are equivalent.

Proposition 8.15 *There are only finitely many families of left-symmetric products for* $\mathfrak{g}$*. Up to isomorphism each family has a unique product with* 1*.*

Sketch of proof: From propositions 7.8 and 7.11 it follows that $(\mathfrak{g}, *)$ and $(\mathfrak{g}, \star)$ belong to the same family if and only if they induce the same special modules for S.

From theorem 8.12 we see that there is a unique equivalence class of étale representations such that $d\varrho(a) = \mathrm{id}_V$ for a fixed $a \in \mathfrak{gl}_1$. For every linear étale representation of $\mathfrak{g}$, the element $r = \mathrm{ev}_x^{-1}(x)$ is the right-identity. This is a neutral element 1 if and only if r is central, i.e. $r = a \in \mathfrak{gl}_1$. ∎

The left-symmetric product with 1 is called the **canonical representative** of its family.

Proposition 8.16 *Let* $(\mathfrak{g}, *)$ *be a left-symmetric algebra with* 1 *and* $\mathrm{Aut}(\mathfrak{g}, *)$ *its group of automorphisms. The isomorphism classes of left-symmetric algebras in the family of* $(\mathfrak{g}, *)$ *correspond to the orbits of the group* $\mathrm{GL}_1 \cdot \mathrm{Aut}(\mathfrak{g}, *)$ *in the set*

$$\mathfrak{f}(\mathfrak{g}, *) = \{X \in \mathfrak{nuc}(\mathfrak{g}, *) \mid X \notin \mathfrak{s}\}.$$

Proof: This follows from theorem 8.12, using the results from section 2.4 in Baues [2] which were not presented here. ∎

From propositions 13.2 and 13.3, we immediately see that there are only two kinds of special modules for GL_n (resp. $\mathfrak{gl}_n$): The module Ks I, where GL_n acts by matrix (left-)multiplication on Mat_n, and for $n = 2$ the module SK I-4 where GL_2 acts on $\mathrm{Sym}^3\mathbb{C}^2$, the binary forms of degree 3.

So we arrive at the main result of Baues [2].

Theorem 8.17 *The families I and II described below determine all left-symmetric algebras for* $\mathfrak{gl}_n$*.*

8.3.1 Family I

The first family of left-symmetric products for $\mathfrak{gl}_n$ arises from the special module Ks I. Here, the left-symmetric product is the associative matrix multiplication, in particular $\mathfrak{nuc}(\mathfrak{gl}_n, \cdot) = \mathfrak{gl}_n$. The neutral element is $1 = I_n$. The left-regular representation of $\mathfrak{s} = \mathfrak{sl}_n$ is given by left multiplication on $\mathfrak{gl}_n$. The set $\mathfrak{f}$ from proposition 8.16 is

$$\mathfrak{f}(\mathfrak{gl}_n, \cdot) = \{X \in \mathfrak{gl}_n \mid \operatorname{tr}(X) \neq 0\}.$$

It can be shown that the automorphism group of the associative product consists of the conjugations with elements of GL_n. Then it follows from proposition 8.16 that the isomorphism classes of left-symmetric algebras in this family correspond to the conjugacy classes of elements $X \in \mathfrak{gl}_n$ with $\operatorname{tr}(X) \neq 0$.

In section 5.2 of Baues [2] it is shown that every left-symmetric product in this family is obtained by a Helmstetter transform of the associative product.

8.3.2 Family II

The second family of left-symmetric products for $\mathfrak{gl}_2$ arises from the special module SK I-4. The left-symmetric product $*$ is induced by the étale representation of $\mathfrak{gl}_2$ on $\mathrm{Sym}^3\mathbb{C}^2$, cf. proposition 7.9. Since the module $\mathrm{Sym}^3\mathbb{C}^2$ is irreducible, the centraliser of the representation of $\mathfrak{gl}_n$ on $\mathrm{Sym}^3\mathbb{C}^2$ is $\mathfrak{gl}_1$. By a result from section 2.4 in Baues it follows that $\mathfrak{nuc}(\mathfrak{g}, *) = \mathfrak{gl}_1$, hence $\mathfrak{f}(\mathfrak{g}, *) = \mathfrak{gl}_1$. So there is only one orbit under the action of $GL_1 \cdot \mathrm{Aut}(\mathfrak{g}, *)$, i.e. only one isomorphism class of left-symmetric products by proposition 8.16.

All Helmstetter transforms of $*$ are trivial.

Part III

Prehomogeneous Modules

9 Basic Properties and Castling Transformations

Naively spoken, a homogeneous space is a variety X that looks the same everywhere with respect to the action of some algebraic group. This means the group acts transitively on each connected component of X. A vector space V cannot have this property for a *linear* and non-trivial group action on V: at least the origin 0 must be an exceptional point, as it is fixed by every group element. But we can get a next best thing to homogeneity when the group has an open orbit, i.e. V is homogeneous "almost everywhere".

Definition 9.1 Let V be a finite-dimensional vector space and $\varrho : G \to \mathrm{GL}(V)$ a rational representation of a connected linear algebraic group G. Then the module (G, ϱ, V) is called **prehomogeneous** if there exists an element $v \in V$ such that the orbit $\varrho(G).v$ is a Zariski-dense subset of V (i.e. V is the closure of $\varrho(G).v$). The elements of $\varrho(G).v$ are called **generic points**, and the elements of the **singular set** $V_{\mathrm{sing}} = V \backslash (\varrho(G).v)$ are called **singular points**.

Note that in this definition, we could equivalently require the existence of an open orbit, as the dense orbit is always an open subset (see lemma 2.1 in Kimura [14]), and any open subset is dense in the Zariski topology.

In the following chapters, we shall study some properties of prehomogeneous modules and one of the most important tools for working with them: the castling transformation. It provides us with a notion of equivalence for prehomogeneous modules and is essential for their classification. Most of the proofs are taken from § 2 in Sato, Kimura [28] or from chapters 2 and 7 in Kimura [14].

To see some examples, the reader might take a peek at the classification in chapter 11, or at section 2.4 in Kimura's book [14].

9.1 Dimension and Generic Isotropy Subgroups

Definition 9.2 For a prehomogeneous module (G, ϱ, V), the isotropy subgroup at a generic point $v \in V$ is called the **generic isotropy subgroup** of G at v. For $\mathfrak{g} = \mathfrak{Lie}(G)$, the **generic isotropy subalgebra** of $\mathfrak{g}$ at v is the isotropy subalgebra $\mathfrak{g}_v$.

Remark 9.3 If $v, w \in V$ are generic points, we have $w = \varrho(g).v$ for some $g \in G$. For $h \in G_{\varrho(g).v}$ we have $\varrho(g)^{-1}\varrho(h)\varrho(g).v = v$. So we have $g^{-1}hg \in G_v$, which implies

$$G_{\varrho(g).v} = gG_v g^{-1}.$$

The generic isotropy subgroups at different generic points are isomorphic to each other, so it is justified to speak of "the" generic isotropy subgroup of (G, ϱ, V).

If we think of the dimension of G as the number of the degrees of freedom to manipulate a point $v \in V$, then the dimension of the generic isotropy subgroup represents the redundant degrees of freedom. This is reflected by the following proposition.

Proposition 9.4 *The following conditions are equivalent:*

1. (G, ϱ, V) *is a prehomogeneous module with generic point* v.
2. $\dim(G_v) = \dim(G) - \dim(V)$.
3. $\dim(\mathfrak{g}_v) = \dim(\mathfrak{g}) - \dim(V)$, *where* $\mathfrak{g} = \mathfrak{Lie}(G)$.
4. $\{\mathrm{d}\varrho(A).v \mid A \in \mathfrak{g}\} = V$.

Proof: We can identify $\varrho(G).v$ with with the coset space G/G_v, where $\varrho(g).v$ corresponds to the coset gG_v. Hence we have

$$\dim(\varrho(G).v) = \dim(G) - \dim(G_v).$$

The condition $\mathrm{clos}(\varrho(G).v) = V$ is equivalent to $\dim(\varrho(G).v) = \dim(V)$, so we have the equivalence of the conditions 1 and 2. The equivalence of the conditions 2 and 3 follows from $\mathfrak{Lie}(G_v) = \mathfrak{g}_v$. Then we have $\mathfrak{g}/\mathfrak{g}_v \cong \mathrm{d}\varrho(\mathfrak{g}).v \subseteq V$ as vector spaces. Now $\dim(\mathfrak{g}) - \dim(\mathfrak{g}_v) = \dim(\mathfrak{g}/\mathfrak{g}_v)$ implies the equivalence of 3 and 4. ■

Corollary 9.5 *If* $\dim(G) < \dim(V)$, *then* (G, ϱ, V) *is not a prehomogeneous module for any representation* ϱ.

Corollary 9.6 *Let* (G, ϱ, V) *be a prehomogeneous module. The set of singular points* $V_{\mathrm{sing}} \subset V$ *is a closed algebraic subset of* V *and for* $w \in V_{\mathrm{sing}}$ *we have* $\dim(G_w) > \dim(G) - \dim(V)$.

Corollary 9.7 *If* $\varrho : G \to \mathrm{GL}(V)$ *is a linear rational étale representation for* G, *then* (G, ϱ, V) *is prehomogeneous.*

From proposition 9.4 we also learn that prehomogeneity can be characterised by Lie algebras. In fact, using Lie algebras often simplifies the analysis of prehomogeneous modules.

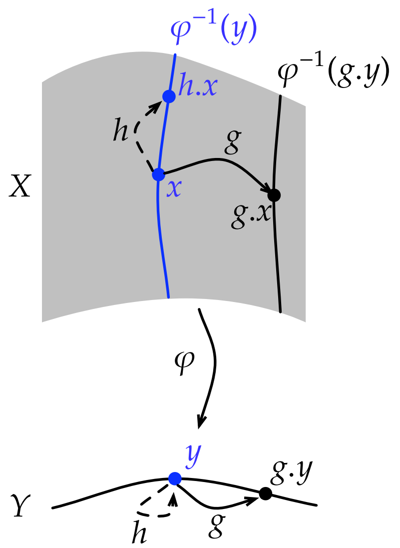
Example 9.8 Let $\varrho : G \to \mathrm{GL}(V^m)$ be a rational representation of an algebraic group G and $n \in \mathbb{N}$ with $\dim(V) = m \leq n$. Then $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V \otimes \mathbb{k}^n)$ is always a prehomogeneous module. If we identify $V^m \otimes \mathbb{k}^n$ with $\mathrm{Mat}_{m,n}$, it is easily seen that the action of $\{1\} \times \mathrm{GL}_n$ alone is sufficient to move the generic point $(I_m\ 0)$ to all matrices of rank m, which form an open subset of $\mathrm{Mat}_{m,n}$. Such a module is called a **trivial prehomogeneous module**.

Next, we will prove a proposition telling us that a sufficiently well-behaved equivariant morphism preserves prehomogeneity between irreducible varieties, where we call any algebraic variety X with a Zariski-dense orbit $G.x$ **prehomogeneous**.

Proposition 9.9 *Let G be an algebraic group and let X, Y be irreducible algebraic varieties on which G acts. Further, let $\varphi : X \to Y$ be a G-equivariant morphism such that $\operatorname{clos}(\varphi(X)) = Y$ and each fiber $\varphi^{-1}(y)$ is irreducible. Then the following conditions are equivalent:*

1. *G has a Zariski-dense orbit on X, i.e. $\operatorname{clos}(G.x) = X$ for some $x \in X$.*
2. *G has a Zariski-dense orbit on Y, i.e. $\operatorname{clos}(G.y) = Y$ for some $y \in Y$, and there exists a point $x \in \varphi^{-1}(y)$ such that $\varphi^{-1}(y) = \operatorname{clos}(G_y.x)$.*

This figure illustrates the situation of the proposition, where the action of $h \in G_y$ is indicated by the dashed lines.



Proof: 1. ⇒ 2.: By the continuity and equivariance of φ we have

$$\varphi(X) = \varphi(\operatorname{clos}(G.x)) \subseteq \operatorname{clos}(\varphi(G.x)) = \operatorname{clos}(G.\varphi(x)) \subseteq Y$$

and taking the Zariski closure yields

$$Y = \operatorname{clos}(\varphi(X)) \subseteq \operatorname{clos}(G.\varphi(x)) \subseteq Y.$$

Hence $Y = \operatorname{clos}(G.y)$ for $y = \varphi(x)$ and in particular

$$\dim(Y) = \dim(G.y) = \dim(G) - \dim(G_y).$$

But we also have

$$\dim(X) = \dim(G.x) = \dim(G) - \dim(G_x)$$

as $X = \mathrm{clos}(G.x)$. Obviously $G_x \cap G_y = G_x$, and we obtain

$$\dim(G_y.x) = \dim(G_y) - \dim(G_x) = \dim(X) - \dim(Y).$$

By proposition 3.45 there exists an open subset $U \subset Y$ such that $\dim(\varphi^{-1}(\tilde{y})) = \dim(X) - \dim(Y)$ for any $\tilde{y} \in U$. Since $U \cap G.y \neq \emptyset$, there exists a $g \in G$ such that $\varphi(g.x) = g.y \in U$. As $g.\varphi^{-1}(y) = \varphi^{-1}(g.y)$, we have

$$\dim(\varphi^{-1}(y)) = \dim(\varphi^{-1}(g.y)) = \dim(X) - \dim(Y) = \dim(G_y.x).$$

As $\varphi^{-1}(y)$ is irreducible by assumption, we have $\varphi^{-1}(y) = \mathrm{clos}(G_y.x)$.

2. $\Rightarrow$ 1.: From $\mathrm{clos}(G.y) = Y$ it follows that

$$\dim(Y) = \dim(G.y) = \dim(G) - \dim(G_y).$$

Using the existence of $x \in \varphi^{-1}(y)$ with $\varphi^{-1}(y) = \mathrm{clos}(G_y.x)$, we obtain

$$\dim(X) - \dim(Y) = \dim(\varphi^{-1}(y)) = \dim(G_y.x) = \dim(G_y) - \dim(G_x),$$

or

$$\dim(G) - \dim(G_x) = \dim(G.x).$$

As X is irreducible, we have $X = \mathrm{clos}(G.x)$. ■

We can generalise this proposition even further.

Proposition 9.10 *Let G be an algebraic group and let X, Y be algebraic varieties on which G acts. Further, let $\varphi : X \to Y$ be a G-equivariant morphism such that $\mathrm{clos}(\varphi(X)) = Y$ and if X_i is an irreducible component of X, then $\mathrm{clos}(\varphi(X_i))$ is an irreducible component of Y. Then the following conditions are equivalent:*

1. *G has a Zariski-dense orbit on X, i.e. $\mathrm{clos}(G.x) = X$ for some $x \in X$.*
2. *G has a Zariski-dense orbit on Y, i.e. $\mathrm{clos}(G.y) = Y$ for some $y \in Y$, and there exists a point $x \in \varphi^{-1}(y)$ such that $\varphi^{-1}(y) = \mathrm{clos}(G_y.x)$.*

The proof of this proposition is rather tedious and we shall omit it here. It can be found following lemma 7.6 in Kimura [14].

As noted before, the dimension of the generic isotropy subgroup H can be considered as the redundant degrees of freedom for the action of an algebraic group G on a module V. Now it is tempting to try and harness these redundant degrees of freedom by composing V with another module W with $\dim(W) \leq \dim(H)$ in such a way that we can combine the action of G on V and the action of H in W to obtain a prehomogeneous module structure on $V \oplus W$. The next proposition gives a precise formulation of this idea.

Proposition 9.11 *The following conditions are equivalent:*

1. $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ *is a prehomogeneous module.*
2. (G, ϱ_1, V_1) *is prehomogeneous and* $(H, \varrho_2|_H, V_2)$ *is also a prehomogeneous module, where* H *denotes the connected component of the generic isotropy subgroup of* (G, ϱ_1, V_1).

Proof: If we set $X = V_1 \oplus V_2$, $Y = V_1$ and let φ be the projection from $V_1 \oplus V_2$ to V_1 in proposition 9.9, we have our result. ■

Most of the time this proposition will be used when we already know (G, ϱ_1, V_1) to be prehomogeneous, so it is sufficient to check $(H, \varrho_2|_H, V_2)$ for prehomogeneity.

Considering dimensions, we get the following important corollary.

Corollary 9.12 *The following conditions are equivalent:*

1. $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ *is a special module.*
2. (G, ϱ_1, V_1) *is prehomogeneous and* $(H, \varrho_2|_H, V_2)$ *is a special module, where* H *denotes the connected component of the generic isotropy subgroup of* (G, ϱ_1, V_1).

Proof: The representation $\varrho_1 \oplus \varrho_2$ is étale if and only if it is prehomogeneous and $\dim(G) = \dim(V_1) + \dim(V_2)$. By proposition 9.11, it follows that (G, ϱ_1, V_1) and $(H, \varrho_2|_H, V_2)$ are prehomogeneous and because $\dim(G) - \dim(H) = \dim(V_1)$ we have

$$\dim(H) = \dim(V_2),$$

so $(H, \varrho_2|_H, V_2)$ is special.

Conversely, if we assume (G, ϱ_1, V_1) to be prehomogeneous and $(H, \varrho_2|_H, V_2)$ to be special, then $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ is obviously prehomogenous and as

$$\dim(G) - \dim(V_1) = \dim(H) = \dim(V_2),$$

it is even special. ■

Note that proposition 9.11 does not say that one can simply patch together (G, ϱ, V) with any prehomogeneous module (H, σ, W) for the isotropy subgroup H to obtain a new prehomogeneous module for G. Rather, it is necessary that σ can be extended to a representation $\tilde{\sigma}$ of G such that the restriction of $\tilde{\sigma}$ to H yields σ.

In the course of this thesis we will be mainly concerned with prehomogeneous modules for a reductive group G, which we will simply call **reductive prehomogeneous modules**. Further, if $G = \mathrm{GL}_1^k \times G_1 \times \cdots \times G_n$ with simple groups

$G_1, \ldots, G_n$, we will speak of a ***n*-simple prehomogeneous module**, or just of an **simple prehomogeneous module**[7)] if $n = 1$.

Reductive groups have some nice properties which will provide us with a wealth of useful results, see also section 10.2.

Proposition 9.13 *Let $\varrho : G \to \mathrm{GL}(V)$ be any finite-dimensional rational representation of a reductive algebraic group G. Then (G, ϱ, V) is equivalent to its dual module (G, ϱ^*, V^*).*

Proof: We identify V and V^* with $\Bbbk^n$, where $n = \dim(V)$. By theorem 7.1 in Mostow [26] we have $\varrho^*(G) = \varrho(G) \subset \mathrm{GL}_n$ with respect to some basis of $\Bbbk^n$. So we can choose $g^* \in G$ for each $g \in G$ such that $\varrho(g) = \varrho^*(g^*)$. Then ϱ and ϱ^* are equivalent via the isomorphism $\varphi : \varrho(G) \to \varrho^*(G), \varrho(g) \mapsto \varrho^*(g^*)$. ■

This proposition implies that $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ is equivalent to $(G, \varrho_1^* \oplus \varrho_2^*, V_1^* \oplus V_2^*)$ for a reductive group. However, in general $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ and $(G, \varrho_1 \oplus \varrho_2^*, V_1 \oplus V_2^*)$ are not equivalent. Nevertheless, we get the following relation.

Corollary 9.14 *Let (G, ϱ_1, V_1) be a prehomogeneous module with a reductive generic isotropy subgroup. Then a module $(G, \varrho_1 \oplus \varrho_2, V_1 \oplus V_2)$ is prehomogeneous if and only if $(G, \varrho_1 \oplus \varrho_2^*, V_1 \oplus V_2^*)$ is prehomogeneous, and their generic isotropy subgroups are isomorphic. In particular, one module is special if and only if the other one is.*

Proof: This follows from propositions 9.11 and 9.13. ■

9.2 The Castling Transformation

In this section we introduce one of our most potent tools in the study of prehomogeneous modules: the castling transformation. It provides us with the means to construct new prehomogeneous modules from a given one, and the question of prehomogeneity for a whole equivalence class of irreducible modules can be reduced to that of certain module which is uniquely determined for that class.

Definition 9.15 Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V^m)$ be a finite-dimensional rational representation of an algebraic group G. Then we say the modules

$$\left(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V^m \otimes \Bbbk^n\right) \quad \text{and} \quad \left(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \Bbbk^{m-n}\right)$$

are **castling transforms** of each other.

We shall also call two modules castling transforms of each other if both modules are equivalent to modules which are castling transforms of each other.

7) This is not to be confused with a prehomogeneous module for a simple group.

Theorem 9.16 (Sato) *Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V^m)$ be a finite-dimensional rational representation of an algebraic group G. Then*

$$\left(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V^m \otimes \Bbbk^n\right)$$

is a prehomogeneous module (with generic isotropy subgroup $H^{(n)}$) if and only if its castling transform

$$\left(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \Bbbk^{m-n}\right)$$

is prehomogeneous (with generic isotropy subgroup $H^{(m-n)}$). Furthermore, $H^{(n)}$ and $H^{(m-n)}$ are isomorphic.

Proof: We write $V = V^m$ and identify $V \otimes \Bbbk^n$ with $\mathrm{Mat}_{m,n}$. Then the action of $(g, h) \in G \times \mathrm{GL}_n$ on $A \in \mathrm{Mat}_{m,n}$ is given by $(g, h).A = \varrho(g)Ah^\top$. Let

$$X = \{(a_1, \dots, a_n) \in \mathrm{Mat}_{m,n} \mid a_1, \dots, a_n \in V \text{ are linearly independent}\}$$

be the set of full-rank matrices in $\mathrm{Mat}_{m,n}$. This is a Zariski-open subset of $\mathrm{Mat}_{m,n}$ and invariant under the action of $G \times \mathrm{GL}_n$.

Recall that the Grassmann variety $\mathrm{Gr}_n(V)$ is the variety of all n-dimensional subspaces of V. The mapping

$$\varphi : X \to \mathrm{Gr}_n(V), \quad (a_1, \dots, a_n) \mapsto \langle a_1, \dots, a_n \rangle$$

assigns to each matrix $(a_1, \dots, a_n) \in X$ the subspace spannend by its column vectors. An element $h \in \mathrm{GL}_n$ acts on $A \in X$ via $Ah^\top$, so the columns of $Ah^\top \in X$ consist of linear combinations of the columns of A. This means that the columns of $Ah^\top$ span the same n-dimensional subspace as the columns of A, or $\varphi(A) = \varphi(Ah^\top)$. For a subspace $\varphi(A) = \langle a_1, \dots, a_n \rangle \in \mathrm{Gr}_n(V)$ we then have

$$\varphi(\varrho(g)Ah^\top) = \varphi(\varrho(g)A).$$

If we define an action of G on $\mathrm{Gr}_n(V)$ by

$$\langle a_1, \dots, a_n \rangle \mapsto \langle \varrho(g)a_1, \dots, \varrho(g)a_n \rangle,$$

then φ is an equivariant mapping. As φ is obviously surjective, the conditions for proposition 9.9 (with $Y = \mathrm{Gr}_n(V)$) are satisfied if the fibers of φ are irreducible. This is the case, since the bases of any n-dimensional subspace $U \in \mathrm{Gr}_n(V)$ are uniquely transformed to one another by the action of GL_n, so the the fiber $\varphi^{-1}(U)$ bijectively corresponds to GL_n.

In particular, the last argument implies $\varphi^{-1}(U) = \mathrm{clos}(\mathrm{GL}_n.A)$ for some $A \in \varphi^{-1}(U)$.

With these prerequisites, we get the equivalence of the following statements from proposition 9.9:

(a) $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V \otimes \Bbbk^n)$ is a prehomogeneous module.

(b) X is a prehomogeneous variety for the action of $G \times \mathrm{GL}_n$.

(c) $Y = \mathrm{Gr}_n(V^m)$ is a prehomogeneous variety for the action of $\varrho(G)$.

A generic point A for $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V \otimes \Bbbk^n)$ is an element of X. Let $H^{(n)} = (G \times \mathrm{GL}_n)_A$ be the isotropy subgroup at A and H_U the isotropy subgroup at $U = \varphi(A)$. We define a projection

$$\pi : H^{(n)} \to H_U, \quad (g, h) \mapsto g.$$

For each element $g \in H_U$, we have $\varphi(\varrho(g)A) = \varrho(g).\varphi(A) = \varrho(g).U = U$, or $\varrho(g)A \in \varphi^{-1}(U)$. As explained above, there is a unique element $h_g \in \mathrm{GL}_n$ such that $\varrho(g)Ah_g^{\top} = A$. So we can define a mapping

$$s : H_U \to H^{(n)}, \quad g \mapsto (g, h_g).$$

Since π and s are inverses of each other, we have

$$H^{(n)} \cong H_U.$$

For any subspace $U \in \mathrm{Gr}_n(V)$ we have its annihilator

$$U^{\perp} = \{v^* \in V \mid \langle u | v^* \rangle = 0 \text{ for all } u \in U\},$$

which is an element of $\mathrm{Gr}_{m-n}(V^*)$. By the correspondence $(U^{\perp})^{\perp} = U$, we can identify $\mathrm{Gr}_n(V)$ and $\mathrm{Gr}_{m-n}(V^*)$. Since

$$\langle \varrho(g)v | \varrho^*(g)v^* \rangle = \langle v | v^* \rangle$$

for any $v \in V$ and $v^* \in V^*$, we have

$$(\varrho(g).U)^{\perp} = \varrho^*(g).U^{\perp}$$

for $g \in G$. This implies that statement (c) is equivalent to

(d) $\mathrm{Gr}_{m-n}(V^*)$ is a prehomogeneous variety for the action of $\varrho^*(G)$.

It also follows that $g \in H_U$ if and only if $g \in H_{U^{\perp}}$ (the generic isotropy subgroup of G at $U^{\perp}$).

Now, we might as well formulate (a) with ϱ, GL_n and V replaced by ϱ^*, GL_{m-n} and V^*. Then, by the equivalenc of (c) and (d), we get our result, namely that $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V \otimes \Bbbk^n)$ is prehomogeneous if and only if $(G \times \mathrm{GL}_{m-n}, \varrho^* \otimes \omega_1, V^* \otimes \Bbbk^{m-n})$ is prehomogeneous, and $H^{(n)} \cong H_U = H_{U^{\perp}} \cong H^{(m-n)}$. ■

Corollary 9.17 *A castling transform of a special module is also special.*

Proof: The generic isotropy subgroup of a special module is finite, and so is the generic isotropy subgroup of its castling transform. ■

Next, we have a corollary to the proof of theorem 9.16.

Corollary 9.18 *In the situation of the proof of theorem 9.16, let $A \in \mathrm{Mat}_{m,n}$ be a generic point for the action $G \times \mathrm{GL}_n$ with isotropy subgroup $H^{(n)}$ at A. Then there is a generic point $B \in \mathrm{Mat}_{m,m-n}$ for the action of $G \times \mathrm{GL}_{m-n}$ with isotropy subgroup $H^{(m-n)}$ at B such that the projections on the G-components of $H^{(n)}$ resp. $H^{(m-n)}$ are identical and isomorphic to $H^{(n)}$ (resp. $H^{(m-n)}$).*

Proof: For $A \in \mathrm{Mat}_n$, we have $U = \varphi(A)$, and the projection on the G-component of $H^{(n)}$ is H_U. Now choose $B \in \mathrm{Mat}_{m,m-n}$ so that B is mapped to $U^\perp \in \mathrm{Gr}_{m-n}(V^*)$. Then the projection of $H^{(m-n)}$ on the G-component is $H_{U^\perp}$. By the proof of theorem 9.16, $H^{(n)} \cong H_U = H_{U^\perp} \cong H^{(m-n)}$. ■

We will now show that GL_n can be replaced by SL_n in theorem 9.16 under some mild assumptions.

Lemma 9.19 *Let G be a connected algebraic group and $\varrho : G \to \mathrm{GL}(V)$ a rational representation. Assume that $(\mathrm{GL}_1 \times G, \mu \otimes \varrho, V)$ is prehomogeneous with generic isotropy subgroup H. Then (G, ϱ, V) is prehomogeneous if and only if the connected component of H is not contained in G.*

Proof: First, note that there is a common generic point for $(\mathrm{GL}_1 \times G, \mu \otimes \varrho, V)$ and (G, ϱ, V) if both are prehomogeneous, because the open dense orbits intersect in a non-empty open subset. So we may assume that in this case $G \cap H$ is the generic isotropy subgroup of (G, ϱ, V).

(G, ϱ, V) is prehomogeneous if and only if

$$\dim(G) - \dim(G \cap H) = \dim(V) = \dim(\mathrm{GL}_1 \times G) - \dim(H),$$

i.e. $\dim(G \cap H) = \dim(H) - 1$. But this is the case if and only if the connected component of H is not contained in G. ■

Theorem 9.20 *Let $m > n \geq 1$ and $\varrho : G \to \mathrm{GL}(V^m)$ be a faithful irreducible representation of an algebraic group G. Then*

$$\left(G \times \mathrm{SL}_n,\ \varrho \otimes \omega_1,\ V^m \otimes \mathbb{k}^n\right)$$

is a prehomogeneous module if and only if

$$\left(G \times \mathrm{SL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \mathbb{k}^{m-n}\right)$$

is prehomogeneous. Further, their generic isotropy subgroups are isomorphic.

Proof: We may assume G to be reductive by theorem 4.62 with at most one-dimensional centre. When G has a one-dimensional centre, i.e. $G = \mathrm{GL}_1 \times G_0$ for semisimple G_0, the representation of $G \times \mathrm{SL}_n$ is equivalent to a representation of $G_0 \times \mathrm{GL}_n$ and this is just the setting of theorem 9.16.

Now assume that G is semisimple. If $(G \times \mathrm{SL}_n, \varrho \otimes \omega_1, V^m \otimes \Bbbk^n)$ is prehomogeneous with generic isotropy subgroup H, then $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V^m \otimes \Bbbk^n)$ is also prehomogeneous with generic isotropy subgroup isomorphic to $\mathrm{GL}_1 \times H$ by lemma 9.19. Then theorem 9.16 tells us that $(G \times \mathrm{GL}_{m-n}, \varrho \otimes \omega_1, V^{m*} \otimes \Bbbk^{m-n})$ is also prehomogeneous with generic isotropy subgroup $\tilde{H}$ isomorphic to $\mathrm{GL}_1 \times H$. Since $\tilde{H} \cap (G \times \mathrm{SL}_{m-n}) \cong H$, it follows again by lemma 9.19 that $(G \times \mathrm{SL}_{m-n}, \varrho \otimes \omega_1, V^{m*} \otimes \Bbbk^{m-n})$ is prehomogeneous. ■

Remark 9.21 In theorem 9.20 we can drop the assumption that ϱ is irreducible if G is of the form $G = \mathrm{GL}_1 \times G_0$, with GL_1 acting by scalar multiplication, for *any* algebraic group G_0, because in this case we have just the setting of theorem 9.16, where irreducibility is not required.

Remark 9.22 If G is reductive, proposition 9.13 tells us that we can replace $\varrho^* \otimes \omega_1$ by $\varrho \otimes \omega_1$ in theorems 9.16 and 9.20.

Remark 9.23 Theorem 9.20 gives us a method to obtain infinitely many new prehomogeneous modules out of a given one. To simplify the notation we assume G is reductive. We can identify a prehomogeneous module (G, ϱ, V) with $m = \dim(V) \geq 2$ with $(G \times \mathrm{SL}_1, \varrho \otimes \omega_1, V)$ and obtain a new prehomogeneous module $(G \times \mathrm{SL}_{m-1}, \varrho \otimes \omega_1, V \otimes \Bbbk^{m-1})$. Repeating this procedure, we obtain $(G \times \mathrm{SL}_{m-1} \times \mathrm{SL}_{m^2-m-1}, \varrho \otimes \omega_1 \otimes \omega_1, V \otimes \Bbbk^{m-1} \otimes \Bbbk^{m^2-m-1})$. Now there are two ways to obtain new prehomogeneous modules, namely

$$(G \times \mathrm{SL}_{m^2-m-1} \times \mathrm{SL}_{m^3-m^2-2m+1}, \varrho \otimes \omega_1 \otimes \omega_1, V \otimes \Bbbk^{m^2-m-1} \otimes \Bbbk^{m^3-m^2-2m+1}),$$
$$(G \times \mathrm{SL}_{m-1} \times \mathrm{SL}_{m^2-m-1} \times \mathrm{SL}_{m^4-2m^3+m-1}, \varrho \otimes \omega_1 \otimes \omega_1 \otimes \omega_1, V \otimes \Bbbk^{m-1} \otimes \Bbbk^{m^2-m-1} \otimes \Bbbk^{m^4-2m^3+m-1}),$$

and so on.

We say that two modules are **castling-equivalent** if one can be obtained from the other by a finite number of castling transformations. Again, we extend this definition to equivalent modules. A module (G, ϱ, V) is called **reduced** if there is no castling transform $(\tilde{G}, \tilde{\varrho}, \tilde{V})$ of (G, ϱ, V) with $\dim(\tilde{V}) < \dim(V)$.

Remark 9.24 For each class of castling-equivalent modules there is up to equivalence exactly one reduced triplet, see proposition 12 on p. 39 in Sato, Kimura [28]. Now the classification of prehomogeneous modules is reduced to the classification of the reduced modules.

Lemma 9.25 *Assume $m > n$. The module*

$$\Big(G \times \mathrm{GL}_n,\ (\sigma \otimes 1) \oplus (\varrho \otimes \omega_1),\ V^k \oplus (V^m \otimes \Bbbk^n)\Big) \qquad (*)$$

is prehomogeneous if and only if

$$\Big(G \times \mathrm{GL}_{m-n},\ (\sigma \otimes 1) \oplus (\varrho^* \otimes \omega_1),\ V^k \oplus (V^{m*} \otimes \Bbbk^{m-n})\Big) \qquad (**)$$

is prehomogeneous, and the generic isotropy subgroups of these modules are isomorphic. If G is reductive with the centre acting by scalar multiplication on $V^m \otimes \Bbbk^n$, then we can replace GL_n by SL_n and $(**)$ by

$$\Big(G \times \mathrm{SL}_{m-n},\ (\sigma^* \otimes 1) \oplus (\varrho \otimes \omega_1),\ V^{k*} \oplus (V^m \otimes \Bbbk^{m-n})\Big).$$

Proof: The modules $(G\times\mathrm{GL}_n, \varrho\otimes\omega_1, V^m\otimes\Bbbk^n)$ and $(G\times\mathrm{GL}_{m-n}, \varrho^*\otimes\omega_1, V^{m*}\otimes\Bbbk^{m-n})$ are castling-equivalent and thus have isomorphic generic isotropy subgroups. By corollary 9.18, we can find generic points such that the projection H on the G-component of the respective isotropy subgroup is the same for both isotropy subgroups. As GL_n acts trivial via $\sigma\otimes 1$, the action of either of the generic isotropy subgroups is given by the action of the projection H via σ. By proposition 9.11, $(*)$ and $(**)$ are both prehomogeneous if and only if $(H, \sigma|_H, V^k)$ is prehomogeneous, and in either case the generic isotropy subgroup is isomorphic to that of $(H, \sigma|_H, V^k)$.

If G is reductive with the centre acting by scalar multiplication, remark 9.21 allows us to replace GL_n by SL_n. Further, any representation of G is equivalent to its dual (proposition 9.13), so

$$(\sigma^* \otimes 1) \oplus (\varrho^{**} \otimes \omega_1) = (\sigma^* \otimes 1) \oplus (\varrho \otimes \omega_1)$$

and we can replace $(\sigma \otimes 1) \oplus (\varrho^* \otimes \omega_1)$ by $(\sigma^* \otimes 1) \oplus (\varrho \otimes \omega_1)$ in $(**)$. ■

In the setting of lemma 9.25, we will also call the modules castling-equivalent.

Now we present some further equivalence results for prehomogeneous modules. Some of them will be used repeatedly throughout this thesis, others are included because they are interesting enough in their own right.

Proposition 9.26 *Let $\varrho_1 : G \to \mathrm{GL}(V^{m_1})$ and $\varrho_2 : G \to \mathrm{GL}(V^{m_2})$ be rational representations of an algebraic group G with $m_1 \geq m_2$. Assume that*

$$\Big(G \times \mathrm{GL}_n,\ (\varrho_1 \otimes \omega_1) \oplus (\varrho_2 \otimes \omega_1^*),\ (V^{m_1} \otimes \Bbbk^n) \oplus (V^{m_2} \otimes \Bbbk^{n*})\Big)$$

is a prehomogeneous module and $n \geq m_2$. Then $(G, \varrho_1 \otimes \varrho_2, V^{m_1} \otimes V^{m_2})$ is also prehomogeneous.

Sketch of proof: Define

$$\varphi : \mathrm{Mat}_{m_1,n} \oplus \mathrm{Mat}_{m_2,n} \to \mathrm{Mat}_{m_1,m_2}, \quad (A, B) \mapsto AB^\top.$$

We have $\varphi((A\ 0), (I_{m_2}\ 0)) = A$ for any $A \in \mathrm{Mat}_{m_1,m_2}$, so φ is surjective. Since

$$\varphi(\varrho_1(g)Ah^\top, \varrho_2(g)Bh^{-1}) = \varrho_1(g)AB^\top\varrho_2(g)^\top$$

for all $(g, h) \in G \times \mathrm{GL}_n$, the map φ is equivariant and we get our result by proposition 9.10.

See Theorem 1.14 in Kimura et al. [17] for a full proof. ■

Proposition 9.27 *Let $\varrho_1 : G \to \mathrm{GL}(V^{m_1})$ and $\varrho_2 : G \to \mathrm{GL}(V^{m_2})$ be rational representations of an algebraic group G with $m_1 \geq m_2$. If $(G, \varrho_1 \otimes \varrho_2, V^{m_1} \otimes V^{m_2})$ is a prehomogeneous module, then*

$$\left(G \times \mathrm{GL}_n,\ (\varrho_1 \otimes \omega_1) \oplus (\varrho_2 \otimes \omega_1^*),\ (V^{m_1} \otimes \Bbbk^n) \oplus (V^{m_2} \otimes \Bbbk^{n*})\right)$$

is also prehomogeneous for any $n \geq m_1$.

Sketch of proof: Let $\mathfrak{g} = \mathfrak{Lie}(G)$. The isotropy Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{gl}_n$ at the generic point $(I_{m_1}\ 0)$ for the action on $\mathrm{Mat}_{m_1,n}$ is given by

$$\mathfrak{h} = \left\{\left(A, \begin{pmatrix} -\mathrm{d}\varrho_1(A)^\top & B \\ 0 & C \end{pmatrix}\right) \,\middle|\, A \in \mathfrak{g}, B \in \mathrm{Mat}_{m_1,n-m_1}, C \in \mathrm{Mat}_{n-m_1}\right\}$$

If Z is a generic point for the action of G on Mat_{m_1,m_2}, we can find an element in $\mathfrak{h}$ that maps $(Z^\top\ 0)$ to any element in $\mathrm{Mat}_{m_2,n-m_1}$, so $((I_{m_2}\ 0), (Z^\top\ 0))$ is a generic point for $G \times \mathrm{GL}_n$.

See Theorem 1.16 in Kimura et al. [17] for a full proof. ■

Corollary 9.28 *Assume $n > m_1 \geq m_2$. Then*

$$\left(G \times \mathrm{SL}_n,\ (\varrho_1 \otimes \omega_1) \oplus (\varrho_2 \otimes \omega_1^*),\ (V^{m_1} \otimes \Bbbk^n) \oplus (V^{m_2} \otimes \Bbbk^{n*})\right)$$

is a prehomogeneous module if and only if $(G, \varrho_1 \otimes \varrho_2, V^{m_1} \otimes V^{m_2})$ is prehomogeneous.

Proof: Corollary of theorem 1.16 in Kimura et al. [17]. ■

Proposition 9.29 *Let $2n \geq m$. Then $(\mathrm{Sp}_n \times G, \omega_1 \otimes \varrho, V^{2n} \otimes V^m)$ is a prehomogeneous module if and only if $(G, \varrho \wedge \varrho, \bigwedge^2 V^m)$ is prehomogeneous, where $(\varrho \wedge \varrho)(g).X = \varrho(g)X\varrho(g)^\top$ for $X \in \bigwedge^2 V^m$.*

Proof: Proposition 13 on p. 40 of Sato, Kimura [28]. ■

Proposition 9.30 *Let $n \geq m$. Then $(\mathrm{SO}_n \times G, \omega_1 \otimes \varrho, V^n \otimes V^m)$ is a prehomogeneous module if and only if $(G, \varrho \cdot \varrho, \mathrm{Sym}^2 V^m)$ is prehomogeneous, where $(\varrho \cdot \varrho)(g).X = \varrho(g)X\varrho(g)^\top$ for $X \in \mathrm{Sym}^2 V^m$.*

Proof: Proposition 14 on p. 41 of Sato, Kimura [28]. ■

10 Relative Invariants

In this chapter we introduce our second major tool in the study of prehomogeneous modules: the relative invariants. With their help, we can define a notion of regularity for prehomogeneous modules that will prove particularly useful in the quest for special modules. As some of the proofs of this chapter's results are rather long and tedious, they have only been sketched here. They can be found in sections 2.2 and 2.3 of Kimura [14].

10.1 Associated Characters and Basic Relative Invariants

Definition 10.1 Let $\varrho : G \to \mathrm{GL}(V)$ be a rational representation of an algebraic group G. A rational function $f : V \to \Bbbk$, $f \neq 0$, is called a **relative invariant** if there exists a rational character $\chi \in \mathcal{X}(G)$ such that

$$f(\varrho(g).v) = \chi(g)f(v)$$

for all $v \in V$ and $g \in G$. We say that χ is **associated** to f.

An invariant f in the sense of chapter 5 is a relative invariant with associated character $\chi = 1$, and we will call it an **absolute invariant** here.

We can use relative invariants to characterise prehomogeneous modules.

Proposition 10.2 *Let $\varrho : G \to \mathrm{GL}(V)$ be a rational representation of an algebraic group G. Then the following conditions are equivalent:*

1. *(G, ϱ, V) is a prehomogeneous module.*
2. *Any absolute invariant is a constant.*
3. *If f_1 and f_2 are relative invariants with the same associated character χ, then there exists a constant c with $cf_1 = f_2$.*

Proof: 1. $\Rightarrow$ 2.: Let f be an absolute invariant. As f is rational, we can write $f(x) = \frac{p(x)}{q(x)}$ for some polynomials $p, q \in \Bbbk[V]$. Since f is constant on the dense orbit $\varrho(G).v$, it is constant on $V = \mathrm{clos}(\varrho(G).v)$ by the continuity of rational functions.

2. $\Rightarrow$ 1.: By theorem 5.10, there exists a Zariski-open G-invariant subset U of V such that the coset space G/U is an algebraic variety and its function field coincides with the field $\Bbbk(V)^G$ of absolute invariants on V. As all absolute invariants are constant, we have $\Bbbk(V)^G = \Bbbk$. We obtain

$$\dim(G/U) = \mathrm{trdeg}_{\Bbbk}(\Bbbk(G/U)) = \mathrm{trdeg}_{\Bbbk}(\Bbbk(V)^G) = 0.$$

This implies the existence of an orbit of dimension $\dim(V)$, so this orbit is open and (G, ϱ, V) is a prehomogeneous module.

2. $\Rightarrow$ 3.: Since $\frac{f_2}{f_1}$ has the associated character 1, it is an absolute invariant, hence constant.

3. $\Rightarrow$ 2.: Let $f_1 = 1$, which is associated to the character 1. If f_2 is an absolute invariant, we have $f_2 = cf_1 = c$ for some constant c. ∎

Corollary 10.3 *If there exists a non-constant absolute invariant for the action of $\varrho(G)$ on V, then (G, ϱ, V) is not prehomogeneous.*

Corollary 10.4 *Any relative invariant is a homogeneous function.*

Proof: If f is a relative invariant with associated character χ, so is f_λ defined by $f_\lambda(v) = f(\lambda v)$, with $\lambda \in \mathbb{k}^\times$. By part 3 of proposition 10.2, we have $f_\lambda(v) = cf(v)$ for some constant c, so f is homogeneous and $c = \lambda^{\deg(f)}$. ■

Definition 10.5 If $\chi_1, \ldots, \chi_k \in \mathcal{X}(G)$ generate a free abelian subgroup of rank k of $\mathcal{X}(G)$, then $\chi_1, \ldots, \chi_k$ are called **multiplicatively independent**.

Lemma 10.6 *Relative invariants corresponding to multiplicatively independent characters are algebraically independent.*

Sketch of proof: Let $\chi_1, \ldots, \chi_k$ multiplicatively independent and associated to relative invariants $f_1, \ldots, f_k$.

If we assume the $f_1, \ldots, f_k$ to be algebraically dependent, then certain monomials $u_1, \ldots, u_m$ in the f_i must be linearly dependent, and any $m-1$ of them linearly independent. The equation $c_1u_1 + \ldots + c_mu_m = 0$ determines a one-dimensional solution space U in $\mathbb{k}^m$.

As the u_i are relative invariants as well, they correspond to certain characters $\tilde{\chi}_1, \ldots, \tilde{\chi}_m$, which are given as products of the $\chi_1, \ldots, \chi_k$. For $(c_1, \ldots, c_m) \in U$, we have $(c_1\tilde{\chi}_1(g), \ldots, c_m\tilde{\chi}_m(g)) \in U$ as well for any $g \in G$. As U is one-dimensional, we have $\tilde{\chi}_1 = \ldots = \tilde{\chi}_m$, which contradicts the fact that $\chi_1, \ldots, \chi_k$ are multiplicatively independent.

For a full proof see lemma 2.8 in Kimura [14]. ■

Theorem 10.7 *Let (G, ϱ, V) be a prehomogeneous module and V_{sing} its singular set. Let*

$$V_i = \{v \in V \mid f_i(v) = 0\}, \quad i = 1, \ldots, k,$$

be the irreducible components of V_{sing} of codimension 1 in V. Then the irreducible polynomials $f_1, \ldots, f_k$ are relative invariants and algebraically independent. Any relative invariant f is uniquely expressed in the form

$$f = cf_1^{m_1} \cdots f_k^{m_k}$$

for some constant $c \in \mathbb{k}^\times$ and $m_1, \ldots, m_k \in \mathbb{Z}$.

Sketch of proof: Since the action of $g \in G$ on V_i can be considered as rational morphism $G \times V_i \to V$, it follows by a result from algebraic geometry that $\varrho(G).V_i$ is irreducible, and as V_i is also irreducible and contained in $\varrho(G).V_i$, we have $V_i = \varrho(G).V_i$. From this and Hilbert's Nullstellensatz it follows that $f_i(\varrho(g).v) = \chi_i(g)f(v)$ for some χ_i, which can be shown to be a rational character. So the f_i are relative invariants.

To show that the f_i are algebraically independent, assume $\chi_1^{a_1} \cdots \chi_k^{a_k} = 1$ for some $a_1, \ldots, a_k \in \mathbb{Z}$. Then $f_1^{a_1} \cdots f_k^{a_k}$ is constant, hence an absolute invariant. A unique factorisation into irreducible factors shows $a_1 = \ldots = a_k = 0$, so the χ_i are multiplicatively independent.

Each polynomial relative invariant f can be written as a product

$$f = h_1 \cdots h_r$$

of irreducible polynomials $h_1, \ldots, h_r$. Then $h_i(\varrho(g).v)$ is also an irreducible polynomial for each $g \in G$, and as the factorisation of $f(\varrho(g).v) = \chi(g)f(v)$ into irreducible factors is unique up to a constant, $h_i(\varrho(g).v)$ must equal some h_j. This means the action of G on f permutes the irreducible factors of f, which gives us a homomorphism to the permutation group, $\psi : G \to \mathrm{S}_r$, and the kernel of ψ is a subgroup of finite index in G. But then the kernel must be G itself, i.e. $\psi = 1$ and hence each h_i is a relative invariant itself. But now the zero set of any h_i is an irreducible subset of codimension 1 and contained in V_{sing}, so h_i must coincide with one of the f_j up to a constant factor.

For a full proof, see theorem 2.9 in Kimura [14]. ■

The polynomials $f_1, \ldots, f_k$ in theorem 10.7 are called the **basic relative invariants** of (G, ϱ, V).

Corollary 10.8 *Let (G, ϱ, V) be a prehomogeneous module. Then there exists a relative invariant if and only if V_{sing} has an irreducible component of codimension 1 in V.*

We define a subgroup

$$\mathcal{X}_{\mathrm{rel}}(G) = \{\chi \in \mathcal{X}(G) \mid \chi \text{ is associated to some relative invariant of } G\}$$

of $\mathcal{X}(G)$.

Proposition 10.9 *Let (G, ϱ, V) be a prehomogeneous module with generic point v. Then we have*

$$\mathcal{X}_{\mathrm{rel}}(G) = \{\chi \in \mathcal{X}(G) \mid \chi|_{G_v} = 1\}.$$

Sketch of proof: If χ is associated to some relative invariant f, we have

$$f(v) = f(\varrho(g).v) = \chi(g)f(v)$$

for $g \in G_v$. As $f(v) \in \mathbb{k}^{\times}$, we have $\chi(g) = 1$ for all $g \in G_v$.

Conversely, assume $\chi|_{G_v} = 1$. We can define a regular function h on the coset space G/G_v by

$$h(gG_v) = \chi(g),$$

and a morphism φ of non-singular varieties by

$$\varphi : G/G_v \to V \setminus V_{\mathrm{sing}}, \quad gG_v \mapsto \varrho(g).v.$$

One can show that $\mathbb{k}(G/G_v)$ equals $\varphi^*(\mathbb{k}(V\backslash V_{\text{sing}}))$, where φ^* is the comorphism of φ. Hence φ is an isomorphism and there exists $f \in \mathbb{k}[V\backslash V_{\text{sing}}]$ such that $\varphi^*(h) = f$. In fact, $f(x) = \chi(g)$, where $x = \varrho(g).v$. Further, f must be a rational function on V and thus a relative invariant corresponding to χ.

For a full proof, see proposition 2.11 in Kimura [14]. ■

Proposition 10.10 *Let (G, ϱ, V) be a prehomogeneous module and H the normal subgroup of G generated by a generic isotropy subgroup of (G, ϱ, V) and the commutator subgroup $[G, G]$ of G. Then H does not depend on the choice of a generic point, and the character group $\mathcal{X}(G/H)$ of the quotient group G/H can be identified with $\mathcal{X}_{\text{rel}}(G)$. This implies that the rank of $\mathcal{X}(G/H)$ equals the number of basic relative invariants, i.e. the number of irreducible components of V_{sing} of codimension 1 in V.*

Proof: Since $G_{\varrho(g).v} = gG_vg^{-1}$, any element $g_0 \in G_{\varrho(g).v}$ can be written as $g_0 = gag^{-1} = (gag^{-1}a^{-1})a \in [G, G]G_v$ for some $a \in G_v$. So we have

$$G_{\varrho(g).v} \subseteq [G, G]G_v \quad \text{and} \quad [G, G]G_{\varrho(g).v} \subseteq [G, G]G_v$$

and by exchanging the roles of v and $\varrho(g).v$, we see that the converse inclusion holds. It follows that $H = [G, G]G_v$ is independent of the choice of v, and it is a normal subgroup as it contains the commutator subgroup $[G, G]$.

For any $\chi \in \mathcal{X}(G)$ we have

$$\chi(aba^{-1}b^{-1}) = \chi(a)\chi(b)\chi(a)^{-1}\chi(b)^{-1} = 1, \quad \text{or} \quad \chi|_{[G,G]} = 1.$$

By proposition 10.9, we have $\mathcal{X}_{\text{rel}}(G) = \{\chi \in \mathcal{X}(G) \mid \chi|_H = 1\}$. As this is independent of a generic point, we can identify $\mathcal{X}_{\text{rel}}(G)$ with $\mathcal{X}(G/H)$. As a consequence, the number of basic relative invariants equals the rank of the free abelian group $\mathcal{X}(G/H)$. ■

10.2 Regular Prehomogeneous Modules

In this section, we introduce the notion of regularity for prehomogeneous modules. Essentially, this means we it is possible to use a relative invariant f to construct a mapping $\varphi : V \to V^*$ such that φ maps the dense orbit in V to a dense orbit in V^*.

Now let (G, ϱ, V) be a prehomogeneous modules with $\dim(V) = n$. For convenience, we identify V and V^* with $\mathbb{k}^n$. Let f be a relative invariant. We define

$$\varphi_f = \operatorname{grad} \log f : V\backslash V_{\text{sing}} \to V^*, \quad x \mapsto \frac{1}{f(x)} \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

See p. 30 in Kimura [14] for a proof that the definition of φ_f does not depend on the choice of a basis of V.

φ_f provides us with an infinitesimal expression for the relative invariance of f.

Lemma 10.11 *Let f be a relative invariant of the prehomogeneous module (G, ϱ, V) and χ its associated character. For all $A \in \mathfrak{Lie}(G)$ and $x \in V \backslash V_{\text{sing}}$, we have*

$$\langle \mathrm{d}\varrho(A).x | \varphi_f(x) \rangle = \mathrm{d}\chi(A).$$

Proof: Let (g_{ij}) denote the matrix coordinates in $\mathrm{GL}(V)$, (a_{ij}) the matrix coordinates in $\mathfrak{gl}(V)$ and $x_1, \dots, x_n$ the coordinates in V. For any $x \in V$, the differential at 1_G of the mapping $g \mapsto \varrho(g).x$ is given by $A \mapsto \mathrm{d}\varrho(A).x$, or in coordinates

$$(\mathrm{d}\varrho(A).x)_k = \sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial(\varrho(g).x)_k}{\partial g_{ij}}\Big|_{\varrho(g)=I_n}.$$

Now define a mapping

$$h : G \to \Bbbk, \quad g \mapsto f(\varrho(g).x) = \chi(g)f(x).$$

The differential $\mathrm{d}h$ at 1_G is given by $\mathrm{d}h(A) = \mathrm{d}\chi(A)f(x)$.

We have

$$\begin{aligned}
\mathrm{d}h(A) &= \sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial f(\varrho(g).x)}{\partial g_{ij}}\Big|_{\varrho(g)=I_n} \\
&= \sum_{i,j=1}^{n} a_{ij} \sum_{k=1}^{n} \frac{\partial f(\varrho(g).x)}{\partial x_k} \cdot \frac{\partial(\varrho(g).x)_k}{\partial g_{ij}}\Big|_{\varrho(g)=I_n} \\
&= \sum_{k=1}^{n} \frac{\partial f(x)}{\partial x_k} \cdot (\mathrm{d}\varrho(A).x)_k \\
&= \langle \mathrm{d}\varrho(A).x | \mathrm{grad} f \rangle,
\end{aligned}$$

from which

$$\langle \mathrm{d}\varrho(A).x | \varphi_f(x) \rangle = \langle \mathrm{d}\varrho(A).x | f^{-1} \mathrm{grad} f \rangle = \mathrm{d}\chi(A)$$

follows. ∎

Proposition 10.12 *Let (G, ϱ, V) be a prehomogeneous module. We have*

$$\varphi_f(\varrho(g).x) = \varrho^*(g)\varphi_f(x)$$

and for $H(x) = \left(\frac{\partial^2 \log(f)}{\partial x_i \partial x_j}(x)\right)_{i,j}$ we have

$$H(\varrho(g).x) = \varrho^*(g)H(x)\varrho^*(g)^\top$$

and in particular

$$\det(H(\varrho(g).x)) = \det(\varrho^*(g))^2 \det(H(x)) = \det(\varrho(g))^{-2} \det(H(x)).$$

Sketch of proof: The proof is essentially done by direct computation: Differentiating both sides of the equality $f(\varrho(g).x) = \chi(g)f(x)$ with respect to x_i and then dividing by the respective other side yields the first equality $\varphi_f(\varrho(g).x) = \varrho^*(g)\varphi_f(x)$.

Differentiating again with respect to x_j yields the second equation $H(\varrho(g).x) = \varrho^*(g)H(x)\varrho^*(g)^\top$.

For a full proof, see propostion 2.13 in Kimura [14]. ■

The following corollary will lead us to the definition of regularity.

Corollary 10.13 *Let (G, ϱ, V) be a prehomogeneous module with generic point v. Then $\varphi_f(V \backslash V_{\text{sing}})$ is an orbit for the module (G, ϱ^*, V^*).*

Proof: $\varphi_f(V \backslash V_{\text{sing}}) = \varphi_f(\varrho(G).v) = \varrho^*(G).\varphi_f(v)$ by proposition 10.12. ■

Definition 10.14 Let (G, ϱ, V) be a prehomogeneous module with a relative invariant f. If the image $\varphi_f(V \backslash V_{\text{sing}})$ is dense in V^*, then f is called **non-degenerate**. A prehomogeneous module which has a non-degenerate invariant is called a **regular prehomogeneous module**.

Remark 2.1 in Kimura [14] contains an example of a non-regular prehomogeneous module with non-prehomogeneous dual module.

Theorem 10.15 *If (G, ϱ, V) is a regular prehomogeneous module, then (G, ϱ^*, V^*) is also prehomogeneous. Let $\mathcal{X}^*_{\text{rel}}(G)$ denote the group of characters associated to a relative invariant of (G, ϱ^*, V^*). Then we have*

$$\mathcal{X}_{\text{rel}}(G) = \mathcal{X}^*_{\text{rel}}(G).$$

Further, $V \backslash V_{\text{sing}}$ and $V^* \backslash V^*_{\text{sing}}$ are birationally equivalent.

Sketch of proof: The prehomogeneity of (G, ϱ^*, V^*) is immediate from the definition of regularity.

First, we note that for $x \in V$, $y \in V^*$ and $A \in \mathfrak{Lie}(G)$,

$$\langle \mathrm{d}\varrho(A).x|y\rangle + \langle x|\mathrm{d}\varrho^*(A).y\rangle = 0, \tag{$*$}$$

because it is the differential expression of $\langle \varrho(g).x|\varrho^*(g).y\rangle = \langle x|y\rangle$. Using $(*)$ and lemma 10.11, we get

$$\langle v - w|\mathrm{d}\varrho^*(A).\varphi_f(v)\rangle$$

for generic points v, w with $\varphi_f(v) = \varphi_f(w)$. But as $\varphi_f(v)$ is a generic point for V^*, we get $\mathrm{d}\varrho^*(\mathfrak{Lie}(G)) = V^*$, so $v = w$ holds and φ_f is injective.

As $G_v \subseteq G_{\varphi_f(v)}$ and φ_f is injective, we have $G_v = G_{\varphi_f(v)}$. Now $\mathcal{X}^*_{\text{rel}}(G) = \mathcal{X}_{\text{rel}}(G)$ by proposition 10.9.

For a relative invariant f of (G, ϱ, V) with associated character χ, there is a relative invariant f^* of (G, ϱ^*, V^*) with character χ^{-1}. Using $(*)$ again, we obtain

$$\langle \varphi_{f^*}(\varphi_f(v)) - v | \mathrm{d}\varrho^*(A).\varphi_f(v)\rangle = 0$$

for a generic point v and any $A \in \mathfrak{Lie}(G)$. It follows that $\varphi_{f^*}(\varphi_f(v)) = v \in V\backslash V_{\mathrm{sing}}$, i.e. $V\backslash V_{\mathrm{sing}}$ and $V^*\backslash V^*_{\mathrm{sing}}$ are birationally equivalent.

For a full proof, see theorem 2.16 in Kimura [14]. ■

Corollary 10.16 *Let (G, ϱ, V) be a regular prehomogeneous module. Then there exists a relative invariant with associated character* $\det(\varrho(g))^2$.

Proof: Let f be a non-degenerate relative invariant. The differential of φ_f at a generic point x is bijective following theorem 10.15 and given by

$$\mathrm{d}\varphi_f|_x = \left(\frac{\partial^2 \log(f)}{\partial x_i \partial x_j}(x)\right)_{i,j} = H(x).$$

Hence we have $\det(H(x)) \neq 0$ and $\det(H(x))$ is a relative invariant with character $\det(\varrho(g))^{-2}$ by proposition 10.12. ■

Proposition 10.17 *Let (G, ϱ, V) be a regular prehomogeneous module. Assume $\mathcal{X}_{\mathrm{rel}}(G)$ is of rank 1, i.e. there exists (up to a constant factor) only one irreducible relative invariant f with associated character χ. Let $n = \dim(V)$ and $d = \deg(f)$. Then d divides $2n$ and $\det(\varrho(g))^2 = \chi(g)^{2n/d}$.*

Proof: By corollary 10.16, there exists a relative invariant h with associated character $\det(\varrho(g))^2$. By theorem 10.7, h is of the form

$$h = cf^m$$

for some constant c and $m \in \mathbb{Z}$. Hence we have $\chi(g)^m = \det(\varrho(g))^2$. Note that the set of relative invariants does not change if we consider $\mathrm{GL}_1 \times G$ instead of G, so with out loss of generality we may assume $\mathrm{GL}_1 \subset G$. Choose a $g \in G$ such that $\varrho(g) = \lambda I_n$ with $\lambda \in \Bbbk^\times\backslash\{1\}$. Then $\det(\varrho(g))^2 = \lambda^{2n}$ and $\chi(g)f(x) = f(\varrho(g).x) = f(\lambda x) = \lambda^d f(x)$ by the homogeneity of f. Hence $\lambda^{2n} = \lambda^{dm}$, or

$$\det(\varrho(g))^2 = \chi(g)^{2n/d},$$

and $m = \frac{2n}{d} \in \mathbb{Z}$. ■

Using this proposition, we can derive a degree formula for irreducible relative invariants.

Proposition 10.18 *Let (G, ϱ, V) be a prehomogeneous module and assume the rank of $\mathcal{X}_{\mathrm{rel}}(G)$ is 1, i.e. there exists up to a constant factor only one irreducible f relative* *invariant. Moreover, assume that there exists an orbit* $\varrho(G).x_0$ *of codimension* 1*, i.e.* $\mathrm{clos}(\varrho(G).x_0) = \{x \in V \mid f(x) = 0\}$*. Let* $\mathfrak{g} = \mathfrak{Lie}(G)$*. Then*

$$\deg(f) = \frac{\mathrm{tr}(\mathrm{d}\varrho(A)) + \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}_{x_0}}(A)) - \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(A))}{\mathrm{tr}(\mathrm{d}\varrho(A))} \cdot \dim(V),$$

for $A \in \mathfrak{g}_{x_0}$ *and* $\mathrm{tr}(\mathrm{d}\varrho(A)) \neq 0$*, where* $\mathrm{ad}_{\mathfrak{g}_{x_0}} A$ *is the adjoint representation in the isotropy subalgebra at* x_0*. In particular, if G is reductive,*

$$\deg(f) = \frac{\mathrm{tr}(\mathrm{d}\varrho(A)) + \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}_{x_0}}(A))}{\mathrm{tr}(\mathrm{d}\varrho(A))} \cdot \dim(V).$$

Sketch of proof: Let $n = \dim(V)$ and $d = \deg(f)$. By proposition 10.17, $\chi(g)^{2n/d} = \det(\varrho(g))^2$.

The differential at 1_G of $g \mapsto \chi(g)^{2n/d}$ is given by $A \mapsto \frac{2n}{d}\mathrm{d}\chi(A)$, and the differential at 1_G of $g \mapsto \det(\varrho(g))$ is given by $A \mapsto \mathrm{tr}(\mathrm{d}\varrho(A))$. From this, we derive

$$\deg(f) = \frac{\mathrm{d}\chi(A)}{\mathrm{tr}(\mathrm{d}\varrho(A))} \cdot \dim(V)$$

for $A \in \mathfrak{g}$ with $\mathrm{tr}(\mathrm{d}\varrho(A)) \neq 0$.

The isotropy subgroup G_{x_0} acts trivially on $\mathrm{d}\varrho(\mathfrak{g}).x_0$. Hence, G_{x_0} acts on the one-dimensional quotient space $V_0 = V/(\mathrm{d}\varrho(\mathfrak{g}).x_0)$.

Let S be the set of points in $\{x \in V \mid f(x) = 0\}$ satisfying $\frac{\partial f}{\partial x_i}(x) = 0$ for $i = 1, \dots, n$. If $x_0 \in S$, then $S = \{x \in V \mid f(x) = 0\}$ would follow, which leads to a contradiction. Hence, $\mathrm{d}f(x_0) = \sum_i \frac{\partial f}{\partial x_i}(x_0)\mathrm{d}x_i \neq 0$. By some clever manipulations, using the Taylor expansion of $\langle x | \mathrm{d}f(x_0)\rangle$ and the relative invariance of f, one arrives at

$$\langle \varrho(g).x | \mathrm{d}f(x_0)\rangle = \langle \chi(g)x | \mathrm{d}f(x_0)\rangle.$$

Assuming $\varrho(g).x - \chi(g)x \notin \mathrm{d}\varrho(\mathfrak{g}).x_0$ for $x \in V$ leads to a contradiction, so $\varrho(g).x = \chi(g)x + y$ with $y \in \mathrm{d}\varrho(\mathfrak{g}).x_0$. This implies $\chi(g) = \det_{V_0}(g)$ for $g \in G_{x_0}$. The differential of the mapping

$$G_{x_0} \to \mathbb{k}^\times, \quad g \mapsto \det_{V_0}(g)$$

is given by

$$\mathfrak{g}_{x_0} \to \mathbb{k}, \quad A \mapsto \mathrm{tr}_{V_0}(A) = \mathrm{tr}(\mathrm{d}\varrho(A)) - \mathrm{tr}_{\mathrm{d}\varrho(\mathfrak{g}).x_0}(A).$$

Restricting the mapping $x \mapsto \mathrm{d}\varrho(A).x$ wit $A \in \mathfrak{g}_{x_0}$ to $\mathrm{d}\varrho(\mathfrak{g}).x_0$ leads to

$$\mathrm{d}\varrho(A).x = \mathrm{d}\varrho([A, B]).x_0,$$

where $x = \mathrm{d}\varrho(B).x_0$ for $B \in \mathfrak{g}$. If we identify $\mathrm{d}\varrho(\mathfrak{g}).x_0$ with the quotient space $\mathfrak{g}/\mathfrak{g}_{x_0}$, we can identify $\mathrm{d}\varrho(A).x$ with $\mathrm{ad}(A)B + C$ for some $C \in \mathrm{ad}(A)\mathfrak{g}_{x_0}$. We obtain

$$\mathrm{tr}_{\mathrm{d}\varrho(\mathfrak{g}).x_0}(A) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(A)) - \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}_{x_0}}(A)),$$

or

$$d\chi(A) = \mathrm{tr}(d\varrho(A)) + \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(A)) - \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}_{x_0}}(A)),$$

and plugging this into the formula for $\deg(f)$ yields the first equation. For the second equation, use that $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(A)) = 0$ if $\mathfrak{g}$ is reductive.

For a full proof, see proposition 2.19 in Kimura [14]. ■

Remark 10.19 If (G, ϱ, V) is an irreducible prehomogeneous module, then there is up to a constant only one irreducible relative invariant, cf. proposition 12 on p. 64 in Sato, Kimura [28].

Proposition 10.20 *Let (G, ϱ, V) be a special module, i.e. prehomogeneous with $\dim(G) = \dim(V)$. Then there exists a polynomial relative invariant f with $\deg(f) = \dim(V)$.*

Proof: Identify V with $\mathbb{k}^n$, where $n = \dim(V)$, and let $\mathfrak{g} = \mathfrak{Lie}(G)$. By the prehomogeneity there exists a generic point $v \in V$ such that $d\varrho(\mathfrak{g}).v = V$. For a basis $A_1, \dots, A_n$ of $d\varrho(\mathfrak{g})$ we see that $A_1 v, \dots, A_n v$ are linearly independent. Define f by

$$f(x) = \det(A_1 x | \dots | A_n x).$$

We have $f \neq 0$ as $f(v) \neq 0$.

For $g \in \varrho(G)$, let $(c_{ij}(g))$ denote the coefficients of the matrix representation of $\mathrm{Ad}(g)$ with respect to the basis $A_1, \dots, A_n$. Then we have

$$\begin{aligned} f(gx) &= \det((gg^{-1})A_1 gx | \dots |(gg^{-1})A_n gx) \\ &= \det(g) \cdot \det(\mathrm{Ad}(g^{-1})A_1 x | \dots |\mathrm{Ad}(g^{-1})A_n x) \\ &= \det(g) \cdot \det(c_{ij}(g^{-1})) \cdot f(x), \end{aligned}$$

so f is a relative invariant. Plugging in $g = \lambda I_n$ with $\lambda \neq 1$ yields

$$f(gx) = \lambda^n f(x),$$

and by the homogeneity of relative invariants we have $\deg(f) = n$. ■

Note that the relative invariant in proposition 10.20 is not necessarily irreducible.

Proposition 10.21 *Let (G, ϱ, V) be an irreducible prehomogeneous module, $\mathfrak{g} = \mathfrak{Lie}(G)$ and $\mathfrak{g}_x = \mathfrak{Lie}(G_x)$ a generic isotropy subalgebra. Then the following holds:*

1. *If $d\varrho(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$, there exists no non-constant relative invariant and hence (G, ϱ, V) is not regular.*
2. *If $d\varrho(\mathfrak{g}_x) \not\subseteq \mathfrak{sl}(V)$, there exists no no-constant relative invariant and hence (G, ϱ, V) is not regular.*
3. *If $d\varrho(\mathfrak{g}) \not\subseteq \mathfrak{sl}(V)$ and $d\varrho(\mathfrak{g}_x) \subseteq \mathfrak{sl}(V)$, then there exists a non-constant relative invariant of (G, ϱ, V).*

Proof:

1. As $\mathrm{d}\varrho$ is irreducible and $\mathrm{d}\varrho(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$, theorem 4.62 implies that $\mathrm{d}\varrho(\mathfrak{g})$ is semisimple. If there exists a non-constant relative invariant with associated character $\chi \neq 1$, then $\mathrm{d}\chi \neq 0$ since G is connected. Hence the kernel of $\mathrm{d}\chi$ is ideal in $\mathrm{d}\varrho(\mathfrak{g})$ of codimension 1, which is a contradiction to the semisimplicity of $\mathrm{d}\varrho(\mathfrak{g})$.
2. Assume there exists a relative invariant with associated character $\chi \neq 1$. By Cartan's theorem we have $\mathrm{d}\chi = \lambda \cdot \mathrm{tr}$ (the trace on $\mathrm{d}\varrho(\mathfrak{g})$) with $\lambda \in \Bbbk$. Hence, $\mathrm{d}\chi|_{\mathfrak{g}_x} \neq 0$ and consequently $\chi|_{G_x} \neq 1$. This contradicts proposition 10.9.
3. In this case $\mathrm{tr}|_{\mathrm{d}\varrho(\mathfrak{g})} \neq 0$ and $\det(\varrho(G_x))$ is a finite group. Hence some power $(\det \circ \varrho)^k$ satisfies $(\det \circ \varrho)^k \neq 1$ and $(\det \circ \varrho)^k|_{G_x} = 1$. By proposition 10.9, there exists a relative invariant corresponding to $(\det \circ \varrho)^k$. ■

At the end of this section, we present a result from Kimura et al. [17] which will be used frequently in part IV.

Definition 10.22 For $i = 1, \ldots, k$, let (G_i, ϱ_i, V_i) be modules.

1. We call (G, ϱ, V) the **direct composition** of the (G_i, ϱ_i, V_i), written
$$(G, \varrho, V) = \bigoplus_{i=1}^{k} (G_i, \varrho_i, V_i),$$
if $G = G_1 \times \cdots \times G_k$, $\varrho = (\varrho_1 \oplus 1^{\oplus k-1}) \oplus \ldots \oplus (1^{\oplus k-1} \oplus \varrho_k)$ and $V = V_1 \oplus \ldots \oplus V_k$. A module is called **indecomposable** if it cannot be written as a direct composition.
2. We call (G, ϱ, V) a $\mathbf{GL}_n$**-composition** of the (G_i, ϱ_i, V_i) if
$$\begin{aligned} G &= G_1 \times \cdots \times G_k \times \mathrm{GL}_n, \\ \varrho &= ((\varrho_1 \oplus 1^{\oplus k-1}) \oplus \ldots \oplus (1^{\oplus k-1} \oplus \varrho_k) \otimes 1) \oplus (\sigma \otimes \omega_1), \\ \text{and } V &= V_1 \oplus \ldots \oplus V_k \oplus (W \otimes \Bbbk^n), \end{aligned}$$
where $\sigma : G_1 \times \cdots \times G_k \to \mathrm{GL}(W)$ is any finite-dimensional rational representation and $n \geq \dim(W)$. [8)]

Proposition 10.23 *A* GL_n*-composition* (G, ϱ, V) *of* (G_i, ϱ_i, V_i) *with* $i = 1, \ldots, k$ *is a regular prehomogeneous module if and only if each* (G_i, ϱ_i, V_i) *is a regular prehomogeneous module and* $n = \dim(W)$ *(cf. definition 10.22).*

Proof: Proposition 1.9 in Kimura et al. [17]. ■

[8)] Kimura et al. [17] use the terms *direct sum* and *generalised direct sum* for direct compositions and GL_n-compositions, resp.

10.3 Reductive Prehomogeneous Modules

Recall that we call a prehomogeneous module reductive when its group is a reductive group.

From the degree formula in proposition 10.18 and from proposition 10.20 we get the following result.

Corollary 10.24 *Let (G, ϱ, V) be a reductive special module, i.e. prehomogeneous with* $\dim(G) = \dim(V)$*, and assume there exists an orbit of codimension* 1 *and the singular set V_{sing} is an irreducible hypersurface. Then we have*

$$V_{\text{sing}} = \{x \in V \mid f(x) = 0\},$$

where f is the relative invariant of degree $\dim(V)$ *from proposition 10.20, and f is an irreducible polynomial.*

Proof: As V_{sing} is an irreducible hypersurface, there exists exactly one irreducible relative invariant h by proposition 10.10.

Let $\varrho(G).x_0$ be an orbit of codimension 1. By the second degree formula in proposition 10.18,

$$\deg(h) = \frac{\operatorname{tr}(\mathrm{d}\varrho(A)) + \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}_{x_0}}(A))}{\operatorname{tr}(\mathrm{d}\varrho(A))} \cdot \dim(V)$$

for $A \in \mathfrak{g}_{x_0}$ with $\operatorname{tr}(\mathrm{d}\varrho(A)) \neq 0$. Since $n - 1 = \dim(\varrho(G).x_0) = \dim(G) - \dim(\mathfrak{g}_{x_0}) = n - \dim(\mathfrak{g}_{x_0})$, we have $\dim(\mathfrak{g}_{x_0}) = 1$. This implies that $\mathfrak{g}_{x_0}$ is abelian, hence $\operatorname{ad}_{\mathfrak{g}_{x_0}} = 0$ and $\operatorname{tr}_{\operatorname{ad}_{\mathfrak{g}_{x_0}}}(A) = 0$. So we have $\deg(h) = \dim(V)$. But the relative invariant f from proposition 10.20 is also of degree $\dim(V)$, and for some constant c we have $f(x) = ch(x)^m$ with $m = 1$, so f is irreducible. ■

For the following results on reductive modules, we must assume $\Bbbk = \mathbb{C}$.

Proposition 10.25 *Let (G, ϱ, V) be a reductive prehomogeneous module with a polynomial relative invariant f of degree d and associated character χ. Then the dual module (G, ϱ^*, V^*) is also a prehomogeneous module with a polynomial relative invariant f^* of degree d with associated character χ^{-1}.*

Proof: For $c \in \mathbb{C}$, let $\bar{c}$ denote the complex conjugate.

Let $n = \dim(V)$. As G is reductive, it is the Zariski closure of a maximal compact subgroup K (cf. section 4.6). Every compact subgroup of GL_n is conjugate to subgroup of the unitary group

$$\mathrm{U}_n = \{g \in \mathrm{GL}_n \mid \overline{g}^{\top} g = I_n\},$$

so we may assume $\varrho(K) \subseteq \mathrm{U}_n$. For $g \in K$, we have $\varrho^*(g) = \overline{\varrho(g)}$.

If we identify V and V^* with $\mathbb{C}^n$ by choosing a basis and its dual basis, we can define $f^* \in \mathbb{C}[V^*]$ by

$$f^*(y) = \overline{f(\overline{y})}.$$

For $g \in K$ we then have

$$f^*(\varrho^*(g).y) = f^*(\overline{\varrho(g)}.y) = \overline{f(\varrho(g).\overline{y})} = \overline{\chi(g)} \cdot \overline{f(\overline{y})} = \overline{\chi(g)} \cdot f^*(y).$$

As K is compact, so is $|\chi(K)| \subset \mathbb{R}^\times$, i.e. $|\chi(K)| = 1$ and hence $\overline{\chi(g)} = \chi^{-1}(g)$. Thus we have

$$K \subseteq \{g \in G \mid f^*(\varrho^*(g).y) = \chi^{-1}(g)f^*(y)\} \subseteq G,$$

and as $\operatorname{clos}(K) = G$ and by continuity, f^* is a relative invariant.

To see that (G, ϱ^*, V^*) is prehomogeneous, let $v_0 \in \mathbb{C}^n$ be a generic point for V and note that

$$\varrho(G).v_0 = \varrho(\operatorname{clos}(K).v_0) \subseteq \operatorname{clos}(\varrho(K).v_0) \subseteq V$$

and taking the closure in V yields $\operatorname{clos}(\varrho(K).v_0) = V$. Hence, $\varrho(K).v_0$ is dense in $\mathbb{C}^n \cong V$, so the conjugat $\overline{\varrho(K).v_0} = \varrho^*(K).\overline{v}_0$ is dense in $\mathbb{C}^n \cong V^*$. Taking the closure in G, we have that $\varrho^*(G).v_0$ is dense in V^*, so (G, ϱ^*, V^*) is prehomogeneous. ∎

Proposition 10.26 *Let (G, ϱ, V) be a reductive prehomogeneous module. Assume that the singular set V_{sing} is a hypersurface,*

$$V_{\text{sing}} = \{x \in V \mid f(x) = 0\}.$$

Then f is a non-degenerate relative invariant, i.e. (G, ϱ, V) is a regular prehomogeneous module.

Sketch of proof: Identify V and V^* with $\mathbb{C}^n$ by choosing a basis and its dual basis.

Let χ be the character associated to f and f^* the relative invariant of (G, ϱ^*, V^*) corresponding to χ^{-1}. Choose $v_0 \in \mathbb{C}^n$ such that $f^*(\overline{v_0}) = \overline{f(v_0)} \neq 0$. Then $f(v_0) \neq 0$ and so v_0 is a generic point for V. As shown in the proof of proposition 10.25, v_0 is also a generic point for V^*, and the singular set

$$V^*_{\text{sing}} = \{v \in V^* \mid f^*(v) = 0\}$$

for (G, ϱ^*, V^*) is a hypersurface.

By proposition 2.22 in Kimura [14], there exists a constant $b_0 \neq 0$ for f with

$$f^*(\varphi_f(x)) = \frac{b_0}{f(x)}$$

for $x \in V \backslash V_{\text{sing}}$. This implies $\varphi_f(x) \in V^* \backslash V^*_{\text{sing}}$, and by the equivariance of φ_f we obtain

$$\varphi_f(V \backslash V_{\text{sing}}) = \varphi_f(\varrho(G).x) = \varrho^*(G).\varphi_f(x) = V^* \backslash V^*_{\text{sing}},$$

so that f is non-degenerate. ∎

Proposition 10.26 does not hold when G is not reductive, see remark 2.1 in Kimura [14] for a counter-example.

Proposition 10.27 *Let (G, ϱ, V) be a regular reductive prehomogeneous module and f a non-degenerate polynomial relative invariant. Then the singular set is a hypersurface given by*

$$V_{\text{sing}} = \{x \in V \mid \det(H(x)) = 0\},$$

where $H(x) = \left(\frac{\partial^2 \log(f)}{\partial x_i \partial x_j}(x)\right)_{i,j}$.

Proof: From proposition 10.12 it follows that

$$\varrho(g)^\top H(\varrho(g).x)\varrho(g) = H(x).$$

In particular, this holds for $g \in G_x$, so we obtain a G_x-invariant symmetric form given by $Q_x(u, v) = u^\top H(x)v$. Since f is non-degenerate, we have $\det(H(x)) \neq 0$ and Q_x is non-degenerate.

The set

$$X = \{x \in V \mid \det(H(x)) \neq 0\}$$

is an affine variety by proposition 3.42. Since $\det(H(x))$ is a relative invariant by proposition 10.12, we have $X \supset V \backslash V_{\text{sing}}$. By theorem 4.68, X itself is an orbit, so $X = V \backslash V_{\text{sing}}$. Now

$$V_{\text{sing}} = \{x \in V \mid \det(H(x)) = 0\}$$

follows. ■

Even for reductive groups, proposition 10.27 is not always true when the module is not regular, see remark 2.5 in Kimura [14].

Combining the results in this section, we arrive at the following theorem.

Theorem 10.28 *Let (G, ϱ, V) be a reductive prehomogeneous module. Then the following conditions are equivalent:*

1. *(G, ϱ, V) is a regular prehomogeneous module.*
2. *The singular set V_{sing} is a hypersurface.*
3. *The open orbit $\varrho(G).v = V \backslash V_{\text{sing}}$ is an affine variety.*
4. *Each generic isotropy subgroup G_v is reductive.*
5. *Each generic isotropy subalgebra $\mathfrak{Lie}(G_v)$ is reductive in $\mathfrak{Lie}(G)$.*

Proof: 1. $\Leftrightarrow$ 2.: This follows from propositions 10.26 and 10.27.

2. $\Leftrightarrow$ 3.: The condition that V_{sing} is a hypersurface is equivalent to the condition that the Zariski dense orbit $\varrho(G).v$ (and by identification G/G_v) is an affine variety.

3. $\Leftrightarrow$ 4.: This follows immediately from theorem 4.69.

4. $\Leftrightarrow$ 5.: Follows from the definition of reductivity and proposition 4.60. ■

10.4 Relative Invariants and Castling Transformations

Now we investigate the relations between the regularity and relative invariants of a prehomogeneous module and its castling transforms.

As the generic isotropy subgroups of castling equivalent modules are isomorphic, we immediately get from theorem 10.28 that regularity of a reductive prehomogeneous module is invariant under castling transformations. But we have an even stronger result.

Theorem 10.29 *The regularity of a prehomogeneous module is invariant under castling transformations.*

Proof: See theorem 1.30 in Kimura et al. [17]. ■

Proposition 10.30 *Let* $m = \dim(V) > n \geq 1$*. Then the degree of each polynomial relative invariant of*

$$\left(G \times \mathrm{GL}_n,\ \varrho \otimes \omega_1,\ V \otimes \mathbb{k}^n\right)$$

is a multiple of n*. For a relative invariant* f *with* $\deg(f) = kn$*, the castling transform*

$$\left(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^* \otimes \mathbb{k}^{m-n}\right)$$

has a polynomial relative invariant f^* *satisfying* $\deg(f^*) = k(m-n)$*. The correspondence* $f \mapsto f^*$ *is a bijection between the polynomial relative invariants of the two modules.*

Proof: We identify $V \otimes \mathbb{k}^n$ with $V^{\oplus n}$ and $V^* \otimes \mathbb{k}^{m-n}$ with $V^{*\oplus m-n}$.

Let f be a polynomial relative invariant of the $G \times \mathrm{GL}_n$-module. Then f is an absolute invariant under the action of SL_n (by the semisimplicity of SL_n). Define a rational map φ (of degree n) by

$$\varphi : V^{\oplus n} \to \bigwedge^n V, \quad (v_1, \dots, v_n) \mapsto v_1 \wedge \cdots \wedge v_n.$$

For the action of G on $\bigwedge^n V$, there exists a polynomial relative invariant $h : \bigwedge^n \to \mathbb{k}$ by the first main theorem of invariant theory 5.19, and this invariant satisfies $f = h \circ \varphi$. Let $k = \deg(h)$. Then we have $\deg(f) = kn$, i.e. the degree of a polynomial relative invariant is a multiple of n.

If we fix a basis $b_1, \dots, b_m$ of V, we can identify $\bigwedge^{m-n} V$ with $(\bigwedge^n V)^*$ by remark 2.26, and we identify $\bigwedge^{m-n} V^*$ with $(\bigwedge^{m-n} V)^*$ by remark 2.25. Thus, we can identify $\bigwedge^{m-n} V^*$ with $\bigwedge^n V$, and the polynomial h can be regarded as a map $h : \bigwedge^{m-n} V^* \to \mathbb{k}$. Now, define a map φ^* (of degree $m-n$) by

$$\varphi^* : V^{\oplus m-n} \to \bigwedge^{m-n} V^* \cong \bigwedge^n V, \quad (v_1, \dots, v_{m-n}) \mapsto v_1^* \wedge \cdots \wedge v_{m-n}^*.$$

Then $f^* = h \circ \varphi^*$ is a relative invariant of degree $k(m-n)$ of the $G \times \mathrm{GL}_{m-n}$-module, and $f \mapsto f^*$ is a bijection. ■

11 Classification

The classification of reductive prehomogeneous modules proves to be a hard task and is far from complete. The first step was taken by Sato and Kimura [28] in 1975 by classifying the irreducible and reduced reductive prehomogeneous modules. In later works, classifications for simple (Kimura [15]), 2-simple (Kimura et al. [16], [17]) and 3-simple (Kimura et al. [19]) reductive prehomogeneous modules were given.

In this chapter, we will present these classfications, which will be useful in finding new examples of special modules of semisimple groups later on. For the reader's convenience, the lists in this chapter can be found in a separate document [9]. For the notation used in the classification, see section 4.4. We assume $\Bbbk$ to be an algebraically closed field of characteristic 0 in this section, and for the modules involving Spin_n or an exceptional simple group, we must even assume $\Bbbk = \mathbb{C}$.

11.1 Irreducible Reduced Prehomogeneous Modules

The irreducible and reduced prehomogeneous modules were classified by Sato and Kimura, thus we will label each class by SK n, where n is the number given to the class in § 7 of the the original work by Sato and Kimura [28]. Along with each module, we will state the connected component of the generic isotropy group, denoted by G_v°, and in some cases the irreducible relative invariant, denoted by f.

Theorem 11.1 (Sato, Kimura) *Let G be a reductive group and (G, ϱ, V) an irreducible and reduced prehomogeneous module. Then it is equivalent to one of the following prehomogeneous modules:*

SK I *Regular irreducible reduced prehomogeneous modules.*

1. $(G \times \mathrm{GL}_m, \varrho \otimes \omega_1, V^m \otimes \Bbbk^m)$,
   where $\varrho : G \to \mathrm{GL}(V^m)$ is an m-dimensional irreducible representation of a connected semisimple algebraic group G (or $G = \{1\}$ and $m = 1$). We have $G_v^\circ \cong G$ and $f(x) = \det(x)$ for $x \in \mathrm{Mat}_m \cong V^m \otimes \Bbbk^m$, $\deg(f) = m$.
2. $(\mathrm{GL}_n, 2\omega_1, \mathrm{Sym}^2\Bbbk^n)$ *for $n \geq 2$.*
   We have $G_v^\circ \cong \mathrm{SO}_n$ and $f(x) = \det(x)$ for $x \in \{A \in \mathrm{Mat}_n \mid A^\top = A\} \cong \mathrm{Sym}^2\Bbbk^n$, $\deg(f) = n$.
3. $(\mathrm{GL}_{2n}, \omega_2, \bigwedge^2 \Bbbk^{2n})$ *for $n \geq 3$.*
   We have $G_v^\circ \cong \mathrm{Sp}_n$ and $f(x) = \mathrm{Pf}(x)$ for $x \in \{A \in \mathrm{Mat}_{2n} \mid A^\top = -A\} \cong \bigwedge^2 \Bbbk^{2n}$, $\deg(f) = n$.
4. $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\Bbbk^2)$.
   We have $G_v^\circ \cong \{1\}$ and $f(a) = a_2^2a_3^2 + 18a_1a_2a_3a_4 - 4a_1a_3^3 - 4a_2^3a_4 - 27a_1^2a_4^2$ for $a = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \in \mathrm{Sym}^3\Bbbk^2$ (so f is the discriminant of a binary cubic form $a(x, y)$).

5. $(\mathrm{GL}_6, \omega_3, \bigwedge^3 \Bbbk^6)$.
We have $G_v^\circ \cong \mathrm{SL}_3 \times \mathrm{SL}_3$ *and* $f(x) = (x_0 y_0 - \mathrm{tr}(XY))^2 + 4x_0 \det(Y) + 4y_0 \det(X) - 4\sum_{i,j} \det(X_{ij}) \det(Y_{ji})$ *(see § 5, p. 83 in [28] for a definition),* $\deg(f) = 4$.

6. $(\mathrm{GL}_7, \omega_3, \bigwedge^3 \Bbbk^7)$.
We have $G_v^\circ \cong \mathrm{G}_2$ *and* $\deg(f) = 7$.

7. $(\mathrm{GL}_8, \omega_3, \bigwedge^3 \Bbbk^2)$.
We have $G_v^\circ \cong \mathrm{SL}_3$ *and* $\deg(f) = 16$.

8. $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2)$.
We have $G_v^\circ \cong \{1\}$ *and* $f(A, B) = \mathrm{dis}(\det(xA + yB))$ *for* $(A, B) \in \{(X, Y) \mid X, Y \in \mathrm{Mat}_3, X^\top = X, Y^\top = Y\} \cong \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2$, $\deg(f) = 12$.

9. $(\mathrm{SL}_6 \times \mathrm{GL}_2, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^6 \otimes \Bbbk^2)$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ *and* $f(A, B) = \mathrm{dis}(\mathrm{Pf}(xA + yB))$ *for* $(A, B) \in \{(X, Y) \mid X, Y \in \mathrm{Mat}_6, X^\top = -X, Y^\top = -Y\} \cong \bigwedge^2 \Bbbk^6 \otimes \Bbbk^2$, $\deg(f) = 12$.

10. $(\mathrm{SL}_5 \times \mathrm{GL}_3, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^3)$.
We have $G_v^\circ \cong \mathrm{SL}_2$ *and* $\deg(f) = 15$.

11. $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.
We have $G_v^\circ \cong \{1\}$ *and* $\deg(f) = 40$.

12. $(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_2, \omega_1 \otimes \omega_1 \otimes \omega_1, \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^2)$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{GL}_1$ *and* $f(A, B) = \mathrm{dis}(\det(xA + yB))$ *for* $(A, B) \in \mathrm{Mat}_3 \oplus \mathrm{Mat}_3 \cong \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^2$, $\deg(f) = 12$.

13. $(\mathrm{Sp}_n \times \mathrm{GL}_{2m}, \omega_1 \otimes \omega_1, \Bbbk^{2n} \otimes \Bbbk^{2m})$ *for* $n \geq 2m \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}$ *and* $f(X) = \mathrm{Pf}(X^\top J X)$ *for* $X \in \mathrm{Mat}_{2n,2m}$, $\deg(f) = 2m$.

14. $(\mathrm{GL}_1 \times \mathrm{Sp}_3, \mu \otimes \omega_3, \Bbbk \otimes V^{14})$.
We have $G_v^\circ \cong \mathrm{SL}_3$ *and* $\deg(f) = 4$*, where* f *is given by the restriction of the relative invariant of SK I-5.*

15. $(\mathrm{SO}_n \times \mathrm{GL}_m, \omega_1 \otimes \omega_1, \Bbbk^n \otimes \Bbbk^m)$ *for* $n \geq 3$, $\frac{1}{2}n \geq m \geq 1$.
We have $G_v^\circ \cong \mathrm{SO}_m \times \mathrm{SO}_{n-m}$ *and* $f(X) = \det(X^\top Q X)$ *for* $X \in \mathrm{Mat}_{n,m} \cong \Bbbk^n \otimes \Bbbk^m$, $\deg(f) = 2m$*, where* $Q = g^\top Q g$ *for* $g \in \mathrm{SO}_n$.

16. $(\mathrm{GL}_1 \times \mathrm{Spin}_7, \mu \otimes \mathrm{spinrep}, \Bbbk \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{G}_2$ *and* $\deg(f) = 2$*, where* f *is the relative invariant of SK I-15 for* $m = 1, n = 8$.

17. $(\mathrm{GL}_2 \times \mathrm{Spin}_7, \omega_1 \otimes \mathrm{spinrep}, \Bbbk^2 \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{SO}_2 \times \mathrm{SL}_3$ *and* $\deg(f) = 4$*, where* f *is the relative invariant of SK I-15 for* $m = 2, n = 8$.

18. $(\mathrm{GL}_3 \times \mathrm{Spin}_7, \omega_1 \otimes \mathrm{spinrep}, \Bbbk^3 \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{SO}_3 \times \mathrm{SL}_2$ *and* $\deg(f) = 6$*, where* f *is the relative invariant of SK I-15 for* $m = 3, n = 8$.

19. $(\mathrm{GL}_1 \times \mathrm{Spin}_9, \mu \otimes \mathrm{spinrep}, \Bbbk \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{Spin}_7$ *and* $\deg(f) = 2$.

20. $(\mathrm{GL}_2 \times \mathrm{Spin}_{10}, \omega_1 \otimes \text{halfspinrep}, \Bbbk^2 \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{G}_2$ *and* $\deg(f) = 4$.
21. $(\mathrm{GL}_3 \times \mathrm{Spin}_{10}, \omega_1 \otimes \text{halfspinrep}, \Bbbk^2 \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{SO}_3 \times \mathrm{SL}_2$ *and* $\deg(f) = 12$.
22. $(\mathrm{GL}_1 \times \mathrm{Spin}_{11}, \mu \otimes \text{spinrep}, \Bbbk \otimes V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_5$ *and* $\deg(f) = 4$.
23. $(\mathrm{GL}_1 \times \mathrm{Spin}_{12}, \mu \otimes \text{halfspinrep}, \Bbbk \otimes V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_6$ *and* $\deg(f) = 4$.
24. $(\mathrm{GL}_1 \times \mathrm{Spin}_{14}, \mu \otimes \text{halfspinrep}, \Bbbk \otimes V^{64})$.
We have $G_v^\circ \cong \mathrm{G}_2 \times \mathrm{G}_2$ *and* $\deg(f) = 8$.
25. $(\mathrm{GL}_1 \times \mathrm{G}_2, \mu \otimes \omega_2, \Bbbk \otimes V^7)$.
We have $G_v^\circ \cong \mathrm{SL}_3$ *and* $\deg(f) = 2$*, where* f *is the relative invariant of SK I-15 for* $m = 1$*,* $n = 7$.
26. $(\mathrm{GL}_2 \times \mathrm{G}_2, \omega_1 \otimes \omega_2, \Bbbk^2 \otimes V^7)$.
We have $G_v^\circ \cong \mathrm{GL}_2$ *and* $\deg(f) = 4$*, where* f *is the relative invariant of SK I-15 for* $m = 2$*,* $n = 7$.
27. $(\mathrm{GL}_1 \times \mathrm{E}_6, \mu \otimes \omega_1, \Bbbk \otimes V^{27})$.
We have $G_v^\circ \cong \mathrm{F}_4$ *and* $\deg(f) = 4$.
28. $(\mathrm{GL}_2 \times \mathrm{E}_6, \omega_1 \otimes \omega_1, \Bbbk^2 \otimes V^{27})$.
We have $G_v^\circ \cong \mathrm{SO}_8$ *and* $\deg(f) = 12$.
29. $(\mathrm{GL}_1 \times \mathrm{E}_7, \mu \otimes \omega_6, \Bbbk \otimes V^{56})$.
We have $G_v^\circ \cong \mathrm{E}_6$ *and* $\deg(f) = 4$.

SK II *Non-regular irreducible reduced prehomogeneous modules with non-constant relative invariant.*

1. $(\mathrm{GL}_1 \times \mathrm{Sp}_n \times \mathrm{SO}_3, \mu \otimes \omega_1 \otimes \omega_1, \Bbbk \otimes \Bbbk^{2n} \otimes \Bbbk^3)$.
We have $G_v^\circ \cong (\mathrm{Sp}_{n-2} \times \mathrm{SO}_2) \cdot \mathrm{Un}_{2n-3}$ *and* $f(X) = \mathrm{tr}(X^\top JXQ)^2$ *for* $X \in \mathrm{Mat}_{2n,3} \cong \Bbbk \otimes \Bbbk^{2n} \otimes \Bbbk^3$.

SK III *Non-regular irreducible reduced prehomogeneous modules without non-constant relative invariants.*

1. $(G \times \mathrm{GL}_m, \varrho \otimes \omega_1, V^n \otimes \Bbbk^m)$,
where $\varrho : G \to \mathrm{GL}(V^n)$ *is an* n*-dimensional irreducible representation of a semisimple algebraic group* $G(\neq \mathrm{SL}_n)$ *with* $m > n \geq 3$*. We have* $G_v^\circ \cong (G \times \mathrm{GL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$*. The module* $(G \times \mathrm{SL}_m, \varrho \otimes \omega_1)$ *is prehomogeneous with* $G_v^\circ \cong (G \times \mathrm{SL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$.
2. $(\mathrm{SL}_n \times \mathrm{GL}_m, \omega_1 \otimes \omega_1, \Bbbk^n \otimes \Bbbk^m)$ *for* $\frac{1}{2}m \geq n \geq 1$.
We have $G_v^\circ \cong (\mathrm{SL}_n \times \mathrm{GL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$*. The module* $(\mathrm{SL}_n \times \mathrm{SL}_m, \omega_1 \otimes \omega_1)$ *is prehomogeneous with* $G_v^\circ \cong (\mathrm{SL}_n \times \mathrm{SL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$.
3. $(\mathrm{GL}_{2n+1}, \omega_2, \bigwedge^2 \Bbbk^{2n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{Sp}_n \times \mathrm{GL}_1) \cdot \mathrm{G}^+_{2n}$*. The module* $(\mathrm{SL}_{2n+1}, \omega_2)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_n \cdot \mathrm{G}^+_{2n}$.

4. $(\mathrm{GL}_2 \times \mathrm{SL}_{2n+1}, \omega_1 \otimes \omega_2, \Bbbk^2 \otimes \bigwedge^2 \Bbbk^{2n+1})$ *for* $n \geq 2$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_2) \cdot \mathrm{G}_{2n}^+$ *(see lemma 1.4 in Kimura et al. [16]). The module* $(\mathrm{SL}_2 \times \mathrm{SL}_{2n+1}, \omega_1 \otimes \omega_2)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{SL}_2 \cdot \mathrm{G}_{2n}^+$.
5. $(\mathrm{Sp}_n \times \mathrm{GL}_{2m+1}, \omega_1 \otimes \omega_1, \Bbbk^{2n} \otimes \Bbbk^{2m+1})$ *for* $n > 2m + 1 \geq 1$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}) \cdot \mathrm{Un}_{2n-1}$. *The module* $(\mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, \omega_1 \otimes \omega_1)$ *is prehomogeneous with* $G_v^\circ \cong (\mathrm{Sp}_m \times \mathrm{Sp}_{n-m}) \cdot \mathrm{Un}_{2n-1}$.
6. $(\mathrm{GL}_1 \times \mathrm{Spin}_{10}, \mu \otimes \text{halfspinrep}, \Bbbk \otimes V^{16})$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Spin}_7) \cdot \mathrm{G}_8^+$. *The module* $(\mathrm{Spin}_{10}, \text{halfspinrep})$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Spin}_7 \cdot \mathrm{G}_8^+$.

Preceding the proof of this classification, Sato and Kimura determine the irreducible representations of algebraic groups admitting $\dim(G) \geq \dim(V)$ (see § 3, p. 41, of [28]) by some combinatorial considerations. The proof of the classification itself is given in § 5, pp. 73-141, of Sato and Kimura [28], where in most of the above cases the essential idea is to find a point $v \in V$ and show that the generic isotropy algebra $\mathfrak{g}_v$ is of dimension $\dim(\mathfrak{g}) - \dim(V)$. This list can also be found in the appendix of Kimura's book [14], providing some additional properties of these modules.

11.2 Non-Irreducible Simple Prehomogeneous Modules

The simple prehomogeneous, including the non-irreducible ones, were classified by Kimura, thus we will label them by Ks n, where n is the number of the module in § 3 of Kimura's article [15].

In this section, it is understood that each representation ϱ_i of the simple group is composed with a scalar multiplication μ of GL_1^k. We shall simply write ϱ_i instead of $\mu \otimes \varrho_i$. In some cases, a module $V_1 \oplus \ldots \oplus V_k$ will be prehomogeneous even with fewer than k scalar multiplications, in which case we will state this fact explicitely. We shall also state the connected component G_v° of the generic isotropy subgroup and the relative invariants $f_1, \ldots, f_l$ where they exist.

Theorem 11.2 (Kimura) *Let* $G = \mathrm{GL}_1^k \times G_{\mathrm{s}}$ *be a reductive group, where* G_{s} *is a simple algebraic group, let* (ϱ_1, V_1), …, (ϱ_k, V_k) *be irreducible* G_{s}*-modules and* $V = V_1 \oplus \ldots \oplus V_k$ *a* G_{s}*-module with representation* $\varrho = \varrho_1 \oplus \ldots \oplus \varrho_k$. *Then* (G, ϱ, V) *is equivalent to one of the following:*

Ks I *Regular non-irreducible simple prehomogeneous modules.*

1. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, \omega_1 \oplus \omega_1^*, \Bbbk^n \oplus \Bbbk^{n*})$ *for* $n \geq 3$.
   We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{SL}_{n-1}$ *and* $f_1(x, y) = \langle x|y \rangle$, *where* $(x, y) \in \Bbbk^n \oplus \Bbbk^{n*}$ *and* $\langle \cdot \mid \cdot \rangle$ *is the dual pairing. The module* $(\mathrm{GL}_1 \times \mathrm{SL}_n, (\mu \otimes \omega_1) \oplus \omega_1^*)$ *is prehomogeneous with* $G_v^\circ = \mathrm{SL}_{n-1}$.

2. $(\mathrm{GL}_1^n \times \mathrm{SL}_n, \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1^{n-1}$ *and* $f_1(X) = \det(X)$ *for* $X \in \mathrm{Mat}_n \cong (\Bbbk^n)^{\oplus n}$. *The module* $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n})$ *is prehomogeneous with* $G_v^\circ = \{1\}$.

3. $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\Bbbk^n)^{\oplus n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \{1\}$ *and* $f_i(X) = \det(x_1, \ldots, \not{x_i}, \ldots, x_{n+1})$ *for* $X = (x_1, \ldots, x_{n+1}) \in \mathrm{Mat}_{n,n+1} \cong (\Bbbk^n)^{\oplus n+1}$.

4. $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*})$ *for* $n \geq 3$.
We have $G_v^\circ \cong \{1\}$ *and* $f_1(X) = \langle x_1|y\rangle, \ldots, f_n(X) = \langle x_n|y\rangle$, $f_{n+1}(X) = \det(x_1, \ldots, x_n)$ *for* $X = (x_1, \ldots, x_n, y) \in (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*}$.

5. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ *and* $f_1(X, y, z) = \mathrm{Pf}(X)$, $f_2(X, y, z) = y^\top X^\# z$, *where* $(X, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n}$ *and* $X^\#$ *is the cofactor matrix of* X. *The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1 \oplus \omega_1)))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

6. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ *and* $f_1(X, y, z) = \mathrm{Pf}(X)$, $f_2(X, y, z) = \langle y|z\rangle$, *where* $(X, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*}$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1 \oplus \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

7. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ *for* $n \geq 3$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ *and* $f_1(X, y, z) = \mathrm{Pf}(X)$, $f_2(X, y, z) = y^\top X z$, *where* $(X, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1^* \oplus \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

8. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_n$. *The module* $(\mathrm{GL}_1 \times \mathrm{Sp}_{2n+1}, \mu \otimes (\omega_2 \oplus \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_n$, *see p. 94 in [15] for the relative invariant.*

9. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1}$, *see p. 94 in [15] for the relative invariants.*

10. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1}$ *and* $f_1(X) = \mathrm{Pf}\begin{pmatrix} A & x \\ x^\top & 0 \end{pmatrix}$, $f_2(X) = \langle x|y\rangle$, $f_3(X) = \langle x|z\rangle$, $f_4 = y^\top A z$ *for* $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*}$.

11. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n)$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{SO}_{n-1}$ *and* $f_1(X) = \det(A)$, $f_2(X) = x^\top A^\# x$ *for* $X = (A, x) \in \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n$.

12. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, 2\omega_1 \oplus \omega_1^*, \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^{n*})$ *for* $n \geq 3$.
We have $G_v^\circ \cong \mathrm{SO}_{n-1}$ *and* $f_1(X) = \det(A)$, $f_2(X) = x^\top A x$ *for* $X = (A, x) \in \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n$.

13. $(\mathrm{GL}_1^2 \times \mathrm{SL}_7, \omega_3 \oplus \omega_1, \bigwedge^3 \Bbbk^7 \oplus \Bbbk^7)$.
We have $G_v^\circ \cong \mathrm{SL}_3$, *see p. 96 in [15] for the relative invariants.*

14. $(\mathrm{GL}_1^2 \times \mathrm{SL}_7, \omega_3 \oplus \omega_1^*, \bigwedge^3 \Bbbk^7 \oplus \Bbbk^{7*})$.
We have $G_v^\circ \cong \mathrm{SL}_3$*, see p. 96 in [15] for the relative invariants.*
15. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8, \text{spinrep} \oplus \text{halfspinrep}, V^8 \oplus V^8)$.
We have $G_v^\circ \cong \mathrm{G}_2$ *and two quadratic invariants* $f_1(x,y) = q_1(x)$*,* $f_2(x,y) = q_2(y)$ *for* $(x,y) \in V^8 \oplus V^8$*.*
16. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7, \text{vecrep} \oplus \text{spinrep}, V^7 \oplus V^8)$.
We have $G_v^\circ \cong \mathrm{SL}_3$ *and two quadratic invariants* $f_1(x,y) = q_1(x)$*,* $f_2(x,y) = q_2(y)$ *for* $(x,y) \in V^7 \oplus V^8$*.*
17. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10}, \text{halfspinrep}_{\text{even}} \oplus \text{halfspinrep}_{\text{even}}, V^{16} \oplus V^{16})$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{G}_2$*, see p. 96 in [15] for the relative invariants. The module* $(\mathrm{GL}_1 \times \mathrm{Spin}_{10}, \mu \otimes (\text{halfspinrep}_{\text{even}} \oplus \text{halfspinrep}_{\text{even}}))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{G}_2$*.*
18. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10}, \text{vecrep} \oplus \text{halfspinrep}, V^{10} \oplus V^{16})$.
We have $G_v^\circ \cong \mathrm{Spin}_7$*, see p. 97 in [15] for the relative invariants.*
19. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{12}, \text{vecrep} \oplus \text{halfspinrep}, V^{12} \oplus V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_5$*, see p. 97 in [15] for the relative invariants.*
20. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n, \omega_1 \oplus \omega_1, \Bbbk^{2n} \oplus \Bbbk^{2n})$ *for* $n \geq 2$*.*
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$*, see p. 97 in [15] for the relative invariant. The module* $(\mathrm{GL}_1 \times \mathrm{Sp}_n, \mu \otimes (\omega_1 \oplus \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1}$*.*
21. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_3, \omega_3 \oplus \omega_1, V^{14} \oplus \Bbbk^6)$.
We have $G_v^\circ \cong \mathrm{SL}_2$*, see p. 97 in [15] for the relative invariants.*

Ks II *Non-regular non-irreducible simple prehomogeneous modules.*

1. $(\mathrm{GL}_1^k \times \mathrm{SL}_n, \omega_1^{\oplus k}, (\Bbbk^n)^{\oplus k})$ *for* $2 \leq k \leq n-1$*.*
We have $G_v^\circ \cong (\mathrm{GL}_1^k \times \mathrm{SL}_{n-k}) \cdot \mathrm{G}^+_{k(n-k)}$*. The module* $(\mathrm{SL}_n, \omega_1^{\oplus k})$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{SL}_{n-k} \cdot \mathrm{G}^+_{k(n-k)}$*.*
2. $(\mathrm{GL}_1^k \times \mathrm{SL}_n, \omega_1^{\oplus k-1} \oplus \omega_1^*, (\Bbbk^n)^{\oplus k-1} \oplus \Bbbk^{n*})$ *for* $3 \leq k \leq n$*.*
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_{n-k+1}) \cdot \mathrm{G}^+_{(n-k+1)(k-2)}$ *and* $f_1(X) = \langle x_1 | y \rangle, \ldots,$ $f_{k-1}(X) = \langle x_{k-1} | y \rangle$ *for* $X = (x_1, \ldots, x_{k-1}, y) \in (\Bbbk^n)^{\oplus k-1} \oplus \Bbbk^{n*}$*. The module* $(\mathrm{GL}_1^{k-1} \times \mathrm{SL}_n, (\mu \otimes \omega_1^{\oplus k-1}) \oplus \omega_1^*)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{SL}_{n-k+1} \cdot \mathrm{G}^+_{(n-k+1)(k-2)}$*.*
3. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_2, \bigwedge^2 \Bbbk^{2n+1} \oplus \bigwedge^2 \Bbbk^{2n+1})$ *for* $n \geq 2$*.*
We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}^+_{2n}$*. The module* $(\mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_2)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{G}^+_{2n}$*.*
4. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n})$ *for* $n \geq 2$*.*
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{n-1}$ *and* $f_1(X) = \mathrm{Pf}(X)$ *where* $X \in \bigwedge^2 \Bbbk^{2n}$*. The module* $(\mathrm{GL}_1 \times \mathrm{SL}_{2n}, \mu \otimes (\omega_2 \oplus \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{n-1}$*.*
5. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*})$ *for* $n \geq 3$*.*
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{n-1}$ *and* $f_1(X) = \mathrm{Pf}(X)$ *where* $X \in \bigwedge^2 \Bbbk^{2n}$*. The module* $(\mathrm{GL}_1 \times \mathrm{SL}_{2n}, \mu \otimes (\omega_2 \oplus \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{n-1}$*.*

6. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ *and* $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A^\# y$, $f_3(X) = y^\top A^\# z$, $f_4(X) = z^\top A x$ *for* $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n}$.

7. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ *and* $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A^\# y$, $f_3(X) = \langle x|z\rangle$, $f_4(X) = \langle y|z\rangle$ *for* $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*}$.

8. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ *for* $n \geq 3$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ *and* $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = \langle x|y\rangle$, $f_3(X) = \langle x|z\rangle$, $f_4(X) = y^\top A z$ *for* $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$.

9. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^* \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ *for* $n \geq 3$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ *and* $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A y$, $f_3(X) = y^\top A z$, $f_4(X) = z^\top A x$ *for* $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$.

10. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1^2 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{4n-2}$. *The module* $(\mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^*)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{4n-2}$.

11. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$, *see p. 99 in [15] for the relative invariants. The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, (\mu \otimes (\omega_2 \oplus \omega_1)) \oplus (\mu \otimes \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-1}$.

12. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$, *see p. 99 in [15] for the relative invariants. The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, (\mu \otimes (\omega_2 \oplus \omega_1)) \oplus (\mu \otimes \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-1}$.

13. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$ *and* $f_1(X) = x^\top A y$ *for* $X = (A, x, y) \in \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*}$. *The module* $(\mathrm{GL}_1 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus (\mu \otimes (\omega_1^* \oplus \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-2}$.

14. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^* \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-2}) \cdot \mathrm{Un}_{4n-6}$ *and* $f_1(X) = x^\top A y$, $f_2(X) = y^\top A z$, $f_3(X) = z^\top A x$. *The module* $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus (\mu \otimes \omega_1^*) \oplus (\mu \otimes \omega_1^*) \oplus (\mu \otimes \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{4n-6}$.

15. $(\mathrm{GL}_1^2 \times \mathrm{SL}_6, \omega_3 \oplus \omega_1, \bigwedge^3 \Bbbk^6 \oplus \Bbbk^6)$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{SL}_2) \cdot \mathrm{G}_4^+$, *see p. 100 in [15] for the relative invariant. The module* $(\mathrm{GL}_1 \times \mathrm{SL}_6, \mu \otimes (\omega_3 \oplus \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong (\mathrm{SL}_2 \times \mathrm{SL}_2) \cdot \mathrm{G}_4^+$.

16. $(\mathrm{GL}_1^3 \times \mathrm{SL}_6, \omega_3 \oplus \omega_1 \oplus \omega_1, \bigwedge^3 \Bbbk^6 \oplus \Bbbk^6 \oplus \Bbbk^6)$.
We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}_4^+$, *see p. 100 in [15] for the relative invariant. The module* $(\mathrm{GL}_1 \times \mathrm{SL}_6, \mu \otimes (\omega_3 \oplus \omega_1 \oplus \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{G}_4^+$.

17. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n, \omega_1 \oplus \omega_1 \oplus \omega_1, \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$, *see p. 100 in [15] for the relative invariants.*

18. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2, \omega_2 \oplus \omega_1, V^5 \oplus \Bbbk^4)$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_2$*, see p. 100 in [15] for the relative invariant. The module* $(\mathrm{GL}_1 \times \mathrm{Sp}_2, (\mu \otimes \omega_2) \oplus \omega_1)$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Un}_2$.
19. $(\mathrm{GL}_1^3 \times \mathrm{SL}_5, \omega_2 \oplus \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^5 \oplus \bigwedge^2 \Bbbk^5 \oplus \Bbbk^5)$.
See proposition 1.1 in [16].

11.3 2-Simple Prehomogeneous Modules of Type I

In this and the following chapter we shall give a classification of the non-irreducible 2-simple prehomogeneous modules, i.e. modules of the form

$$\Big(\mathrm{GL}_1^l \times G_1 \times G_2, \\ (\varrho_1 \otimes \tilde{\varrho}_1) \oplus \ldots \oplus (\varrho_k \otimes \tilde{\varrho}_k) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t), \\ V_1 \oplus \ldots \oplus V_l\Big),$$

where G_1 and G_2 are simple algebraic groups, $l = k + s + t$, and the ϱ_i, σ_j (resp. $\tilde{\varrho}_i$, τ_j) are irreducible representations of G_1 (resp. G_2). As in the previous chapter, it is understood that each of these representations is composed with a scalar multiplication of GL_1^k. First, we give the classification of the type I-modules, i.e. at least one of the modules $(\mathrm{GL}_1 \times G_1 \times G_2, \varrho_i \otimes \tilde{\varrho}_i)$ is a non-trivial prehomogeneous module (see example 9.8). These were classified by Kimura et al. [16], thus we shall refer to them as KI n, where n is the number of the module in § 3 of [16]. We shall state the non-irreducible modules only, as the irreducible ones already appear in theorem 11.1 or as castling transformas of those (see also theorem 1.5 in [16]). In the next chapter, we shall classify the remaining 2-simple modules of type II.

Theorem 11.3 (Kimura, Kasai, Inuzuka, Yasukura) *Let* (G, ϱ, V) *be a 2-simple prehomogeneous module of type I. Then it is equivalent to one of the following:*

KI I *Regular 2-simple prehomogeneous modules of type I.*

1. $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
We have $G_v^\circ \cong \{1\}$.
2. $(\mathrm{GL}_1^3 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_1$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$ *is prehomogeneous with* $G_v^\circ \cong \{1\}$.
3. $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_3, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SO}_3$.
4. $(\mathrm{GL}_1^3 \times \mathrm{SL}_4 \times \mathrm{SL}_3, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.
5. $(\mathrm{GL}_1^3 \times \mathrm{SL}_4 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.

6. $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1))$.
We have $G_v^\circ \cong \{1\}$.

7. $(\mathrm{GL}_1^2 \times \mathrm{SL}_5 \times \mathrm{SL}_3, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.

8. $(\mathrm{GL}_1^2 \times \mathrm{SL}_5 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.

9. $(\mathrm{GL}_1^2 \times \mathrm{SL}_5 \times \mathrm{SL}_9, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. *The module* $(\mathrm{GL}_1 \times \mathrm{SL}_5 \times \mathrm{SL}_9, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SL}_2$.

10. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-m} \times \mathrm{Sp}_{m-1}$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{Sp}_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_{n-m} \times \mathrm{Sp}_{m-1}$.

11. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1} \times \mathrm{SO}_2$.

12. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

13. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_2) \oplus (1 \otimes \omega_1))$. *We have* $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

14. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m-1}$. *The module* $(\mathrm{GL}_1 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{Sp}_m \times \mathrm{Sp}_{n-m-1}$.

15. $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes (\omega_1 \oplus \omega_1)^{(*)}))$.
We have $G_v^\circ \cong \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-1}$.

16. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{GL}_1$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*))$ *is prehomogeneous with* $G_v^\circ \cong \{1\}$.

17. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.

18. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \{1\}$.

19. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \{1\}$.

20. $(\mathrm{GL}_1^2 \times \mathrm{SO}_n \times \mathrm{SL}_m, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SO}_{m-1} \times \mathrm{SO}_{n-m}$.

21. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.

22. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_3, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

23. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_6, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

24. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_7, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.

25. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_2$.

26. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.

27. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_7 \times \mathrm{SL}_6, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.

28. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_3 \times \mathrm{SO}_2$.

29. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8 \times \mathrm{SL}_3, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_3$.

30. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.

31. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_3, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

32. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_6, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

33. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_7, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.

34. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2 \times \mathrm{SO}_3$.

35. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

36. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{G}_2$. *The module* $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{G}_2$.

37. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 2\omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

38. $(\mathrm{GL}_1^4 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

39. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_3, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

40. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{14}, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

41. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{15}, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{SL}_4$. *The module* $(\mathrm{GL}_1 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{15}, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$ *is prehomogeneous with* $G_v^\circ \cong \mathrm{SL}_4$.

42. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

43. $(GL_1^2 \times Spin_{10} \times SL_3, (\text{halfspinrep} \otimes \omega_1) \oplus (\text{vecrep} \otimes 1))$.
We have $G_v^\circ \cong SL_3 \times SO_2$.

44. $(GL_1^2 \times Spin_{10} \times SL_4, (\text{halfspinrep} \otimes \omega_1) \oplus (\text{vecrep} \otimes 1))$.
We have $G_v^\circ \cong SL_2 \times SL_2$.

45. $(GL_1^2 \times G_2 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong SL_2$.

46. $(GL_1^2 \times G_2 \times SL_6, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong SL_2$.

KI II *Non-regular 2-simple prehomogeneous modules of type I.*

1. (a) $(GL_1^2 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong GL_1^2 \cdot G_1^+$.
(b) $(GL_1^2 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot G_{2n}^+$.
(c) $(GL_1^2 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong GL_1 \cdot G_n^+$.

2. (a) $(GL_1^3 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong GL_1^2 \cdot G_{2n}^+$.
(b) $(GL_1^3 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong GL_1 \cdot G_{2n}^+$.

3. $(GL_1^4 \times SL_{2n+1} \times SL_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong GL_1 \cdot G_{2n}^+$.

4. $(GL_1^2 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot Un_2$.

5. $(GL_1^3 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.

6. $(GL_1^3 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.

7. (a) $(GL_1^2 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot Un_2$.
(b) $(GL_1^2 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong GL_1^2 \cdot Un_2$.

8. (a) $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
(b) $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1^2 \cdot Un_2$.

9. $(GL_1^3 \times SL_5 \times SL_9, (\omega_2 \otimes \omega_1) \oplus (\omega_1^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.

10. $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.

11. $(\mathrm{GL}_1^4 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_2$.

12. $(\mathrm{GL}_1^2 \times \mathrm{SL}_6 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^{(*)} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_3$.

13. *(a)* $(\mathrm{GL}_1^2 \times \mathrm{SL}_7 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_3$.
(b) $(\mathrm{GL}_1^2 \times \mathrm{SL}_7 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SO}_2) \cdot \mathrm{Un}_2$.

14. $(\mathrm{GL}_1^3 \times \mathrm{SL}_7 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1)$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_2$.

15. $(\mathrm{GL}_1^2 \times \mathrm{SL}_9 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_3$.

16. *(a)* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2}$.
(b) $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2}$.

17. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2m-2}$.

18. *(a)* $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m \geq 2$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-2} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-4}$.
(b) $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$.
(c) $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m \geq 2$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-2} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-4}$.

19. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes 2\omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$.

20. *(a)* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2m-1}$.
(b) $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1^2 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2m-3}$.
(c) $(\mathrm{GL}_1^2 \times \mathrm{Sp}_{m+1} \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ *for* $m \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1^2 \times \mathrm{Sp}_{m-1}) \cdot \mathrm{Un}_{4m-1}$.
(d) $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2))$ *for* $n > m+1 \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SO}_2^m \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2m-1}$.

21. *(a)* $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$ *for* $n > m+1$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-2}) \cdot \mathrm{Un}_{2n-4}$.
(b) $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m+1$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-2}) \cdot \mathrm{Un}_{2n-4}$.

(c) $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ *for* $n > m + 1$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-2}) \cdot \mathrm{Un}_{2n-4}$.

22. (a) $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes (\omega_1 \oplus \omega_1)^{(*)}))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-1} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n-2m-1}$.
(b) $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$ *for* $n > m \geq 1$.
We have $G_v^\circ \cong (\mathrm{Sp}_{m-2} \times \mathrm{Sp}_{n-m-1}) \cdot \mathrm{Un}_{2n+2m-7}$.
(c) $(\mathrm{GL}_1^4 \times \mathrm{Sp}_n \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$.

23. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{SO}_2 \times \mathrm{Sp}_{n-2}) \cdot \mathrm{Un}_{2n-3}$.

24. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n \times \mathrm{SL}_5, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*))$ *for* $n \geq 3$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-3}) \cdot \mathrm{Un}_{2n-4}$.

25. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n \times \mathrm{SL}_2, (\omega_1 \otimes 2\omega_1) \oplus (1 \otimes \omega_1))$ *for* $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$.

26. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{G}_2) \cdot \mathrm{G}_1^+$.

11.4 2-Simple Prehomogeneous Modules of Type II

In this chapter we give a classification of the 2-simple prehomogeneous modules of type II, i.e. modules of the form

$$\Big(\mathrm{GL}_1^l \times G_1 \times G_2,$$
$$(\varrho_1 \otimes \tilde{\varrho}_1) \oplus \ldots \oplus (\varrho_k \otimes \tilde{\varrho}_k) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t),$$
$$V_1 \oplus \ldots \oplus V_l\Big),$$

where all of the modules $(\mathrm{GL}_1 \times G_1 \times G_2, \varrho_i \otimes \tilde{\varrho}_i)$ are trivial prehomogeneous modules (see example 9.8).[9] Note that we consider non-irreducible modules only. These were classified by Kimura et al. [17], thus we shall refer to them as KII n, where n is the number of the module in § 5 of [17]. Unfortunately, it is not always obvious from the classification in which cases a module would be prehomogeneous even with fewer than l scalar multiplications.

Theorem 11.4 (Kimura, Kasai, Taguchi, Inuzuka) *Any indecomposable 2-simple prehomogeneous module of type II is equivalent to one of the following:*

KII I *2-simple prehomogeneous modules of type II obtained directly from any given simple module* $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ *(cf. theorem 11.2).*

[9] Ironically, the classification of modules containing only trivial modules proves to be much harder than the case in which non-trival ones appear.

1. *For any representation* $\sigma_1 \oplus \ldots \oplus \sigma_s$ *of* G *and* $n \geq \sum_{i=1}^{s} \dim(\sigma_i)$*:*

$$\Big(\mathrm{GL}_1^{l+s} \times G \times \mathrm{SL}_n, \\ (\sigma_1 \otimes \omega_1) \oplus \ldots \oplus (\sigma_s \otimes \omega_1) \;\oplus\; (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1)\Big).$$

2. *For* $t \geq 0$*,* $1 \leq k \leq l$ *and* $n = t - 1 + \sum_{i=1}^{k} \dim(\varrho_i)$*:*

$$\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n, \\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\varrho_{k+1}^* \otimes 1) \oplus \ldots \oplus (\varrho_l^* \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t})\Big).$$

3. *For* $t \geq 1$*,* $1 \leq k \leq l$ *and* $n \geq t - 1 + \sum_{i=1}^{k} \dim(\varrho_i)$*:*

$$\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n, \\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\varrho_{k+1} \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t-1}) \oplus (1 \otimes \omega_1^*)\Big).$$

KII II *2-simple prehomogeneous modules of type II of the form*

$$\Big(\mathrm{GL}_1^{k+s+t} \times G \times \mathrm{SL}_n, \\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t)\Big),$$

with $2 \leq \dim(\varrho_i) \leq n$ *for all* i *and at least one* $\tau_j \neq \omega_1^{(*)}$*.*

4. $G = \mathrm{SL}_m$ *with* $2 \leq m < n$*.*
   - *4-i*
     - *(a)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)})$.
     - *(b)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$.
     - *(c)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)})$.
     - *(d)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$.
     - *(e)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.
     - *(f)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (1 \otimes \omega_1^{(*)})$.
     - *(g)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.
   - *4-ii* n *even.*
     - *(a)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.
     - *(b)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
     - *(c)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$.
     - *(d)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$*,* m *even.*
     - *(e)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$*,* m *even.*
     - *(f)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_2 \otimes 1)$*,* m *odd.*
     - *(g)* $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)})$*,* m *odd.*
   - *4-iii* n *odd.*

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*)$, $m \geq 3$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$, $m \geq 3$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1)$, m even.
(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1)$, m even.
(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)$, m even.
(h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m even.
(i) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m odd.
(j) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, m odd.
(k) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^*)$, m odd.
(l) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(m) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$.
(n) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.
(o) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m even.
(p) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m even.
(q) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$, m even.
(r) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_2 \otimes 1)$, m odd.
(s) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, m odd.
(t) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)$, m odd.

5. $G = \mathrm{SL}_2$, $n > 2$.

5-i (a) $(2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)})$.
(b) $(2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1)$.

5-ii (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (3\omega_1 \otimes 1)$, n even.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$, n even.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*)$, n even.

5-iii (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (3\omega_1 \otimes 1)$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.

5-iv $n = 5$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2^*)$.

5-v $n = 6$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1^*)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (2\omega_1 \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (3\omega_1 \otimes 1)$.
(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.
(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.
(h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.

5-vi $n = 7$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)})$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)}) \oplus (\omega_1 \otimes 1)$.

6. $G = \mathrm{SL}_3$, $n > 3$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (2\omega_1^{(*)} \otimes 1)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2)$, $n = 5$.

7. $G = \mathrm{SL}_4$, $n > 4$.

7-i n odd.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*)$.

7-ii $n = 5$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2)$.

7-iii $n = 6$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_2^{(*)} \otimes 1)$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1)$.

8. $G = \mathrm{SL}_5$, $n > 5$.

8-i n even.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$.

8-ii n odd.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1) \oplus (\omega_1 \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1^*)$.

8-iii $n = 6$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^{(*)} \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_2^* \otimes 1)$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1^*)$.

8-iv $n = 7$.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)})$.

9. $G = \mathrm{SL}_{2j}$, $n = 2j + 1$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$, $j = 3$ (i.e. $n = 7$).

10. $G = \mathrm{SL}_n$.

$$(\omega_1 \otimes \omega_1) \;\oplus\; (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_k \otimes 1) \;\oplus\; (1 \otimes \varrho_{k+1}^*) \oplus \ldots \oplus (1 \otimes \varrho_r^*),$$

where $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ *is a simple prehomogeneous module.*

11. $G = \mathrm{Sp}_m$, $2m < n$.

11-i n *odd.*

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1)$, $m = 2$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1)$, $m = 2$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*)$, $m = 2$.

11-ii $n = 6$, $m = 2$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.

KII III *2-simple prehomogeneous modules of type II of the form*

$$\Big(\mathrm{GL}_1^{k+s+t} \times G \times \mathrm{SL}_n,$$
$$(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t})\Big),$$

with $2 \le \dim(\varrho_i) \le n$ *for all* i *and*

$$(G,\ \varrho_1, \ldots, \varrho_k,\ \sigma_1, \ldots, \sigma_s) \neq (\mathrm{SL}_m,\ \omega_1, \ldots, \omega_1,\ \omega_1^{(*)}, \ldots, \omega_1^{(*)}).$$

12. $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.

13. $G = \mathrm{SL}_m$.

13-i $m < n$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (\omega_1 \otimes 1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*)$.

(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1)$.

(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m *even.*

(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1)$, m *even.*

(h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_3 \otimes 1)$, $m = 6$.

(i) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_3 \otimes 1) \oplus (1 \otimes \omega_1)$, $m = 6$.

13-ii $n = m + 1$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2 \otimes 1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2^* \otimes 1)$, m *even.*

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_3 \otimes 1)$, $m = 6$.

13-iii $n \geq \frac{1}{2}m(m-1)$.

(a) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m odd.

(b) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m odd, $n > \frac{1}{2}m(m-1)$.

(c) $(\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, $m = 5$.

(d) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 2j+1$, $n = 2j^2 + j + 1$.

(e) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 2j$, $n = 2j^2 + j$.

(f) $(\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 5$, $n = 10$.

14. $G = \mathrm{Sp}_m$, $n \geq 2m$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_1 \otimes 1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1)$, $n > 2m$.

(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (\omega_1 \otimes 1)$, $n = 2m$.

(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^{(*)})$, $n = 2m$.

15. $G = \mathrm{Spin}_{10}$, $n \geq 16$.

(a) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.

(b) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, $n \geq 17$.

(c) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $n = 17$.

(d) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $n = 16$.

KII IV *2-simple prehomogeneous modules of type II of the form*

$$\Big(\mathrm{GL}_1^{k+s_1+s_2+t_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,$$
$$(\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \;\oplus\; (\omega_1 \otimes \omega_1)^{\oplus k} \;\oplus\; (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\Big),$$

where $n \geq m \geq 2$ *and* $k \geq 1$.

16. $n \geq km$.

16-i $n = m$. Then $k = 1$ and $1 \leq (s_1 + t_2) + (s_2 + t_1) \leq n + 1$, where one of $s_1 + t_2$ or $s_2 + t_1$ is 0 or 1.

16-ii $n = km$, $k \geq 2$.

(a) $t_1 = 0$, $2 \leq t_2 \leq n$, $s_2 = 0$, $s_1 + kt_2 \leq m$.

(b) $t_2 = 0$, $2 \leq t_1 \leq n$, $s_1 = 0$, $s_2 + kt_1 \leq m$.

16-iii $n = km + 1$. Then $t_1 \geq 3$, $t_2 = s_1 = 0$, $s_2 + k(t_1 - 1) \leq m$.

16-iv $n \geq km + t_1$, $n > km$.

(a) $k = 1$, $t_1 = 0$, $2 \leq t_2 \leq n$ and $1 \leq (s_1 + t_2) + s_2 \leq m + 1$, where s_2 is 0 or 1.

(b) $k \geq 2$, $t_1 = 0$, $2 \leq t_2 \leq n$, $s_2 = 0$, $s_1 + kt_2 \leq m$.

(c) $k \geq 1$, $t_1 = 1$, $2 \leq t_2 \leq n$, $s_2 = 0$, $s_1 + kt_2 \leq m$.

17. *$km > n$. These are the cases (17)-(25) in § 5.4 of [17], but to keep things simple we subsume them under the case KII IV-17 here. See the following definition 11.5 for the definition of T, $\nu(k,m,n)$ and (a_i). Also, we write $b_i = \frac{a_i}{a_{i+1}}$.*

17-i (a) $t_2 \geq 1, s_2 = t_1 = 0, s_1 + kt_2 \leq m - b_j(n - t_2)$, where $(k,m,n) \in T$ and $j = \nu(k,m,n)$.

(b) $s_2 = t_2 = 0$ and let $p = km + t_1 - n (< m), q = kp - m (< n), (k,p,m) \in T$ (resp. $(k,q,p) \in T$) and $j = \nu(k,p,m)$ (resp. $j = (k,q,p)$).

i. $m \geq kp, s_1 = 0$ and $t_1 \leq p + 1$.

ii. $m \geq kp, s_1 = 1$ and $k + t_1 \leq p + 1$.

iii. $m \geq kp, 2 \leq s_1 \leq m$ and $t_1 + ks_1 \leq p$.

iv. $kp > m, s_1 \geq 1$ and $t_1 + ks_1 \leq p - b_j(m - s_1)$.

v. $kp > m, p \geq kq, s_1 = 0, t_1 = 1$ and $k \leq q + 1$.

vi. $kp > m, p \geq kq, s_1 = 0, 2 \leq t_1 \leq p$ and $kt_1 \leq q$.

vii. $kp > m, kq > p, s_1 = 0, t_1 \geq 1$ and $kt_1 \leq q - b_j(p - t_1)$.

(c) $t_2 = 0, s_2 \geq 1, s_1 = 0$ and let $p = km + t_1 - n (< m), q = kp + s_2 - m (< p), r = kq - p (< q), (k,q,p) \in T$ (resp. $(k,q,p) \in T, (k,r,q) \in T$) and $j = \nu(k,q,p)$ (resp. $j = \nu(k,r,q)$).

i. $m \geq kp + s_2$ and $t_1 \leq p + 1$.

ii. $m = kp + s_2 - 1, t_1 = 0, 1$ and $k + t_1 \leq p + 1$.

iii. $m = kp + 1, m \geq s_2 \geq 3, t_1 = 0$ and $k(s_2 - 1) \leq p$.

iv. $m = kp, m \geq s_2 \geq 2$ and $ks_2 \leq p$.

v. $kp > m, p \geq kq, t_1 = 0$ and $s_2 \leq q + 1$.

vi. $kp > m, p \geq kq, t_1 = 1$ and $s_2 + kt_1 \leq q + 1$.

vii. $kp > m, p \geq kq, p \geq t_1 \geq 2$ and $s_2 + kt_1 \leq q$.

viii. $kp > m, kq > p, t_1 \geq 1$ and $s_2 + kt_1 \leq q - b_j(p - t_1)$.

ix. $kp > m, kq > p, q \geq kr, t_1 = 0, s_2 = 1$ and $k \leq r + 1$.

x. $kp > m, kq > p, q \geq kr, t_1 = 0, q \geq s_2 \geq 2$ and $ks_2 \leq r$.

xi. $kp > m, kq > p, kr > q, t_1 = 0, s_2 \geq 1$ and $ks_2 \leq r - b_j(q - s_2)$.

17-ii (a) $t_2 = 1, s_2 = 0, t_1 \geq 1$ and let $p = km + t_1 - n - 1, (k,p,m) \in T$ and $j = \nu(k,p,m)$.

i. $m \geq kp$ and $(t_1 - 1) + k(k + s_1) \leq p$.

ii. $kp > m$ and $(t_1 - 1) + k(k + s_1) \leq p - b_j(m - k - s_1)$.

(b) $t_2 = 0, s_2 \geq 1, s_1 = 1$ and let $p = km + t_1 - n, q = kp + s_2 - m - 1, (k,q,p) \in T$ and $j = \nu(k,q,p)$.

i. $kp > m, p \geq kq, (s_2 - 1) + k(k + t_1) \leq q$.

ii. $kp > m, kq > n, (s_2 - 1) + k(k + t_1) \leq q - b_j(p - k - t_1)$.

17-iii (a) $t_2 \geq 0, s_2 = 0, t_1 = 1, (s_1 + k) + k(t_2 - 1) \leq m - b_j(n - t_2)$ where $(k,m,n-1) \in T$ and $j = \nu(k,m,n-1)$.

(b) $t_2 = 0, s_2 = 1, s_1 \geq 2$ and let $p = km + t_1 - n (< m), (k,p,m-1) \in T$ and $j = \nu(k,p,m-1)$.

i. $m \geq kp$ and $t_1 + ks_1 \leq p$.
ii. $kp > m$ and $t_1 + ks_1 \leq p - b_j(m - s_1)$.

17-iv (a) $t_2 = s_2 = 1$ and let $p = km + t_1 - n$, $(k, p, m-1) \in T$ and $j = \nu(k, p, m-1)$.
i. $m - 1 \geq kp$ and $(k + t_1 - 2) + k(k + s_1 - 2) \leq p$.
ii. $kp > m$ and $(k + t_1 - 2) + k(k + s_1 - 2) \leq p - b_j(n - k - s_1)$.

Definition 11.5 Let T be the set of triplets $(k, m, n) \in \mathbb{N}^3$ satisfying $k \geq 2$, $n > m \geq 2$ and $k + m^2 + n^2 - 2 > kmn$. For $(k, m, n) \in T$ there exists a $j \in \mathbb{N}$ such that $(\mathrm{GL}_1^k \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k})$ is transformed to a trivial prehomogeneous module by j castling transformations. This number j is uniquely determined if we use only castling transformations decreasing the module's dimension. This unique j will be denoted by $\nu(k, m, n)$. Thus we obtain a map $\nu : T \to \mathbb{N}$. For example, $\nu(k, m, n) = 0$ if and only if $mk \leq n$. We define (a_i) to be the sequence

$$a_{-1} = -1, \qquad a_0 = 0, \qquad a_i = ka_{i-1} - a_{i-2} \text{ for } i > 0.$$

Remark 11.6 There are some cases of the form KII IV belonging neither to KII-16 nor KII-17, but to KII I instead. These are the cases (4.1-i), (4.1-ii), (4.7) and (4.8) from section 4.2 in Kimura et al. [17]. We will list them here for the sake of completeness.

1. $(\mathrm{GL}_1^{1+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{(*)\oplus s} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1})))$ with $k = 1$, $t \geq 1$, $n \geq m + t - 1$ and $s \leq m$. This is the case KII I-3.

2. $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^{(*)}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1})))$ with $k \geq 2$, $t \geq 1$, $n \geq km + t - 1$ and $s + k \leq m + 1$. This is the case KII I-3.

3. $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^{(*)} \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$ with $n \geq km + t$, and $(\mathrm{GL}_1^s \times \mathrm{SL}_m, \omega_1^{(*)\oplus s})$ is a simple prehomogeneous module. This is the case KII I-1.

4. $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^{(*)} \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$ with $t \geq 3$, $n = km + t - 1$, and $(\mathrm{GL}_1^{k+s} \times \mathrm{SL}_m, \omega_1^{*\oplus k} \oplus \omega_1^{(*)\oplus s})$ is a simple prehomogeneous module. This is the case KII I-2.

Part IV

Étale Representations of Algebraic Groups

Special modules were introduced in chapter 8. In the terminology of prehomogeneous modules, a module (G, ϱ, V) is special if it is prehomogeneous and if $\dim(G) = \dim(V)$, or equivalently, if $\varrho : G \to \mathrm{GL}(V)$ is a rational étale representation.

Certain classification results for special modules are immediately obtained from the classification of prehomogeneous modules. For example, all irreducible special modules must appear in theorem 11.1, see also proposition 13.2. Further, all 1-simple special modules must appear in theorem 11.1 (if irreducible) or in theorem 11.2 (if non-irreducible), see also propositions 13.2 and 13.3.

In the following chapters we will use our knowledge on reductive prehomogeneous modules to derive some general results on étale representations and present some examples of special modules. We assume $\Bbbk$ to be algebraically closed.

12 Some Conditions for (Non-)Speciality

We will begin our investigation of special modules by stating some results on which modules *cannot* be special modules and by analysing the structure of special modules with one-dimensional centre.

12.1 Non-Regular Prehomogeneous Modules

First, we note that non-regular prehomogeneous modules are not special modules.

Lemma 12.1 *If* $(\mathrm{GL}_1^k \times G, \varrho, V)$ *is a reductive special module, then it is a regular prehomogeneous module.*

Proof: The generic isotropy subgroup of a special module is finite, hence reductive. By theorem 10.28 the module is regular. ■

This lemma does not imply that any irreducible component of a special module must be regular. In fact, it will follow from theorem 12.14 that for groups with one-dimensional centre, a special module containing a regular irreducible component must be irreducible itself.

12.2 Groups with Trivial Character Group

Oliver Baues suggested the following proposition:

Proposition 12.2 *Let G be an algebraic group with $\mathcal{X}(G) = \{1\}$. Then G does not admit a rational étale representation.*

Proof: Assume that $\varrho : G \to V$ is an étale representation. Let $n = \dim(G) = \dim(V) > 0$. By proposition 10.20, the prehomogeneous module (G, ϱ, V) has a relative invariant f of degree n, so f is not constant. As $\mathcal{X}(G) = \{1\}$, the associated character χ of f must be $\chi = 1$, which means that f is an absolute invariant. But this is a contradiction to the fact that prehomogeneous modules do not admit non-constant absolute invariants by proposition 10.2. ■

We conclude that semisimple groups do not admit étale representations.

Corollary 12.3 *There is no rational étale representation for a semisimple algebraic group G.*

Proof: We show that $\mathcal{X}(G) = \{1\}$ and use proposition 12.2. If G is not connected, replace it by its connected component G°, which is also semisimple.

Assume that $\chi \in \mathcal{X}(G)$ is a non-trivial character. Let $\mathfrak{g} = \mathfrak{Lie}(G)$. As G is connected, the infinitesimal character $d\chi : \mathfrak{g} \to \Bbbk$ is also non-trivial, so the kernel of $d\chi$ is an ideal of codimension 1. This is a contradiction, as any non-trivial ideal of $\mathfrak{g}$ is a direct sum of some of the simple ideals of $\mathfrak{g}$, so its dimension must differ from $\dim(\mathfrak{g})$ by at least $\dim(\mathfrak{sl}_2) = 3$. ■

A different proof can be found in corollary 3.5 in Baues [2]. Is uses the fact that the open orbit must be closed so that G acts transitively on V, which contradicts the fact that 0 is a fixed point.

Corollary 12.4 *There is no rational étale representation for a unipotent algebraic group U.*

Proof: Any morphism of algebraic groups maps unipotent elements to unipotent elements (see remark 10.5 in Milne [25]). The only unipotent element in $\Bbbk^\times$ is 1. So for any $\chi \in \mathcal{X}(U)$ we have $\chi(U) = \{1\}$, hence $\mathcal{X}(U) = \{1\}$. By proposition 12.2, there is no étale representation for U. ■

12.3 Reductive Groups with Centre of Dimension 1

In this section we investigate special modules for algebraic groups $\mathrm{GL}_1 \times G$, where G is semisimple.

Consider $(\mathrm{GL}_1 \times G, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$, where the (ϱ_i, V_i) are the irreducible components of the module. By corollary 9.12, this module is special if and only if $(\mathrm{GL}_1 \times G, \varrho_1, V_1)$ is an irreducible prehomogeneous module with generic isotropy subgroup H, and if $(H, \varrho_2|_H \oplus \ldots \oplus \varrho_k|_H, V_2 \oplus \ldots \oplus V_k)$ is a special module. Then $(\mathrm{GL}_1 \times G, \varrho_1, V_1)$ must appear in theorem 11.1, and we also get all possible candidates for H from this theorem.

By looking at theorem 11.1, we immediately see that several regular irreducible modules cannot appear as irreducible components of a special module, because their generic isotropy subgroups are semisimple and thus do not admit any special modules by corollary 12.3.

Remark 12.5 The regular irreducible modules with *non*-semisimple isotropy subgroup $H \neq \{1\}$ are

- SK I-2 for $n = 2$ with $H \cong \mathrm{SO}_2$.
- SK I-12 with $H \cong \mathrm{GL}_1 \times \mathrm{GL}_1$.
- SK I-15 for $n = 3$ and $m = 1$ with $H \cong \mathrm{SO}_2$.
- SK I-15 for $n = 4$ and $m = 2$ with $H \cong \mathrm{SO}_2 \times \mathrm{SO}_2$.
- SK I-17 with $H \cong \mathrm{SO}_2 \times \mathrm{SL}_3$.
- SK I-26 with $H \cong \mathrm{GL}_2$.

Even for these modules we can show that they cannot appear as components of a special module with centre GL_1. Note that $\mathrm{SO}_2 \cong \mathrm{GL}_1$, so it does not seem plausible to distinguish between GL_1 and SO_2 in this list unless GL_1 was the centre of the reductive group (which it is not in these cases). But we adopt the notation from Sato, Kimura [28] in order to avoid confusion when using this reference.

The next lemma will be used frequently. It is based on lemma 8 on p. 38 of Sato, Kimura [28].

Lemma 12.6 *Let G be an algebraic group and $(\mathrm{GL}_1 \times G, \mu \otimes \varrho, V)$ a prehomogeneous module, such that (G, ϱ, V) is not prehomogeneous. Then the connected component H of the generic isotropy subgroup of $\mathrm{GL}_1 \times G$ is contained in G.*

Proof: See lemma 9.19. ∎

Corollary 12.7 *Let G be a semisimple group with irreducible representations $\varrho_1, \ldots, \varrho_k$, and let $(\mathrm{GL}_1^k \times G, (\mu \otimes \varrho_1) \oplus \ldots \oplus (\mu \otimes \varrho_k), V_1 \oplus \ldots \oplus V_k)$ be a prehomogeneous module, such that $(\mathrm{GL}_1^{k-1} \times G, (\mu \otimes \varrho_1) \oplus \ldots \oplus (\mu \otimes \varrho_{k-1}), V_1 \oplus \ldots \oplus V_{k-1})$ is also prehomogeneous, but $(\mathrm{GL}_1^{k-1} \times G, (\mu \otimes \varrho_1) \oplus \ldots \oplus (\mu \otimes \varrho_{k-1}) \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$ is not. Then the connected component H of the generic isotropy subgroup of $\mathrm{GL}_1^k \times G$ is contained in G.*

Proof: This follows from lemma 12.6, using the fact the fact that only one scalar multiplication acts on each irreducible component (ϱ_i, V_i). ■

Corollary 12.8 *Let G be a semisimple algebraic group and* $(\mathrm{GL}_1 \times G, \mu \otimes \varrho, V)$ *a regular prehomogeneous module. Then the connected component H of the generic isotropy subgroup is contained in G.*

Proof: Assume $(G, \varrho|_G, V)$ is prehomogeneous. As there are only trivial characters for G, any relative invariant is absolute and thus constant. So there are no non-constant relative invariants for the action $\mathrm{GL}_1 \times G$, which contradicts the regularity of $(\mathrm{GL}_1 \times G, \varrho, V)$.

So $(G, \varrho|_G, V)$ is not prehomogeneous and lemma 12.6 concludes the proof. ■

Lemma 12.9 *Let $\mathfrak{g}$ be a semisimple Lie algebra and* $\mathrm{d}\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ *a finite-dimensional representation of $\mathfrak{g}$. Then* $\mathrm{d}\sigma(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$.

Proof: As $\mathfrak{g}$ is semisimple, so is its image $\mathrm{d}\sigma(\mathfrak{g})$. This implies that any $X \in \mathrm{d}\sigma(\mathfrak{g})$ can be written as $X = [Y, Z]$ for some $Y, Z \in \mathrm{d}\sigma(\mathfrak{g})$. So we have $\mathrm{tr}(X) = \mathrm{tr}([Y, Z]) = 0$, or $X \in \mathfrak{sl}(V)$. ■

Proposition 12.10 *Let G be a semisimple algebraic group and let* $(\mathrm{GL}_1 \times G, \varrho, V)$ *be a regular irreducible prehomogeneous module whose connected component H of the generic isotropy subgroup is abelian. Then any module* $(\mathrm{GL}_1 \times G, \varrho \oplus \sigma, V \oplus W)$ *with* $\dim(W) > 0$ *is not a prehomogeneous module (hence not special).*

Proof: In order to obtain a special module, the module $(H, \sigma|_H, W)$ must be special by corollary 9.12. We use Lie algebras to show that this is not the case.

Let $\mathfrak{g} = \mathfrak{Lie}(G)$ and $\mathfrak{h} = \mathfrak{Lie}(H)$. From corollary 12.8 we have $\mathfrak{h} \subseteq \mathfrak{g}$. By lemma 12.9 it follows that $\mathrm{d}\sigma(\mathfrak{h}) \subseteq \mathrm{d}\sigma(\mathfrak{g})$ is contained in $\mathfrak{sl}(W)$. As $\mathrm{d}\sigma(\mathfrak{h})$ is abelian, it must already be contained in a Cartan subalgebra of $\mathfrak{sl}(W)$, which is of dimension $\dim(W) - 1$ (see definition 4.39 and theorem 4.43). In particular, $\dim(\mathrm{d}\sigma(\mathfrak{h})) < \dim(W)$, so the module $(H, \sigma|_H, W)$ is not prehomogeneous (hence not special). ■

With the help of this proposition, we can exclude the regular irreducible prehomogeneous modules with abelian isotropy subgroups from appearing as components of a special module with one-dimensional centre:

Corollary 12.11 *The modules SK I-2, SK I-12 and SK I-15 do not appear as irreducible components of a special module with centre* GL_1.

As for SK I-12 the subgroup H is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_1$, one might be tempted to think that one of the two factors is the centre GL_1 of the reductive group. But by the proof of proposition 12.10 (or by looking at the explicit form of $\mathfrak{Lie}(H)$ on p. 99 of Sato, Kimura [28]), we see that this is not the case.

It remains to be shown that SK I-17 and SK I-26 do not appear as irreducible components of a special module with one-dimensional centre.

Lemma 12.12 *Let G be a semisimple algebraic group and let $(\mathrm{GL}_1 \times G, \varrho, V)$ be a regular irreducible prehomogeneous module, and H the connected component of its generic isotropy subgroup. If $(\mathrm{GL}_1 \times G, \varrho \oplus \sigma, V \oplus W)$ with $\dim(W) > 0$ is prehomogeneous, then $(G, \sigma|_G, W)$ is a non-regular prehomogeneous module.*

Proof: As $(\mathrm{GL}_1 \times G, \varrho, V)$ is regular, the module $(G, \varrho|_G, V)$ is not prehomogeneous. Now corollary 12.8 tells us that $H \subseteq G$, and by proposition 9.11, $(H, \sigma|_H, W)$ is prehomogeneous, and so is $(G, \sigma|_G, W)$. As G is semisimple, the latter module is not regular. ■

Proposition 12.13 *The modules SK I-17 $(\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{Spin}_7, \omega_1 \otimes \mathrm{spinrep}, \Bbbk^2 \otimes V^8)$ and SK I-26 $(\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{G}_2, \omega_1 \otimes \omega_2, \Bbbk^2 \otimes V^7)$ cannot appear as an irreducible component of a special module with centre GL_1.*

Proof: Let G stand for either $\mathrm{SL}_2 \times \mathrm{Spin}_7$ or $\mathrm{SL}_2 \times \mathrm{G}_2$.

Both modules SK I-17 and SK I-26 are not special modules themselves, so any special module would have to be non-irreducible, say $(\varrho \oplus \sigma, V \oplus W)$ with regular irreducible (ϱ, V). By lemma 12.12, $(\sigma|_G, W)$ is a non-regular prehomogeneous module for G and must be (equivalent to) one of the modules in theorem 11.1 or theorem 11.3. But in these theorems there are no non-regular modules for G. ■

From corollary 12.11, proposition 12.13 and the fact that any regular irreducible prehomogeneous module other than those in remark 12.5 has a semisimple generic isotropy subgroup, we derive the following theorem:

Theorem 12.14 *Let $k \geq 2$ and let $(\mathrm{GL}_1 \times G, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$ be a special module for a semisimple algebraic group G, where the (ϱ_i, V_i) are the irreducible components. Then any module $(\mathrm{GL}_1 \times G, \varrho_i, V_i)$ is a non-regular prehomogeneous module (in particular, $(G, \varrho_i|_G, V_i)$ is prehomogeneous).*

We give a second proof of this theorem using the proof of lemma 3.7 in Baues [2].

Second proof: Denote the special module by $(\mathrm{GL}_1 \times G, \varrho, V)$ and let W be a non-trivial G-submodule and U its complement in V.

Assume $\dim(W) = \dim(G)$. Then $\dim(U) = 1$, so the action of G on U is trivial by the semisimplicity of G. It follows that G has an open orbit on W, which contradicts the fact that semisimple groups do not admit special modules. So $\dim(W) < \dim(G)$ must hold. By proposition 8.4, the submodule W must be contained in the zero fiber of the quotient map. Consider $h \in \Bbbk[W]^G$. Then h is also an element of $\Bbbk[V]^G$ and by proposition 8.2 we have $h = f_0 + c$ with $c \in \Bbbk$ and $f_0 \in \langle f \rangle$, where f is the generator of $\Bbbk[V]^G$. As any $w \in W$ is contained in the zero fiber, we have $h(w) = f_0(w) + c = c$, i.e. $h = c$ and thus $\Bbbk[W]^G = \Bbbk$. Now proposition 5.9 implies

$$\dim(W) = \max\{\dim(\varrho(G).w) \mid w \in W\},$$

so W is a prehomogeneous module for G.

As G is semisimple, any non-constant relative invariant of $\mathrm{GL}_1 \times G$ on W is an absolute invariant for G on W. Then $\Bbbk[W]^G = \Bbbk$ implies that there are no non-constant relative invariants for $\mathrm{GL}_1 \times G$ on W, so W is a non-regular prehomogeneous module. ■

Remark 12.15 Note that the second proof of theorem 12.14 implies that not only the irreducible components, but every proper submodule of a special module is non-regular for the action of G.

13 Examples of Special Modules

A special module is prehomogeneous, so in order to find new examples of special modules, we shall take a look at the classification of prehomogeneous modules in chapter 11 and try to find modules with $\dim(G) - \dim(V) = 0$ or equivalently, modules whose generic isotropy subgroup is finite.

By lemma 12.1 we can restrict our search to the regular prehomogeneous modules.

Remark 13.1 In the papers by Kimura et al. [15], [16], [17], the prehomogeneous modules are always stated with one scalar multiplication acting on each irreducible component, i.e. $(\mathrm{GL}_1^k \times G, \varrho_1 \oplus \ldots \oplus \varrho_k)$, and in this case we do not explicitely state the scalar multiplications, as it is understood that each ϱ_i stands for $\mu \otimes \varrho_i$. But in some cases, we do not need an independent scalar multiplication on each component to obtain a prehomogeneous module. Consider for example the prehomogeneous module Ks I-2 from theorem 11.2, $(\mathrm{GL}_1^n \times \mathrm{SL}_n, \omega_1^{\oplus n})$. For $\omega_1^{\oplus n}$ we need only the operation of SL_n and one scalar multiplication GL_1 acting on all components to obtain a prehomogeneous module, i.e. $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n})$. It is in this sense that we say we "replace" $\mathrm{GL}_1^k \times \mathrm{SL}_n$ by $\mathrm{GL}_1 \times \mathrm{SL}_n$, or that we say we "discard" $n-1$ scalar multiplications. We will denote the module by the same identifier as the original module. For this particular example, we can even write GL_n instead of $\mathrm{GL}_1 \times \mathrm{SL}_n$. Even though these groups are not the same (they differ by a finite subgroup), their linear representations coincide when GL_1 acts by scalar multiplication on the whole module. Of course, we do not write $\mu \otimes \omega_1$ for GL_n, as for this group the scalar multiplication is already contained in the standard representation ω_1. In the following, whenever we obtain a special module by replacing a group $\mathrm{GL}_1^k \times G$ by a group $\mathrm{GL}_1^j \times G$ with fewer scalar multiplications, we shall explicitely state on which irreducible component a scalar multiplication acts. However, we shall not state the scalar multiplications explicitely when proving that a module is not special for dimension reasons, or when we are considering some unspecified module $(\mathrm{GL}_1^j \times G, \varrho)$.

13.1 Special Modules from SK, Ks and KI

Finding the special modules in theorem 11.1 (SK), theorem 11.2 (Ks) and theorem 11.3 (KI) is rather easy, as the generic isotropy subgroup is known in each case. Thus we can just pick out the modules with $G_v^\circ \cong \{1\}$ from these theorems.

Proposition 13.2 *The following irreducible reduced prehomogeneous modules are special modules:*

- *SK I-4:* $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{k}^2)$.
- *SK I-8:* $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\mathbb{k}^3 \otimes \mathbb{k}^2)$.
- *SK I-11:* $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{k}^5 \otimes \mathbb{k}^4)$.

Proposition 13.3 *The following non-irreducible simple prehomogeneous modules are special modules:*

- *Ks I-1 for $n = 2$: This is equivalent to Ks I-4 for $n = 2$.*
- *Ks I-2:* $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{k}^n)^{\oplus n})$.
- *Ks I-3:* $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\mathbb{k}^n)^{\oplus n+1})$.
- *Ks I-4:* $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\mathbb{k}^n)^{\oplus n} \oplus \mathbb{k}^{n*})$.
- *Ks I-11 for $n = 2$:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\mathbb{k}^2 \otimes \mathbb{k}^2)$.
- *Ks I-12 for $n = 2$: This is equivalent to Ks I-11 for $n = 2$.*
- *Ks I-20 for $n = 1$: This is equivalent to Ks I-2 for $n = 2$.*

Corollary 13.4 *If $(\mathrm{GL}_1^k \times G, \varrho, V)$ for $k \geq 1$ and a simple group G is a special module, then $G = \mathrm{SL}_n$ for some n.*

Proposition 13.5 *The following 2-simple prehomogeneous modules of type I are special modules:*

- *KI I-1:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \mathbb{k}^4 \otimes \mathbb{k}^2) \oplus (\mathbb{k}^4 \otimes \mathbb{k}^2))$.
- *KI I-2:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\bigwedge^2 \mathbb{k}^4 \otimes \mathbb{k}^2) \oplus \mathbb{k}^4 \oplus \mathbb{k}^2)$.
- *KI I-6:* $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \mathbb{k}^5 \otimes \mathbb{k}^2) \oplus \mathbb{k}^{5*} \oplus \mathbb{k}^{5(*)})$.
- *KI I-16:* $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\mathbb{k}^4 \otimes \mathbb{k}^3) \oplus V^5 \oplus \mathbb{k}^3)$.
- *KI I-18:* $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\mathbb{k}^4 \otimes \mathbb{k}^2) \oplus \mathbb{k}^4 \oplus \mathbb{k}^2)$.
- *KI I-19:* $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\mathbb{k}^4 \otimes \mathbb{k}^4) \oplus \mathbb{k}^4 \oplus \mathbb{k}^4)$.

13.2 Special Modules for KII

Finding special modules for the case KII is not as easy as in the previous cases, as neither the isotropy subgroup nor the regularity is given explicitely in the classification by Kimura et al. [17].

Before proceeding, we prove a lemma which will come in handy a few times.

Lemma 13.6 *Let $n \geq 2$.*

1. *The generic isotropy subgroup of* $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, \omega_1 \oplus \omega_1^*)$ *is*

$$\left\{ \left(\det(A), \det(A)^{-1}, \begin{pmatrix} \det(A)^{-1} & 0 \\ 0 & A \end{pmatrix}\right) \,\middle|\, A \in \mathrm{GL}_{n-1} \right\} \cong \mathrm{GL}_{n-1}.$$

2. *Consider the module*

$$\begin{aligned}\Big(&\mathrm{GL}_1^t \times G \times \mathrm{SL}_n,\\ &(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_r \otimes \omega_1)\\ &\oplus\ (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1)\\ &\oplus\ (\mu \otimes 1 \otimes \omega_1^{\oplus t-1}) \oplus (\mu \otimes 1 \otimes \omega_1^*)\Big)\end{aligned}$$

where the ϱ_i and σ_j are any representations of G and $t \geq 2$. This module is prehomogeneous (resp. special) if and only if

$$\begin{aligned}\Big(&G \times \mathrm{GL}_{n-1},\\ &(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_r \otimes \omega_1)\\ &\oplus (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_r \otimes 1) \oplus (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1)\\ &\oplus (1 \otimes \omega_1^{\oplus t-2})\Big)\end{aligned}$$

is prehomogeneous (resp. special).

Proof:

1. See theorem 11.2, Ks I-1. Identify $\Bbbk^n \oplus \Bbbk^{n*}$ with $\Bbbk^{2n}$ and consider the action on the generic point (u_1, u_1), where u_1 is the first unit vector in $\Bbbk^n$. For a group element $g = (\lambda_1, \lambda_2, B)$ to fix (u_1, u_1), we must have

$$B = \begin{pmatrix} \det(A)^{-1} & 0 \\ 0 & A \end{pmatrix}, \quad \lambda_1 = \det(A), \quad \lambda_2 = \det(A)^{-1}.$$

2. Let M denote the module. We write $\omega_1^{[n]}$ and $\omega_1^{[n-1]}$ to distinguish between the standard representations of $\mathrm{GL}_1 \times \mathrm{SL}_n$ on $\Bbbk^n$ and that of GL_{n-1} on $\Bbbk^{n-1}$.

First, we decompose the representation by extracting a summand $\omega_1^{[n]} \oplus \omega_1^{[n]*}$:

$$\begin{aligned} &1 \otimes (\omega_1^{[n]} \oplus \omega_1^{[n]*}) \\ &\oplus \underbrace{\Big((\varrho_1 \otimes \omega_1^{[n]}) \oplus \ldots \oplus (\varrho_r \otimes \omega_1^{[n]}) \oplus (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \oplus (1 \otimes \omega_1^{[n]})^{\oplus t-2}\Big)}_{=:\varphi_0}. \end{aligned}$$

From part 1 we get that the generic isotropy subgroup on the $1 \otimes (\omega_1^{[n]} \oplus \omega_1^{[n]*})$-part is $H_0 = \mathrm{GL}_1^{t-2} \times G \times \mathrm{GL}_{n-1}$. So by proposition 9.11, the module M is prehomogeneous if and only if $(H_0, \varphi_0|_{H_0})$ is a prehomogeneous module. Under the action of $\mathrm{GL}_1 \times \mathrm{GL}_{n-1} \subset H_0$, each module $(\omega_1^{[n]}, \mathbb{k}^n)$ decomposes to

$$(\mu \oplus \omega_1^{[n-1]},\ \mathbb{k} \oplus \mathbb{k}^{n-1}),$$

and each $\varrho_i \otimes \omega_1^{[n]}$ decomposes to

$$\varrho_i \otimes (1 \oplus \omega_1^{[n-1]}) = (\varrho_i \otimes 1) \oplus (\varrho_i \otimes \omega_1^{[n-1]}),$$

and φ_0 decomposes to $\varphi \oplus (1 \otimes \mu)^{\oplus t-2}$ with

$$\begin{aligned} \varphi = {} & (\varrho_1 \otimes \omega_1^{[n-1]}) \oplus \ldots \oplus (\varrho_r \otimes \omega_1^{[n-1]}) \\ & \oplus (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_r \otimes 1) \oplus (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \\ & \oplus (1 \otimes \omega_1^{[n-1]})^{\oplus t-2}. \end{aligned}$$

The generic isotropy subgroup on the $(1 \otimes \mu)^{\oplus t-2}$-part is $H = G \times \mathrm{GL}_{n-1}$, so by proposition 9.11, $(H, \varphi|_H)$ is prehomogeneous if and only if $(H_0, \varphi_0|_{H_0})$ is prehomogeneous, which again is the case if and only if M is prehomogeneous. Returning to our usual notation (i.e. writing ω_1 instead of $\omega_1^{[n-1]}$), we have our result.

By corollary 9.12, we can replace "prehomogeneous" by "special" throughout the proof. ■

13.2.1 The case KII I

We assume $\dim(\varrho_i) \geq 2$ for all representations ϱ_i of the algebraic group G.

For each of the prehomogeneous modules KII I-1, KII I-2 and KII I-3 we have a component $\varrho \otimes 1$, on which G acts, and a component $\sigma \otimes \omega_1$, on which both G and SL_n act. Additionally, we have a scalar multiplication GL_1 acting on each irreducible component. We will investigate whether special modules can be found from these modules, if necessary after discarding some of the GL_1. First, we prove a lemma that will simplify the investigation of KII I-1 and KII I-3.

Lemma 13.7 *Let $n \geq 2$ and let $(\mathrm{GL}_1^j \times G, \varrho, W)$ be a prehomogeneous module for a simple algebraic group G, such that $(\mathrm{GL}_1^{j-1} \times G, \varrho, W)$ is not prehomogeneous (i.e. j is "minimal"). If*

$$\Big(\mathrm{GL}_1^j \times G \times \mathrm{SL}_n,\ (\sigma \otimes \omega_1) \oplus (\varrho \otimes 1),\ (V \otimes \mathbb{k}^n) \oplus W\Big)$$

is a module with $\dim(V) = n$ *and* GL_1^j *acting by scalar multiplications on* W *and trivially on* $V \otimes \mathbb{k}^n$*, then it is not a prehomogeneous module.*

Proof: Let H be the connected component of the generic isotropy subgroup of $(\mathrm{GL}_1^j \times G, \varrho, W)$. By lemma 12.6, we have $H \subseteq G$. We then have

$$\mathrm{d}\sigma(\mathfrak{Lie}(H)) \subseteq \mathrm{d}\sigma(\mathfrak{Lie}(G)) \subseteq \mathfrak{sl}(V)$$

by lemma 12.9, i.e. $\sigma(H) \subseteq \mathrm{SL}(V) \cong \mathrm{SL}_n$.

If we identify $V \otimes \mathbb{k}^n$ with Mat_n, the action of $(h, g) \in H \times \mathrm{SL}_n$ is given by $\sigma(h)Xg^\top$ for $X \in \mathrm{Mat}_n$. As $\sigma(h) \in \mathrm{SL}_n$,

$$\det(\sigma(h)Xg^\top) = \det(\sigma(h))\det(X)\det(g^\top) = \det(X)$$

is a non-constant absolute invariant, so the module

$$\left(H \times \mathrm{SL}_n,\ \sigma|_H \otimes \omega_1,\ V \otimes \mathbb{k}^n\right)$$

is not prehomogeneous. By proposition 9.11, the proof is complete. ∎

Lemma 13.7 tells us that we cannot "abuse" the isotropy subgroup H of a simple group to play the role of GL_1 in the action on Mat_n.

Proposition 13.8 *Consider KII I-1 from theorem 11.4, i.e.*

$$\Big(\mathrm{GL}_1^{s+l} \times G \times \mathrm{SL}_n,\\ (\sigma_1 \otimes \omega_1) \oplus \ldots \oplus (\sigma_s \otimes \omega_1) \ \oplus\ (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1)\Big),$$

for any representation $\sigma_1 \oplus \ldots \oplus \sigma_s$ *of a simple group* G *and* $n \geq \sum_{i=1}^s \dim(\sigma_i)$. *This is not a special module for* $s \neq 1$. *But when we replace* $\mathrm{GL}_1^{s+l} \times G \times \mathrm{SL}_n$ *by* $\mathrm{GL}_1^j \times G \times \mathrm{GL}_n$ *for some* $0 \leq j \leq l$*, the module*

$$\Big(\mathrm{GL}_1^{j} \times G \times \mathrm{GL}_n,\\ (\sigma_1 \otimes \omega_1) \oplus \ldots \oplus (\sigma_s \otimes \omega_1) \ \oplus\ (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1)\Big),$$

is special if and only if $\sum_{i=1}^s \dim(\sigma_i) = n$ *and* $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ *is a special module (of the form Ks, cf. theorem 11.2). Note that this includes the original module for* $s = 1$. *These are the only special modules obtained from KII I-1.*

Proof: We proceed in several steps.

- Assume the module to be special. Then by lemma 12.1, the module must be regular, and by proposition 10.23, this is the case if and only if $n = \sum \dim(\sigma_i)$ and $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is a regular prehomogeneous module. Furthermore,

$(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ must be special, so we have $l + \dim(G) = \sum \dim(\varrho_i)$ and hence

$$\begin{aligned}\dim(\mathrm{GL}_1^{l+s}) + \dim(G) + \dim(\mathrm{SL}_n) &= l + s + \dim(G) + (n^2 - 1)\\ &= n \cdot \left(\sum \dim(\sigma_i)\right) + \sum \dim(\varrho_i)\\ &= n^2 + \sum \dim(\varrho_i)\\ &= n^2 + \dim(G) + l,\end{aligned}$$

which is equivalent to $s = 1$.

- If we consider $\mathrm{GL}_1^l \times G \times \mathrm{GL}_n$ instead of $\mathrm{GL}_1^{s+l} \times G \times \mathrm{SL}_n$, then by the same reasoning we have that

$$\left(\mathrm{GL}_1^{1+l} \times G \times \mathrm{SL}_n,\ (\sigma_1 \otimes \omega_1) \oplus \ldots \oplus (\sigma_s \otimes \omega_1)\ \oplus\ (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1)\right)$$

  is special if and only if $n = \sum \dim(\sigma_i)$ and $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is special.

- It remains to be shown that we cannot replace the group by $\mathrm{GL}_1^l \times G \times \mathrm{SL}_n$, with GL_1^l acting only on the $\varrho_1 \oplus \ldots \oplus \varrho_l$-part (we assume that all l factors GL_1 are necessary for prehomogeneity, otherwise see the remark at the end of the proof). For $n = 1$ this follows from the fact that semisimple groups have trivial characters only. For $n \geq 2$, this will follow immediately from lemma 13.7 once we have shown that we do not obtain special modules for $\sum \dim(\sigma_i) < n$ in this case. So assume $n \geq 2$, $p := \sum \dim(\sigma_i) < n$ and let h be the dimension of the generic isotropy subgroup of $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$. By corollary 9.12,

$$h + (n^2 - 1) - np = 0$$

  should hold, which is equivalent to

$$0 = h - 1 + n\underbrace{(n - p)}_{\geq 1} \geq n - 1 > 0,$$

  an obvious contradiction.

If $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is a special module for $1 \leq j < l$, we can replace GL_1^l by GL_1^j in the above arguments. ■

Proposition 13.9 *Consider KII I-2 from theorem 11.4, i.e.*

$$\begin{aligned}&\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n,\\ &(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1)\ \oplus\ (\varrho_{k+1}^* \otimes 1) \oplus \ldots \oplus (\varrho_l^* \otimes 1)\ \oplus\ (1 \otimes \omega_1^{\oplus t})\Big)\end{aligned}$$

with $t \geq 0$, $1 \leq k \leq l$ *and* $n = t - 1 + \sum_{i=1}^k \dim(\varrho_i)$. *This module is special if and only if*

$$\left(\mathrm{GL}_1^l \times G,\ \varrho_1 \oplus \ldots \oplus \varrho_k \oplus \varrho_{k+1} \oplus \ldots \oplus \varrho_l\right)$$

is a special module (of the form Ks, cf. theorem 11.2) for the simple group G. If $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ *is special for* $1 \le j < l$*, replace* GL_1^l *by* GL_1^j *in the former statement if at least one scalar multiplication acts on the whole module.*

Proof: Set $\sigma = \varrho_{k+1}^* \oplus \ldots \oplus \varrho_l^*$, $\varrho = \varrho_1 \oplus \ldots \oplus \varrho_k \oplus 1^{\oplus t}$ and $m = t + \sum_{i=1}^k \dim(\varrho_i)$. As $\mathrm{GL}_1^{l+t} \times G$ is reductive, the proposition follows immediately from the isomorphism of the generic isotropy subgroups in lemma 9.25. ■

Proposition 13.10 *Consider KII I-3 from theorem 11.4, i.e.*

$$\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n,$$
$$(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\varrho_{k+1} \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t-1}) \oplus (1 \otimes \omega_1^*)\Big)$$

with $t \ge 1, 1 \le k \le l$ *and* $n \ge t - 1 + \sum_{i=1}^k \dim(\varrho_i)$*. This module is special if and only if*

$$\Big(\mathrm{GL}_1^l \times G,\ \varrho_1 \oplus \ldots \oplus \varrho_k \oplus \varrho_{k+1} \oplus \ldots \oplus \varrho_l\Big)$$

is a special module (of the form Ks, cf. theorem 11.2) and $n = t - 1 + \sum_{i=1}^k \dim(\varrho_i)$*. If* $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ *is special for* $1 \le j < l$*, replace* GL_1^l *by* GL_1^j *in the former statement.*

Proof: Set $p = \sum_{i=1}^k \dim(\varrho_i)$ and $q = \sum_{i=k+1}^l \dim(\varrho_i)$. First, we show that $n = t-1+p$ must hold for any special module obtained from KII I-3, even if we assumed fewer than t scalar multiplications acting on the $1 \otimes \omega_1^{(*)}$-parts. Assume $t - 1 + p < n$. We have $l + \dim(G) - q \ge p \ge 2$ because of prehomogeneity and because $\dim(\varrho_i) \ge 2$. Hence,

$$\begin{aligned} l + \dim(G) + \dim(\mathrm{SL}_n) - np - q - n &= l + \dim(G) + n^2 - 1 - np - q - n \\ &= \underbrace{(l + \dim(G) - q)}_{\ge 2} - 1 + n\underbrace{(n - (t - 1 + p))}_{>0} - n \\ &\ge 1 - 0 > 0. \end{aligned}$$

So, for any special module, $n = t - 1 + p$ must hold.

We treat the cases $t = 1$ and $t > 1$ separately.

- Let $t = 1$. As $p = n$, it follows by dimension reasons that the module is special if and only if $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is special.
- For $t > 1$, consider the operation of $\mathrm{GL}_1^2 \times \mathrm{SL}_n$ via $\omega_1 \oplus \omega_1^*$. Then by lemma 13.6, the module is special if and only if

$$\Big(\mathrm{GL}_1^l \times G \times \mathrm{GL}_{n-1},$$
$$(\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t-2})\Big) \quad (*)$$

is a special module. In particular, the latter must be regular. By proposition 10.23, it is regular if and only if $n-1 = t-2+p$ and $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is regular. Then, for dimension reasons, the module is special if and only if $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is special.

Note that by lemma 13.7, we cannot replace GL_{n-1} by SL_{n-1} in $(*)$ (or equivalently GL_1^t by GL_1^{t-1} in the original module), because the action of GL_1^l is already determined by the action on the $(\varrho_1 \oplus \ldots \oplus \varrho_l) \otimes 1$-part (where we assume that all l factors GL_1 are necessary for prehomogeneity, otherwise see the remark below).

If $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ is a special module for $1 \leq j < l$, we can replace GL_1^l by GL_1^j in the above arguments. ■

Remark 13.11 Note that proposition 13.10 leaves open the case for $t = 1$ if there is no factor GL_1 acting on the $1 \otimes \omega_1^*$-part.

13.2.2 The cases KII II and KII III

The following lemma tells us that we do not have to consider each module from KII II (resp. KII III), as some modules cannot be special modules, because then a module of strictly greater dimension would be prehomogeneous, which is impossible. For example, the module KII II-4-i (a) cannot be special, as it is a proper submodule of KII II-4-i (b).

Lemma 13.12 *Let* $(\mathrm{GL}_1 \times G, \varrho_V \oplus \varrho_W, V \oplus W)$ *be a prehomogeneous module, where* (ϱ_W, W) *is an irreducible submodule of dimension* ≥ 2.

1. (G, ϱ_V, V) *is not a special module.*
2. *If* (ϱ_U, U) *is an irreducible* $\mathrm{GL}_1 \times G$*-module with* $\dim(U) < \dim(W)$*, then* $(\mathrm{GL}_1 \times G, \varrho_V \oplus \varrho_U, V \oplus U)$ *is not a special module.*

Proof:

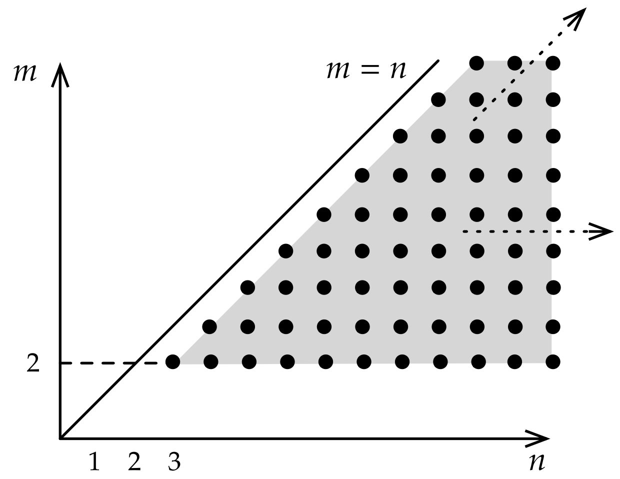
1. We have $1 + \dim(G) \geq \dim(V) + \dim(W)$ and hence
$$\dim(G) \geq \dim(V) + (\dim(W) - 1) \geq \dim(V) + 1 > \dim(V).$$
2. We have $1 + \dim(G) \geq \dim(V) + \dim(W) > \dim(V) + \dim(U)$. ■

In the cases KII II and KII III of theorem 11.4, the groups depend on integer parameters m and n, with $n > m$ in each case. We shall now investigate if the cases

4. to 15. of theorem 11.4 admit special modules for certain values of m and n. To this end, we introduce the function

$$\delta(m,n) = \dim(G) - \dim(V),$$

where $\dim(G)$ and $\dim(V)$ depend on the parameters m and n. In each case, δ will be a quadratic polynomial function in m and n. When $\delta(m,n) = 0$, the module is special. As $n > m \geq 2$ in each case, we only have to consider (m,n)-pairs (indicated by the black dots) lying in the shaded area:



If we fix m and pretend that n is a real variable, then $\delta(m,n)$ is a parabola in n, which we denote by $\delta(n)$. As the modules are assumed to be prehomogeneous, $\delta(n)$ will take only non-negative values for all $n \in \mathbb{N}$ with $n > m$. Because a parabola can have at most two zeros, it must be positive for almost all $n \in \mathbb{N}$ with $n > m$ and thus strictly increasing for $n \to \infty$. In order to determine whether there exist integer values $n > m$ with $\delta(m,n) = 0$, we define

$$\delta'(n) = \frac{\partial}{\partial n}\delta(m,n)$$

and proceed as follows:

1. For fixed m, compute the minimum point $n_0 \in \mathbb{R}$ of the parabola $\delta(n)$, i.e. n_0 with $\delta'(n_0) = 0$.

2. If $n_0 \in \mathbb{N}$ and $n_0 \leq m$, check if $\delta(m+1) = 0$, because in this case we have $0 \leq \delta(m+1) < \delta(m+k)$ for all $k \in \mathbb{N}$, so $n = m+1$ is the only possible integer solution for $\delta(n) = 0$, if one exists.

3. If $n_0 \in \mathbb{N}$ and $n_0 > m$, check if $\delta(n_0) = 0$, because in this case we have $0 \leq \delta(n_0)$ and $\delta(n_0 - k) > \delta(n_0) < \delta(n_0 + k)$ for all $k \in \mathbb{N}$, so $n = n_0$ is the only possible integer solution for $\delta(n) = 0$, if one exists.

4. If $n_0 \notin \mathbb{N}$ and $\lfloor n_0 \rfloor > m$, check if $\delta(\lfloor n_0 \rfloor) = 0$ and if $\delta(\lceil n_0 \rceil) = 0$.
5. If $n_0 \notin \mathbb{N}$ and $\lceil n_0 \rceil \le m + 1$, check if $\delta(m + 1) = 0$.
6. In some cases, we have to take into account additional constraints on m or n, e.g. that that m or n are required to be even (resp. odd). In these cases, a solution for $\delta(n) = 0$ is only valid if the additional constraints are met.

We now apply these tests to the modules of the form KII II and KII III, but exclude some cases a priori by lemma 13.12. We do not check if some modules are special with fewer scalar multiplications than the number of irreducible components.

4. 4-i
   (a) Not special by lemma 13.12.
   (b) $\delta'(n) = n - m - \frac{1}{2}$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (c) Not special by lemma 13.12.
   (d) Not special by lemma 13.12.
   (e) Not special by lemma 13.12.
   (f) Not special by lemma 13.12.
   (g) $\delta'(n) = n - m - \frac{1}{2}$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.

   4-ii
   (a) $\delta'(n) = 2n - 2m - 1$ and $\delta(m, n) = 0$ for $m = 3, n = 4$.
   (b) Not special by lemma 13.12.
   (c) Not special by lemma 13.12.
   (d) Not special by lemma 13.12.
   (e) Not special by lemma 13.12.
   (f) $\delta'(n) = 2n - 2m + 1$ and no integer solution for $\delta(m, n) = 0$.
   (g) $\delta'(n) = 2n - 2m + 4$ and no integer solution for $\delta(m, n) = 0$.

   4-iii
   (a) $\delta'(n) = 2n-2m-1$ and no integer solution with $m \ge 3$ for $\delta(m, n) = 0$.
   (b) $\delta'(n) = 2n - 2m + 1$ and no integer solution for $\delta(m, n) = 0$.
   (c) Not special by lemma 13.12.
   (d) $\delta'(n) = 2n - 2m - 1$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (e) Not special by lemma 13.12.
   (f) $\delta'(n) = 2n - 2m - 1$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (g) $\delta'(n) = 2n - 2m - 3$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (h) $\delta'(n) = 2n - 2m - 3$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (i) Not special by lemma 13.12.
   (j) $\delta'(n) = 2n-2m-3$ and no integer solution with odd m for $\delta(m, n) = 0$.
   (k) $\delta'(n) = 2n-2m-1$ and no integer solution with odd m for $\delta(m, n) = 0$.
   (l) Not special by lemma 13.12.
   (m) Not special by lemma 13.12.
   (n) $\delta'(n) = 2n - 2m - 1$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
   (o) Not special by lemma 13.12.

(p) $\delta'(n) = 2n - 2m - 3$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
(q) $\delta'(n) = 2n - 2m - 1$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
(r) Not special by lemma 13.12.
(s) $\delta'(n) = 2n-2m-3$ and no integer solution with odd m for $\delta(m, n) = 0$.
(t) $\delta'(n) = 2n-2m-3$ and no integer solution with odd m for $\delta(m, n) = 0$.

5. 5-i (a) Not special by lemma 13.12.
(b) $\delta(2, n) = 0$ for $n = 3$.

5-ii (a) Not special by lemma 13.12.
(b) $\delta(2, n) = 0$ for $n = 3$.
(c) No integer solution with $n > m$ for $\delta(2, n) = 0$.
(d) Not special by lemma 13.12.
(e) No integer solution with even n for $\delta(2, n) = 0$.

5-iii (a) Not special by lemma 13.12.
(b) Not special by lemma 13.12.
(c) No integer solution with $n > m$ for $\delta(2, n) = 0$.
(d) Not special by lemma 13.12.
(e) $\delta(2, n) = 0$ for $n = 3$.

5-iv (a) $\delta(2, 5) = 0$ by direct computation.

5-v In all cases, we have $\delta(2, 6) \neq 0$ by direct computation.

5-vi In all cases, we have $\delta(2, 7) \neq 0$ by direct computation.

6. (a) No integer solution with $n > m$ for $\delta(3, n) = 0$.
(b) $\delta(3, 5) = 0$ by direct computation.

7. 7-i (a) No integer solution for $\delta(4, n)$.
(b) Not special by lemma 13.12.
(c) No integer solution for $\delta(4, n)$.

7-ii (a) $\delta(4, 5) \neq 0$ by direct computation.

7-iii In all cases, we have $\delta(4, 6) \neq 0$ by direct computation.

8. 8-i (a) Not special by lemma 13.12.

8-ii (a) Not special by lemma 13.12.
(b) Not special by lemma 13.12.
(c) No integer solution for $\delta(5, n) = 0$.

8-iii In all cases, we have $\delta(5, 6) \neq 0$ by direct computation.

8-iv In all cases, we have $\delta(5, 7) \neq 0$ by direct computation.

9. (a) $\delta'(n) = n - m - \frac{3}{2}$ and $\delta(m, n) = 0$ for $m = 2, n = 3$.
(b) $\delta(6, 7) \neq 0$ by direct computation.

10. For dimension reasons, this module is special if and only if $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ is special.

11. 11-i (a) Not special by lemma 13.12.
    (b) Not special by lemma 13.12.
    (c) No integer solution for $\delta(2,n) = 0$.
    (d) No integer solution for $\delta(2,n) = 0$.
    11-ii (a) $\delta(2,6) \neq 0$ by direct computation.

12. $\delta(4,8) = 0$ by direct computation.

13. 13-i (a) Not special by lemma 13.12.
    (b) Not special by lemma 13.12.
    (c) Not special by lemma 13.12.
    (d) $\delta'(n) = 2n - m - 3$ and $\delta(m,n) = 0$ for $m = 2, n = 3$.
    (e) $\delta'(n) = 2n - m - 3$ and $\delta(m,n) = 0$ for $m = 2, n = 3$.
    (f) Not special by lemma 13.12.
    (g) $\delta'(n) = 2n - m - 3$ and $\delta(m,n) = 0$ for $m = 2, n = 3$.
    (h) Not special by lemma 13.12.
    (i) No integer solution for $\delta(6,n) = 0$.
    13-ii (a) $\delta(m,n) = 0$ for $m = 2, n = 3$.
    (b) $\delta(m,n) = 0$ for $m = 2, n = 3$.
    (c) No integer solution for $\delta(n-1,n)$.
    13-iii (a) $\delta'(n) = 2n - \frac{m(m-1)}{2} - 2$ and no integer solution for $\delta(m,n) = 0$.
    (b) $\delta'(n) = 2n - \frac{m(m-1)}{2} - 3$ and no integer solution for $\delta(m,n) = 0$.
    (c) No integer solution for $\delta(5,n)$.
    (d) $\delta'(n) = 2n - \frac{m(m-1)}{2} - 3$ and no integer solution for $\delta(m,n) = 0$.
    (e) $\delta'(n) = 2n - \frac{m(m-1)}{2} - 2$ and no integer solution for $\delta(m,n) = 0$.
    (f) $\delta(5,10) \neq 0$ by direct computation.

14. (a) Not special by lemma 13.12.
    (b) Not special by lemma 13.12.
    (c) $\delta'(n) = 2n - m - 3$ and no integer solution for $\delta(m,n) = 0$.
    (d) $\delta'(n) = 2n - m - 3$ and no integer solution for $\delta(m,n) = 0$.
    (e) Not special by lemma 13.12.
    (f) No integer solution for $\delta(\frac{n}{2}, n) = 0$.

15. (a) No integer solution for $\delta(10,n) = 0$.
    (b) No integer solution for $\delta(10,n) = 0$.
    (c) $\delta(10,17) \neq 0$ by direct computation.
    (d) $\delta(10,16) \neq 0$ by direct computation.

The following proposition summarises the special modules.

Proposition 13.13 *The following 2-simple modules of the form KII II and KII III are special:*

- *KII II-4-i (b):* $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)}) \oplus (\omega_1^{(*)} \otimes 1))$.
- *KII II-4-ii (a):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_3 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- *KII II-4-iii (d):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1))$.
- *KII II-4-iii (f):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1))$.
- *KII II-4-iii (g):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
- *KII II-4-iii (h):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- *KII II-4-iii (n):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- *KII II-4-iii (p):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- *KII II-4-iii (q):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- *KII II-5-i (b):* $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1))$.
- *KII II-5-ii (b):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
- *KII II-5-iii (e):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- *KII II-5-iv (a):* $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2^*))$.
- *KII II-6 (b):* $(\mathrm{GL}_1^3 \times \mathrm{SL}_3 \times \mathrm{SL}_5, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2))$.
- *KII II-9 (a):* $(\mathrm{GL}_1^4 \times \mathrm{SL}_6 \times \mathrm{SL}_7, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
- *KII II-10:* $(\mathrm{GL}_1^{r+1} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_k \otimes 1) \oplus (1 \otimes \varrho_{k+1}^*) \oplus \ldots \oplus (1 \otimes \varrho_r^*))$, *where* $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ *is a special module.*
- *KII III-12:* $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
- *KII III-13-i (d):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
- *KII III-13-i (e):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1))$.
- *KII III-13-i (g):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1))$.
- *KII III-13-ii (a):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2 \otimes 1))$.
- *KII III-13-ii (b):* $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2^* \otimes 1))$.

13.2.3 The case KII IV for $km \leq n$

As a consequence of theorem 4.15 in Kimura et al. [17], we can always assume $n > m$.

First, we consider the module

$$\Big(\mathrm{GL}_1^{k+s_1+s_2+t_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,\\ (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \ \oplus\ (\omega_1 \otimes \omega_1)^{\oplus k} \ \oplus\ (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\Big),$$

with $n \geq m \geq 2$, $k \geq 1$ and $n \geq km$, i.e. the case KII IV-16.

Again, the question arises if some special modules could be found by considering less than $k + s + t$ scalar multiplications, where $k + s + t$ is the number of irreducible components. The following lemmata 13.14 and 13.15 tackle this question.

We investigate a case where some scalar multiplications can be discarded.

Lemma 13.14 *Let $km \leq n$, $k \geq 1$, $t \geq 1$ and $s + kt \leq m$. The module*

$$\Big(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t}\Big)$$

is prehomogeneous if and only if

$$\Big(\mathrm{GL}_1^{j} \times \mathrm{SL}_m \times \mathrm{GL}_n,\ (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t}\Big)$$

is prehomogeneous, where $j = 1$ for $s + kt = m$ (and we cannot replace it by $j = 0$ in this case), and $j = 0$ for $s + kt < m$. If $km < n$, we can replace GL_n by SL_n.

Proof: The "if"-part of the lemma is obvious.

Now assume the $\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n$-module is prehomogeneous. In particular, $\omega_1^{\oplus s} \otimes 1$ and $(\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t}$ must be prehomogeneous. By propositions 9.26 and 9.27 (with $m_1 = km$ and $m_2 = t$), the latter is equivalent to $(\mathrm{GL}_1^{k+s+t-1} \times \mathrm{SL}_m, \omega_1^{\oplus k} \otimes 1^{\oplus t})$ being prehomogeneous. But as $s + kt \leq m$, this is equivalent to

$$\Big(\mathrm{GL}_1^{k+t+s-1} \times \mathrm{SL}_m,\ \omega_1^{\oplus s} \oplus (\omega_1^{\oplus k} \otimes 1^{\oplus t})\Big)$$

and even

$$\Big(\mathrm{GL}_1^{j} \times \mathrm{SL}_m,\ \omega_1^{\oplus s} \oplus (\omega_1^{\oplus k} \otimes 1^{\oplus t})\Big)$$

being prehomogeneous, with $j = 1$ for $s + kt = m$ by Ks I-2 in theorem 11.2, and $j = 0$ by $s + kt < m$ by Ks II-1 in theorem 11.2. For the generic isotropy subgroup H of $(\mathrm{GL}^j \times \mathrm{SL}_m, \omega_1^{\oplus s})$, this is equivalent to $(H, \omega_1|_H^{\oplus k} \otimes 1^{\oplus t})$ being prehomogeneous (cf. proposition 9.11). Again by using proposition 9.27, we have that

$$\Big(H \times \mathrm{GL}_n, (\omega_1|_H \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t}\Big)$$

is prehomogeneous, and because $H \times \mathrm{GL}_n$ is the isotropy subgroup of $(\mathrm{GL}_1^j \times \mathrm{SL}_m \times \mathrm{GL}_n, \omega_1^{\oplus s} \otimes 1)$, we have the prehomogeneity of the $\mathrm{GL}_1^j \times \mathrm{SL}_m \times \mathrm{GL}_n$-module by proposition 9.11.

By corollary 9.28, we can replace GL_n by SL_n for $km < n$. ∎

Next, we have a case where a certain number of scalar multiplications is indespensable.

Lemma 13.15 *Let* $2 \leq s \leq n+1$. *Assume the module*

$$\left(\mathrm{GL}_1^j \times \mathrm{SL}_n \times G,\ ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1)\ \oplus\ \varrho\right)$$

to be prehomogeneous, where ϱ *is any representation of* $\mathrm{GL}_1^j \times \mathrm{SL}_n \times G$. *Then we have* $j \geq s-1$, *with* GL_1^{s-1} *acting on* $\omega_1^{\oplus s-1} \otimes 1$ *(and even* $j \geq s$ *for* $s = n+1$, *with* GL_1^{n+1} *acting on* $(\omega_1^{\oplus n} \oplus \omega_1^*) \otimes 1$*).*

Proof: First assume $s = n+1$. Then we must have $j \geq n+1$, because for $j < n+1$, the component $(\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1$ would not be prehomogeneous by Ks I-4 in theorem 11.2.

For $s \leq n$, the component $(\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1$ would not be prehomogeneous for $j < s-1$ by Ks II-2 in theorem 11.2.

Alternatively, one could proof this by a similar argument as in the proof of lemma 13.6, part 2, by considering the decomposition $\omega_1^{[n] \oplus s-2} = (\omega_1^{[n-1]} \oplus \mu)^{\oplus s-2}$ under the action of the isotropy subgroup GL_{n-1}, where additional scalar multiplications are needed for each of the $s-2$ components $(\mu, \Bbbk)$ to be prehomogeneous. ∎

Remark 13.16 Note that if a module is special for $\mathrm{GL}_1^j \times \mathrm{SL}_m \times \mathrm{SL}_n$, we do not have to check whether it is special for $\mathrm{GL}_1^l \times \mathrm{SL}_m \times \mathrm{SL}_n$ with $l > j$. We can never discard all scalar multiplications, as there are no special modules for semisimple groups (like $\mathrm{SL}_m \times \mathrm{SL}_n$).

Lemma 13.17 *Let* G *be a simple algebraic group and let* ϱ, σ_i *(resp.* τ_i*) be irreducible representations of* G *(resp.* SL_n*),* $\dim(\varrho) = n$. *As* $\varrho(G)$ *is a subgroup of* SL_n, *we can define representations* $\tau_i \circ \varrho^*$ *for* G. *Then, for* $0 \leq j \leq s+t$,

$$\left(\mathrm{GL}_1^{1+j} \times G \times \mathrm{SL}_n,\ (\mu \otimes \varrho \otimes \omega_1)\ \oplus\ (\sigma_1 \oplus \ldots \oplus \sigma_s) \otimes 1\ \oplus\ 1 \otimes (\tau_1 \oplus \ldots \oplus \tau_t)\right)$$

is a special module if and only if

$$\left(\mathrm{GL}_1^j \times G,\ \sigma_1 \oplus \ldots \oplus \sigma_s\ \oplus\ (\tau_1 \oplus \ldots \oplus \tau_t) \circ \varrho^*\right)$$

is special.

Proof: We identify the module for $\mu \otimes \varrho \otimes \omega_1$ with Mat_n. The isotropy subgroup of $\mathrm{GL}_1^j \times G \times \mathrm{SL}_n$ at the generic point I_n of Mat_n is

$$\{(\varrho(A), \varrho^*(A)) \mid A \in G\}.$$

The SL_n-part of this subgroup is $\varrho^*(G)$. By corollary 9.12, we have our result. ■

We also use a non-trivial result from Kimura et al. [17]:

Lemma 13.18 *Let $\varrho_1 : G \to \mathrm{GL}(V^{m_1})$ and $\varrho_2 : G \to \mathrm{GL}(V^{m_2})$ be rational representations of an algebraic group G. If $n > m_1 > m_2$, the module*

$$\left(G \times \mathrm{GL}_n,\ (\varrho_1 \otimes \omega_1) \oplus (\varrho_2 \otimes \omega_1^*),\ (V^{m_1} \otimes \mathbb{k}^n) \oplus (V^{m_2} \otimes \mathbb{k}^{n*})\right)$$

is a non-regular prehomogeneous module.

Proof: See part 1 of proposition 1.22 in Kimura et al. [17]. ■

We now consider the cases 16-i to 16-iv one by one. See p. 100 for the respective constraints on s_1, s_2, t_1 and t_2. In the following, let G denote the group and V the module.

Lemma 13.19 *We cannot obtain special modules from the case IV-16-i by replacing $\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n \times \mathrm{SL}_n$ by $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$, and we also cannot obtain special modules from the case IV-16-i when $(s_1 + t_2) + (s_2 + t_1) < n$.*

Proof: Let $c = (s_1 + t_2) + (s_2 + t_1)$.

If we replaced $\mathrm{GL}_1^{c+1} \times \mathrm{SL}_n \times \mathrm{SL}_n$ by $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$, we would have

$$\dim(G) - \dim(V) = n^2 - cn - 1,$$

so we have $c < n$ if the module is prehomogeneous. But then we have

$$n^2 - cn - 1 \geq n(n - (n-1)) - 1 = n - 1 > 0.$$

So we have $\dim(G) \geq \dim(\mathrm{GL}_n \times \mathrm{GL}_n) = 2n^2$, i.e.

$$\dim(G) - \dim(V) \geq 2n^2 - (cn + n^2) = n^2 - cn,$$

which is > 0 for any $c < n$. Thus we can get special modules for $c \geq n$ only. ■

Proposition 13.20 *From the case IV-16-i, we get the following special modules:*

- $(\mathrm{GL}_1^{n+2} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, *with $s_1 + t_2 = n$ and $s_2 + t_1 = 1$.*
- $(\mathrm{GL}_1^{n+2} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, *with $s_1 + t_2 = n + 1$ and $s_2 + t_1 = 0$.*

- $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, *with* $s_1 + t_2 = n$ *and* $s_2 = t_1 = 0$.

These are the only special modules obtained from IV-16-i.

Proof: Let $a = s_1 + t_2$, $b = s_2 + t_1$ and $c = a + b$.

For $G = \mathrm{GL}_1^{c+1} \times \mathrm{SL}_n \times \mathrm{SL}_n$, we have

$$\dim(G) - \dim(V) = 2n^2 - 2 + c + 1 - n^2 - cn = n^2 + c(1-n) - 1 = 0$$

if and only if

$$c = \frac{n^2 - 1}{n - 1} = n + 1.$$

Taking into account the constraints on a and b, this is true if and only if either $a = n$, $b = 1$, or $a = n + 1$, $b = 0$ (or with a and b exchanged). Thus, we have the first two cases of the proposition. Also, we have no special modules for $\mathrm{GL}_1^{c+1} \times \mathrm{SL}_n \times \mathrm{SL}_n$ when $c < n + 1$.

For $c = a = n$, $b = 0$ and $t_2 > 0$, the module is prehomogeneous for $G = \mathrm{GL}_n \times \mathrm{GL}_n$ by lemma 13.14, and a dimension count shows that it is even a special module. Thus, we have the third case of the proposition.

For $c = a = n$, $b = 0$ and $t_2 = 0$, the module is obviously a special module for $G = \mathrm{GL}_n \times \mathrm{GL}_n$.

Now it remains to be shown that there are no special modules for $\mathrm{GL}_n \times \mathrm{GL}_n$ with $a = n - 1$, $b = 1$, and together with lemma 13.19 it follows that there are no other special modules at all.

Let $a = n - 1$, $b = 1$, e.g. $s_2 = 1$ and $t_1 = 0$ (the argument is the same for $s_2 = 0$, $t_1 = 1$). By lemma 13.17, our module is special if and only if

$$\left(\mathrm{GL}_n,\ (\omega_1^{\oplus s_1} \oplus \omega_1^*) \oplus (\omega_1^{*\oplus t_2} \circ \omega_1^*)\right) = \left(\mathrm{GL}_n,\ \omega_1^{\oplus n-1} \oplus \omega_1^*\right)$$

is special, but by Ks II-2 (resp. Ks I-1 for $n = 2$) in theorem 11.2, the latter is not prehomogeneous (resp. not special for $n = 2$). ■

Proposition 13.21 *From the case IV-16-ii (a), we get the following special modules:*

- $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, *with* $s_1 + kt_2 = m$.

From the case IV-16-ii (b), we get the following special modules:

- $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1)^{\oplus t_1})$, *with* $s_2 + kt_1 = m$.

These are the only special modules obtained from IV-16-ii.

Proof: Note that by corollary 9.14 it suffices to treat the case (a) and get the corresponding results for case (b) by exchanging s_1, t_2 with s_2, t_1.

For the case IV-16-ii (a), the conditions for lemma 13.14 are fulfilled and so we can assume $G = \mathrm{GL}_m \times \mathrm{GL}_n$. We then have

$$\begin{aligned}\dim(G) - \dim(V) &= (km)^2 + m^2 - (s_1 m + (km)^2 + t_2 km) \\ &= (km)^2 + m^2 - (s_1 + kt_2)m - (km)^2 \\ &= m^2 - (s_1 + kt_2)m \geq 0,\end{aligned}$$

with equality if and only if $s_1 + kt_2 = m$.

Furthermore, the module cannot be special for $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$ in the case $s_1 + kt_2 = m$ for dimension reasons, and going to $s_1 + kt_2 < m$, the modules dimension decreases by $m \geq 2$ at least, whereas the group dimension decreases only by 1 compared to $\mathrm{GL}_m \times \mathrm{GL}_n$. So there are no special modules of type IV-16-ii for $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$. ■

Proposition 13.22 *From the case IV-16-iii, we get the following special modules:*

- $(\mathrm{GL}_1^{t_1-1} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1)^{\oplus t_1 - 1}$, *with* $k = 1$, $s_2 + t_1 = m + 1$.

These are the only special modules obtained from IV-16-iii.

Proof: The component

$$(\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1) = (\omega_1^{\oplus k} \oplus 1) \otimes \omega_1$$

is of dimension $(km + 1)n = n^2$, so the isotropy subgroup on this component is (locally isomorphic to) SL_n, hence reductive. We can apply corollary 9.14 and get that our module is special if and only if

$$\left(\mathrm{GL}_1^{k+s_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s_1 + k} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t_1 - 1}))\right)$$

is special.

If we set $\tilde{t}_1 = 1$, $\tilde{t}_2 = t_1 - 1$, $\tilde{s}_1 = s_2$, $\tilde{s}_2 = 0$ and note that $\tilde{s}_1 + \tilde{t}_2 \leq m$, this is the case IV-16-iv (c) with parameters $\tilde{s}_1, \tilde{s}_2, \tilde{t}_1, \tilde{t}_2$. Thus, by the following proposition 13.23, we have that the module is special if and only if $k = 1$, $\tilde{s}_1 + \tilde{t}_2 = m$ (i.e. $s_2 + t_1 = m + 1$) and $\mathrm{GL}_1^{s_2+1+t_1} \times \mathrm{SL}_m \times \mathrm{SL}_n$ is replaced by $\mathrm{GL}_1^{1+t_1} \times \mathrm{SL}_m \times \mathrm{SL}_n$. ■

Proposition 13.23 *From the case IV-16-iv (a), we get the following special modules:*

- $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$, *with* $t_2 = m$, $s_1 = 0$, $s_2 = 1$ *and* $n = m + 1$.
- $(\mathrm{GL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m})$, *with* $t_2 = m$, $s_1 = s_2 = 0$ *and* $n = m + 1$.
- $(\mathrm{GL}_1^{m+2} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m+1})$, *with* $t_2 = m + 1$, $s_1 = s_2 = 0$ *and* $n = m + 1$.
- $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$, *with* $t_2 = m$, $s_1 = 1$, $s_2 = 0$ *and* $n = m + 1$.

From the case IV-16-iv (c), we get the following special modules:

- $(\mathrm{GL}_1^{t_2} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus t_2})$, *with* $k = 1$, $s_1 + t_2 = m$ *and* $n = m + 1$.

These are the only special modules obtained from IV-16-iv.

Proof: Consider the cases 16-iv (a), (b) and (c) separately.

(a) First we assume $s_2 = 1$. We can use propositions 9.26, 9.27 and corollary 9.28 (with $m_1 = m$, $m_2 = t_2$) almost exactly as in the proof of lemma 13.14 to show that

$$\left(\mathrm{GL}_1^{s_1+2+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ ((\omega_1^{\oplus s_1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

is prehomogeneous if and only if

$$\left(\mathrm{GL}_1^{j} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\mu \otimes \omega_1 \otimes 1)^{\oplus s_1} \oplus (\tilde{\mu} \otimes \omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

is, where $j = s_1 + t_2$ for $s_1 + t_2 \le m - 1$ (i.e. $\tilde{\mu} = 1$) and $j = m + 1$ for $s_1 + t_2 = m$ (i.e. $\tilde{\mu} = \mu$), and no smaller j possible in either case. Let $G = \mathrm{GL}_1^j \times \mathrm{SL}_m \times \mathrm{SL}_n$. For $s_1 + t_2 \le m - 1$, note that $s_1(1-m) + t_2(1-n)$ is minimal for $s_1 = 0$, $t_2 = m - 1$, as $n > m$. Then we have

$$\begin{aligned}
\dim(G) - \dim(V) &= s_1 + t_2 + m^2 - 1 + n^2 - 1 - s_1 m - m - mn - t_2 n \\
&= m^2 + n^2 - 2 + s_1(1 - m) + t_2(1 - n) - m - mn \\
&\ge m^2 + n^2 - 2 + (m - 1)(1 - n) - m - mn \\
&= m^2 + n^2 - 2m + n - 2 \\
&= (n - m)^2 + (n - 2) > 0,
\end{aligned}$$

as $n > m \ge 2$. So there are no special modules. For $s_1 + t_2 = m$, note that $-s_1 m - t_2 n$ is minimal for $t_2 = m$, $s_1 = 0$. We have

$$\begin{aligned}
\dim(G) - \dim(V) &= m + 1 + m^2 - 1 + n^2 - 1 - s_1 m - m - mn - t_2 n \\
&\ge m^2 + n^2 - 1 - 2mn \\
&= (n - m)^2 - 1 \ge 0,
\end{aligned}$$

with equality if and only if $n = m + 1$ and $t_2 = m$.

Now let $s_2 = 0$. If $s_1 + t_2 \leq m$ and the module

$$\left(\mathrm{GL}_1^{s_1+1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

is prehomogeneous, then

$$\left(\mathrm{GL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

must be prehomogeneous by lemma 13.14, so we only need to check whether the latter module is special. Let $G = \mathrm{GL}_m \times \mathrm{SL}_n$. Arguing similarly as in the case $s_2 = 1$, we have

$$\begin{aligned} \dim(G) - \dim(V) &= n^2 + m^2 - 1 - s_1 m - mn - t_2 n \\ &\geq n^2 + m^2 - 2mn - 1 \\ &= (n-m)^2 - 1 \geq 0, \end{aligned}$$

with equality if and only if $n = m + 1$ and $t_2 = m$.

For $s_1 = s_2 = 0$ and $t_2 = m + 1$, we consider $G = \mathrm{GL}_1^{1+m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n$, and we have

$$\begin{aligned} \dim(G) - \dim(V) &= m + n^2 + m^2 - (m+1)n - mn \\ &= n^2 + m^2 + (m-n) - 2nm \\ &= (n-m)^2 - (n-m) \geq 0, \end{aligned}$$

with equality if and only if $n = m + 1$.

For $s_2 = 0$, $s_1 + t_2 = m + 1$ and $s_1 > 0$, we can use propositions 9.26, 9.27 and corollary 9.28 (with $m_1 = m$, $m_2 = t_2$) almost exactly as in the proof of lemma 13.14 to show that

$$\left(\mathrm{GL}_1^{1+m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

is prehomogeneous if and only if

$$\left(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\mu \otimes \omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus t_2}\right)$$

is, with no less than $m+1$ scalar multiplications possible. Similarly as before, we have

$$\begin{aligned} \dim(G) - \dim(V) &= m + 1 + n^2 + m^2 - 2 - s_1 m - mn - t_2 n \\ &\geq n^2 + m^2 - 1 - 2mn \\ &= (n-m)^2 - 1 \geq 0, \end{aligned}$$

with equality if and only if $m = n - 1$, $t_2 = m$ and $s_1 = 1$.

(b) For $G = \mathrm{GL}_1^j \times \mathrm{SL}_m \times \mathrm{SL}_n$ with $1 \leq j \leq k + s_1 + t_2$, we set

$$\delta(n) = \dim(G) - \dim(V).$$

Then we have $\delta'(n) = 2n - t_2 - km$, i.e. as a function over the real numbers, δ is minimal at $n_0 = \frac{t_2+km}{2}$ and it is strictly increasing for $n > n_0$.

Note that $t_2 < m$ because $s_1 + kt_2 \leq m$ and $k \geq 2$. We then have

$$n_0 = \frac{km + t_2}{2} < \frac{km + m}{2} = \frac{k+1}{2}m < km < n,$$

so all valid values for n (i.e. the integers with $km < n$) are contained in the interval $[n_0, \infty)$ where δ is strictly increasing, and thus $\delta(km + 1)$ is the minimal value of δ at a valid integer. We show that even in the case $j = 1$, we still have $\delta(km + 1) > 0$, so this will also be the case for any $j \geq 2$.

$$\begin{aligned}\delta(km+1) &= m^2 + (km^2 + 2km + 1) - 1 - (s_1 m + km^2 + km + t_2 km + t_2)\\ &= m^2 + km - (s_1 + kt_2)m - t_2\\ &\geq m^2 + km - m^2 - t_2\\ &\geq km - t_2 \geq 2m - t_2 \geq 2(t_2+1) - t_2 \geq 2 > 0.\end{aligned}$$

In particular, $\delta(n) > 0$ for any integer $n > km + 1$. So there are no special modules to be obtained from IV-16-iv (b).

(c) By part 2 of lemma 13.6, the module

$$\left(\mathrm{GL}_1^{s_1+k+1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t_2}))\right)$$

is special if and only if

$$\left(\mathrm{GL}_1^{s_1+k} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1},\ (\mu \otimes \omega_1 \otimes 1)^{\oplus s_1+k} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t_2-1}\right)$$

is special.

If $k \geq 2$, this is equivalent to the case IV-16-iv (b) with $\mathrm{GL}_1^{s_1+k+t_2-1}$ replaced by $\mathrm{GL}_1^{s_1+k+1}$, and so there are no special modules by part (b) of this proposition.

Now assume $k = 1$. Set $\tilde{n} = n - 1$, $\tilde{m} = m$, $\tilde{s}_1 = s_1 + 1$, $\tilde{s}_2 = 0$, $\tilde{t}_1 = 0$ and $\tilde{t}_2 = t_2 - 1$. We have $\tilde{n} \geq \tilde{m}$ and $\tilde{s}_1 + \tilde{t}_2 = (s_1 + 1) + (t_2 - 1) = s_1 + t_2 \leq m = \tilde{m}$.

For $\tilde{m} = m = n - 1 = \tilde{n}$, this is the case IV-16-i with parameters $\tilde{n} = \tilde{m}$, $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{t}_1$, $\tilde{t}_2$ and the condition $\tilde{s}_1 + \tilde{s}_1 + \tilde{t}_1 \leq \tilde{n}$, $\tilde{s}_2 = \tilde{t}_1 = 0$, but with at most $\tilde{s}_1 + 1$ scalar multiplications. By proposition 13.20, the module is special only for $\tilde{s}_1 + \tilde{t}_2 = \tilde{n}$ (i.e. $s_1 + t_2 = n - 1 = m$), if we replace $\mathrm{GL}_1^{s_1+1} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1}$ by $\mathrm{GL}_{n-1} \times \mathrm{GL}_{n-1}$. This means we must discard s_1 scalar multiplications of the original module to obtain a special module.

For $\tilde{m} < \tilde{n} = n - 1$, this is the case IV-16-iv (a) with parameters $\tilde{n}$, $\tilde{m}$, $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{t}_2$ and the condition $\tilde{s}_1 + \tilde{t}_1 \leq \tilde{m}$, $\tilde{s}_2 = 0$. By part (a) of this proposition, in order to obtain a special module, we must have $\tilde{m} = \tilde{n} - 1 = n - 2$, $\tilde{t}_2 = \tilde{m}$, $\tilde{s}_1 = 0$, but we have $\tilde{s}_1 = s_1 + 1 > 0$, so it is not possible to fulfil these conditions. Thus, there are no special modules for $\tilde{m} < \tilde{n}$. ■

13.2.4 The case KV IV for $km > n$

Now we consider the module

$$\Big(\mathrm{GL}_1^{k+s_1+s_2+t_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,$$
$$(\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \;\oplus\; (\omega_1 \otimes \omega_1)^{\oplus k} \;\oplus\; (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\Big),$$

with $n > m \geq 2$, $k \geq 2$ and $n < km$, i.e. the case KII IV-17.

To handle this case, we have to take a close look at the proofs in section 4.2 of Kimura et al. [17], where the prehomogeneity of this module is investigated. The essential part for this case is to determine when a module of the form

$$\Big(\mathrm{GL}_m \times \mathrm{GL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t})\Big)$$

is special. In order to investigate this, we consider the generic isotropy algebra $\mathfrak{g}_X$ of

$$\Big(\mathrm{GL}_m \times \mathrm{GL}_n,\ (\omega_1 \otimes \omega_1)^{\oplus k}\Big).$$

To understand the structure of $\mathfrak{g}_X$, we use the fact that $\mathfrak{g}_X$ is isomorphic to the generic isotropy algebra of a castling-equivalent module

$$\Big(\mathrm{GL}_{m_0} \times \mathrm{GL}_{n_0},\ (\omega_1 \otimes \omega_1)^{\oplus k}\Big)$$

with $km_0 \leq n_0$. Then we will show that

$$\dim(\mathfrak{g}_X) = \dim((\omega_1^{\oplus s} \otimes 1) \oplus (1 \otimes \omega_1^{*\oplus t}))$$

if and only if $km_0 = n_0$.

Recall definition 11.5. By theorem 4.5 in [17], we have a uniquely determined (i.e. smallest) $j = \nu(k, m, n)$ such that $(\mathrm{GL}_1^k \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k})$ is transformed to a trivial prehomogeneous module of $\mathrm{GL}_1^k \times \mathrm{SL}_{m_0} \times \mathrm{SL}_{n_0}$ (i.e. $km_0 \leq n_0$) or a simple prehomogeneous module (i.e. $m_0 = 1$) by j castling transformations. Further, we have the sequence (a_i) defined by

$$a_{-1} = -1, \qquad a_0 = 0, \qquad a_i = ka_{i-1} - a_{i-2} \text{ for } i > 0.$$

If we set

$$\begin{pmatrix} n_i \\ m_i \end{pmatrix} = \begin{pmatrix} a_{i+1} & -a_i \\ a_i & -a_{i-1} \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ m_0 \end{pmatrix}$$

for the above m_0 and n_0, lemma 4.6 in [17] tells us that we obtain the following sequence of modules by the above mentioned j castling transformations:

$$\Big(\mathrm{GL}_1^k \times \mathrm{SL}_{m_0} \times \mathrm{SL}_{n_0}, (\omega_1 \otimes \omega_1)^{\oplus k}\Big)$$
$$\vdots$$
$$\Big(\mathrm{GL}_1^k \times \mathrm{SL}_{m_i} \times \mathrm{SL}_{n_i}, (\omega_1 \otimes \omega_1)^{\oplus k}\Big)$$
$$\vdots$$
$$\Big(\mathrm{GL}_1^k \times \mathrm{SL}_{m_j} \times \mathrm{SL}_{n_j}, (\omega_1 \otimes \omega_1)^{\oplus k}\Big)$$

In particular, we have

$$n = n_j = a_{j+1} n_0 - a_j m_0, \quad m = m_j = a_j n_0 - a_{j-1} m_0.$$

Of course, this still holds if we replace $\mathrm{GL}_1^k \times \mathrm{SL}_{m_i} \times \mathrm{SL}_{n_i}$ by $\mathrm{GL}_{m_i} \times \mathrm{GL}_{n_i}$.

Remark 13.24 For all $i \geq 1$, the matrix

$$\begin{pmatrix} a_{i+1} & -a_i \\ a_i & -a_{i-1} \end{pmatrix} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_i & -a_{i-1} \\ a_{i-1} & -a_{i-2} \end{pmatrix}$$

has determinant 1, which easily follows by induction from the recurrence relation defining the sequence (a_i).

Remark 13.25 Generic points X_i for $(\mathrm{GL}_{m_i} \times \mathrm{GL}_{n_i}, (\omega_1 \otimes \omega_1)^{\oplus k})$ and $i = 0, \ldots, j$ are inductively defined on p. 481 in Kimura et al. [17], but we do not need their explicit form here. We consider the generic isotropy algebras $\mathfrak{g}_{X_i}$ as different representations of $\mathfrak{g}_X = \mathfrak{g}_{X_j}$. As a consequence of lemma 4.9 in [17], the generic isotropy algebra of $(\mathrm{GL}_{m_i} \times \mathrm{GL}_{n_i}, (\omega_1 \otimes \omega_1)^{\oplus k})$ is given by

$$\mathfrak{g}_{X_i} = \{(-A_{i-1}^\top, A_i)\},$$

with A_i defined by

$$-A_{-1}^\top = A \in \mathfrak{gl}_{m_0}, \quad -A_0^\top = \left(\begin{array}{c|c|c} A & & \\ \hline & A^{\oplus k-1} & \\ \hline C_0 & B_1 & B_2 \end{array}\right), \quad -A_1^\top = \left(\begin{array}{c|c|c} -A_0^{\top \oplus k-1} & & \\ \hline & A^{\oplus k-1} & \\ \hline C_1 & B_1 & B_2 \end{array}\right),$$

$$-A_i^\top = \left(\begin{array}{cccc|c|c} -A_{i-1}^{\top \oplus k-1} & & & & & \\ & -A_{i-2}^{\top \oplus k-2} & & & & \\ & & \ddots & & & \\ & & & -A_0^{\top \oplus k-2} & & \\ \hline & & & & A^{\oplus k-1} & \\ \hline & & C_i & & B_1 & B_2 \end{array}\right) \text{ for } i = 2, \ldots, j.$$

Here, $B_1 \in \mathrm{Mat}_{n_0 - km_0, m_0(k-1)}$, $B_2 \in \mathrm{Mat}_{n_0 - km_0}$ and C_0 are arbitrary matrices, and the C_i depend on C_0. Note that as B_2 is quadratic, all submatrices A lie on the diagonal of $-A_i^\top$. The submatrices of the form $(C_i \ B_1 \ B_2)$ are called **black blocks**. By b_i we denote the number of black blocks in $-A_i^\top$ for the case $n_0 - km_0 > 0$.

The following lemmata 13.26 and 13.27 give us some results on the sequences (a_i) and (b_i).

Lemma 13.26 *Let $-A_i^\top$ be defined as in remark 13.25.*

1. *An explicit formula for the a_i:*

   For $k = 2$: $\quad a_{-1} = -1, \quad a_0 = 0, \quad a_i = i, \ i > 0.$

   For $k > 2$: $\quad a_{-1} = -1, \quad a_0 = 0, \quad a_i = \dfrac{1}{\sqrt{k^2-4}} \left(\left(\dfrac{k + \sqrt{k^2-4}}{2} \right)^i - \left(\dfrac{k - \sqrt{k^2-4}}{2} \right)^i \right), \ i > 0.$

2. *An explicit formula for the* b_i *(for* $n_0 - km_0 > 0$*):*

$$\text{For } k = 2: \quad b_0 = 1, \quad b_1 = 2, \quad b_i = i + 1, \; i > 1.$$
$$\text{For } k > 2: \quad b_0 = 1, \quad b_1 = k, \quad b_i = \frac{k-1}{k-2}(a_{i+1} - a_i) - \frac{1}{k-2}, \; i > 1.$$

Proof:

1. This was shown in lemma 4.6 of [17].

2. Consider the case $k = 2$. By looking at the definition of the $-A_i^\top$, we immediately have $b_0 = 1$ and $b_1 = 2$. Further, we note that for $i > 1$, the matrix $-A_i^\top$ contains all the black blocks from one copy of $-A_{i-1}^\top$, plus one additional black block. By induction, we have our result.

   For the case $k > 2$, an inductive proof would certainly do, but we shall not miss this opportunity to demonstrate the beauty of generating functions. Throughout this proof, we write

   $$g(z) = \frac{1}{1-z} = \sum_{i=0}^{\infty} z^i$$

   for the geometric series. We proceed in two steps.

   - The generating function $a(z)$ for (a_i) is the formal power series

     $$a(z) = \sum_{i=0}^{\infty} a_i z^i.$$

     For simplicity, we ignore a_{-1} in this definition and pretend the sequence (a_i) to be initialised with $a_0 = 0$ and $a_1 = 1$. By looking at the recurrence relation defining (a_i), we get

     $$\begin{aligned} a(z) &= a_0 + a_1 z + \sum_{i=2}^{\infty}(ka_{i-1} - a_{i-2})z^i \\ &= 0 + z + kz\sum_{i=1}^{\infty} a_i z^i - z^2 \sum_{i=0}^{\infty} a_i z^i \\ &= z + kza(z) - \underbrace{kza_0}_{=0} - z^2 a(z) \\ &= z + kza(z) - z^2 a(z), \end{aligned}$$

     which is equivalent to

     $$a(z) = \frac{z}{z^2 - kz + 1}.$$

- By looking at the definition of the $-A_i^\top$, we immediately have $b_0 = 1$ and $b_1 = k$. Also, we see that b_i must satisfy the recurrence relation

$$b_i = (k-1)b_{i-1} - (b_{i-1} - b_{i-2}) + 1 = kb_{i-1} - b_{i-2} + 1.$$

for $i > 1$. For the formal power series

$$b(z) = \sum_{i=0}^{\infty} b_i z^i$$

we then have

$$\begin{aligned} b(z) &= b_0 + b_1 z + \sum_{i=2}^{\infty} (kb_{i-1} - b_{i-2} + 1) z^i \\ &= 1 + kz + kz \sum_{i=1}^{\infty} b_i z^i - z^2 \sum_{i=0}^{\infty} b_i z^i + z^2 \sum_{i=0}^{\infty} z^i \\ &= 1 + kz + kzb(z) - \underbrace{kzb_0}_{=kz} - z^2 b(z) + z^2 g(z) \\ &= 1 + kzb(z) - z^2 b(z) + z^2 g(z), \end{aligned}$$

which is equivalent to

$$b(z) = \frac{1 + z^2 g(z)}{z^2 - kz + 1}.$$

This can be rewritten as

$$b(z) = \frac{k-1}{k-2} \left(\frac{1}{z^2 - kz + 1} - \frac{z}{z^2 - kz + 1} \right) - \frac{1}{k-2} g(z).$$

Comparing with $a(z)$, we see that

$$\frac{1}{z^2 - kz + 1} = \frac{a(z) - a_0}{z} = \frac{a(z)}{z},$$

and for any power series it holds that

$$\frac{a(z) - a_0}{z}$$

is the power series with coefficient a_i replaced by a_{i+1}. Thus we have

$$b_i = \frac{k-1}{k-2}(a_{i+1} - a_i) - \frac{1}{k-2}$$

by comparing coefficients. ■

For more background on generating functions, we refer to the wonderful book by Graham, Knuth and Patashnik [10].

Lemma 13.27 *Let $-A_i^\top$ be defined as in remark 13.25.*

1. *The number of copies of A contained in $-A_i^\top$ is a_{i+2}.*
2. *We have $a_i > a_{i-1}$ for all i.*
3. *For $n_0 - km_0 > 0$, we have $b_i < a_{i+2}$, i.e. the number of black blocks is smaller than the number of copies of A in $-A_i^\top$.*

Proof:

1. This was shown in lemma 4.9 of [17].
2. We have $a_{-1} = -1 < 0 = a_0$. Assume $a_i > a_{i-1}$. As $k \geq 2$, we have
$$a_{i+1} = ka_i - a_{i-1} > ka_i - a_i = (k-1)a_i \geq a_i.$$
3. For $k > 2$, we first note that $\frac{k-1}{k-2} = 1 + \frac{1}{k-2} < 2 < k$. Then we have
$$\begin{aligned} b_i = \frac{k-1}{k-2}(a_{i+1} - a_i) - \frac{1}{k-2} &< \frac{k-1}{k-2}(a_{i+1} - a_i) \\ &< k(a_{i+1} - a_i) = ka_{i+1} - ka_i < ka_{i+1} - a_i = a_{i+2}. \end{aligned}$$
For $k = 2$, we have $b_i = i + 1 < i + 2 = a_{i+2}$. ■

We now turn to the analysis of $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t}))$. This module is special if and only if the operation on $(\omega_1^{\oplus s} \otimes 1) \oplus (1 \otimes \omega_1^{*\oplus t})$ of the generic isotropy subgroup G_X of $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k})$ yields a special module (corollary 9.12). Due to castling equivalence, the group G_X is isomorphic to the generic isotropy subgroup of $(\mathrm{GL}_{m_0} \times \mathrm{GL}_{n_0}, (\omega_1 \otimes \omega_1)^{\oplus k})$, and so we have the following lemma:

Lemma 13.28 *The dimension of the generic isotropy algebra $\mathfrak{g}_X = \mathfrak{Lie}(G_X)$ is*

$$\dim(\mathfrak{g}_X) = \dim(\mathrm{GL}_{m_0}) + \dim(\mathrm{GL}_{n_0}) - \dim((\omega_1 \otimes \omega_1)^{\oplus k}) = n_0^2 + m_0^2 - km_0n_0.$$

Now we take a closer look at the action of $\mathfrak{g}_X$ on $(\omega_1^{\oplus s} \otimes 1) \oplus (1 \otimes \omega_1^{*\oplus t})$.

Lemma 13.29 *Consider $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t}))$ and $j = \nu(k, m, n)$. This module is not special if one of the following conditions holds:*

1. *$sa_{j+1} + ta_{j+2} < m_0$.*
2. *$sb_{j-1} + tb_j < n_0$ (only relevant for the case $n_0 - km_0 > 0$).*

On the other hand, if we have $>$ in one of these cases, then the module is not even prehomogeneous. So for a special module we must require equality in both cases.

Proof: The vector space for the representation $(\omega_1^{\oplus s} \otimes 1) \oplus (1 \otimes \omega_1^{*\oplus t})$ is

$$V = (\mathbb{k}^m)^{\oplus s} \oplus (\mathbb{k}^n)^{\oplus t}.$$

Recall from remark 13.25 that $\mathfrak{g}_X$ consists of pairs of matrices $(-A_{j-1}^\top, A_j)$. If we represent the elements of $\mathbb{k}^m$ (resp. $\mathbb{k}^n$) by column vectors, the action of $\mathfrak{g}_X$ on $\mathbb{k}^m$ via ω_1 is represented by a matrix $-A_{j-1}^\top$ (resp. the action on $\mathbb{k}^n$ via ω_1^* on $\mathbb{k}^n$ is represented by $-A_j^\top$).

If we restrict the action of $\mathfrak{g}_X$ to its subalgebra $\mathfrak{gl}_{m_0}$, the $\mathfrak{gl}_{m_0}$-action is represented by those blocks of $-A_{j-1}^\top$ (resp. $-A_j^\top$) containing a copy of $A \in \mathfrak{gl}_{m_0}$. The number of these blocks is a_{j+1} (resp. a_{j+2}) by part 1 of lemma 13.27, so this is the number of subspaces $\mathbb{k}^{m_0} \subset \mathbb{k}^m$ (resp. $\mathbb{k}^{m_0} \subset \mathbb{k}^n$) that $\mathfrak{gl}_{m_0}$ acts on non-trivially. Let

$$U = (\mathbb{k}^{m_0})^{\oplus s a_{j+1}} \oplus (\mathbb{k}^{m_0})^{\oplus t a_{j+2}}$$

be the direct sum of these subspaces. We chose a complement W of U in V such that W is spanned by those unit vectors whose indices correspond to the row numbers of the black blocks in $-A_{j-1}^\top$ (resp. $-A_j^\top$). Note that $V = U \oplus W$ is a direct sum of vector spaces, but not a direct sum of $\mathfrak{g}_X$-modules.

The image of the action of the black blocks on U is contained in W. So if the action on U is prehomogeneous, then the action of $\mathfrak{gl}_{m_0}$ on U must be prehomogenous already, so we have

$$\dim(\mathfrak{gl}_{m_0}) = m_0^2 \geq (s a_{j+1} + t a_{j+2}) m_0 = \dim(U),$$

and if $\dim(U) < m_0^2$, we have a non-trivial isotropy subalgebra. As this subalgebra acts trivially on the complement W as well, it acts trivially on V, so the module cannot be special in this case.

Now assume $m_0 = s a_{j+1} + t a_{j+2}$ and $n_0 - k m_0 > 0$. By the construction of the matrices in remark 13.25, each black block contains $n_0 - k m_0$ rows, and there are b_{j-1} (resp. b_j) black blocks contained in $-A_{j-1}^\top$ (resp. $-A_j^\top$). So the complement W must have the form

$$W = (\mathbb{k}^{n_0 - k m_0})^{\oplus s b_{j-1}} \oplus (\mathbb{k}^{n_0 - k m_0})^{\oplus t b_j}.$$

If the module is prehomogeneous, we have

$$\begin{aligned} \dim(\mathfrak{g}_X) = n_0^2 + m_0^2 - k m_0 n_0 &\geq \dim(V) \\ &= \dim(U) + \dim(W) \\ &= \underbrace{(s a_{j+1} + t a_{j+2})}_{=m_0} m_0 + (s b_{j-1} + t b_j)(n_0 - k m_0) \\ &= m_0^2 + (s b_{j-1} + t b_j)(n_0 - k m_0), \end{aligned}$$

with equality if and only if $s b_{j-1} + t b_j = n_0$. As a consequence, the module cannot be special if $s b_{j-1} + t b_j < n_0$. ∎

Proposition 13.30 *For $n > m \geq 2$, $k \geq 2$ and $n < km$, the module*

$$\left(\mathrm{GL}_m \times \mathrm{GL}_n,\ (\omega_1 \otimes 1)^{\oplus s} \ \oplus\ (\omega_1 \otimes \omega_1)^{\oplus k} \ \oplus\ (1 \otimes \omega_1^*)^{\oplus t}\right)$$

special if and only if $n_0 = km_0$ and $sa_{j+1} + ta_{j+2} = m_0$ for $j = \nu(k, m, n)$ and

$$\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} a_{j+1} & -a_j \\ a_j & -a_{j-1} \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ m_0 \end{pmatrix}.$$

We give two different proofs for this proposition.

First proof: For $n_0 = km_0$, we have

$$\dim(\mathfrak{g}_X) = (km_0)^2 + m_0^2 - (km_0)^2 \geq \dim(V) = (sa_{j+1} + ta_{j+2})m_0$$

as in the proof of lemma 13.29. Equality holds if and only if $sa_{j+1} + ta_{j+2} = m_0$.

Now let $n_0 - km_0 > 0$. By our assumptions on m_0 and n_0, we have $m_0 < n_0$, and by lemma 13.29 we have $sa_{j+1} + ta_{j+2} \leq m_0$ if the module is prehomogeneous. Then part 3 of lemma 13.27 gives us

$$sb_{j-1} + tb_j < sa_{j+1} + tb_j < sa_{j+1} + ta_{j+2} \leq m_0 < n_0,$$

so the module is not special again by lemma 13.29. ■

Second proof: Recall that $n = a_{j+1}n_0 - a_j m_0$ and $m = a_j n_0 - a_{j-1}m_0$.

Given j, k and m_0, we define

$$\begin{aligned} \delta_X(n_0) &= \dim(\mathfrak{g}_X) - \dim((\omega_1^{\oplus s} \otimes 1) \oplus (1 \otimes \omega_1^{*\oplus t})) \\ &= m_0^2 + n_0^2 - km_0 n_0 - s(a_j n_0 - a_{j-1}m_0) - t(a_{j+1}n_0 - a_j m_0). \end{aligned}$$

For a special module, there must exist an integer n_0 such that $\delta_X(n_0) = 0$. All valid candidates for n_0 are the integers contained in the interval $[km_0, \infty)$. Pretending that δ_X is a function over the real numbers, we take the derivative:

$$\delta_X'(n_0) = 2n_0 - km_0 - sa_j - ta_{j+1}.$$

So the function has its minimum at

$$\frac{km_0 + sa_j + ta_{j+1}}{2}.$$

By lemma 13.29, we have $sa_{j+1} + ta_{j+2} \leq m_0$, and together with part 2 of lemma 13.27, we have $sa_j + ta_{j+1} < m_0$. So we have

$$\frac{km_0 + sa_j + ta_{j+1}}{2} < \frac{km_0 + m_0}{2} \leq \frac{n_0}{2} + \frac{m_0}{2} < n_0$$

for any integer $n_0 \geq km_0$. It follows that δ_X is strictly increasing on $[km_0, \infty)$, and attains its smallest value for a valid integer at $n_0 = km_0$. We have

$$\begin{aligned}\delta_X(km_0) &= (km_0)^2 + m_0^2 - (km_0)^2 - \underbrace{s(a_j km_0 - a_{j-1}m_0)}_{=m_0 s a_{j+1}} - \underbrace{t(a_{j+1}km_0 - a_j m_0)}_{=m_0 t a_{j+2}} \\ &= m_0^2 - m_0 \underbrace{(sa_{j+1} + ta_{j+2})}_{\leq m_0} \\ &\geq 0,\end{aligned}$$

with equality if and only if $sa_{j+1} + ta_{j+2} = m_0$. As δ_X is strictly increasing on $[km_0, \infty)$, we have $\delta_X(n_0) > \delta_X(km_0 + 1) \geq 0$ for all valid integers $n_0 > km_0$. Thus, the module is not special for $n_0 > km_0$. ■

From either of the two proofs of proposition 13.30, we get the following corollary.

Corollary 13.31 *We cannot obtain any special module by replacing* $\mathrm{GL}_m \times \mathrm{GL}_n$ *by* $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$ *in proposition 13.30.*

Proof: Replacing $\mathrm{GL}_m \times \mathrm{GL}_n$ by $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$ decreases the group dimension by 1.

For the case $n_0 = km_0$, the module's dimension must be smaller by at least 2 than the dimension of the special $\mathrm{GL}_m \times \mathrm{GL}_n$-module, as any irreducible component has dimension ≥ 2.

For $n_0 - km_0 > 0$, we have $sb_{j-1} + tb_j < m_0 \leq n_0 - 3$, as $k \geq 2$. Then we have $\dim(\mathfrak{g}_X) - \dim(V) \geq 2$ similarly as in the proof of lemma 13.29. Alternatively, one could show by direct computation that $\delta_X(km_0 + 1) \geq 2$ in the second proof of proposition 13.30. ■

We are now prepared to take a closer look at the prehomogeneous modules of type KII IV-17. The prehomogeneity of these modules is proved in theorems 4.13 to 4.18 in Kimura et al. [17] by relating them to modules of type KII IV-16, KII I or to $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t}))$. Thus, we can determine if one of these modules is special by the case it is related to.

Further, we may assume $(k, m, n) \in T$ (cf. definition 11.5), because for $\dim(G) = \dim((\omega_1 \otimes \omega_1)^{\oplus k})$, the module $(\mathrm{GL}_1^k \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k})$ is castling-equivalent to Ks I-3 by theorem 4.5 in [17]. So, in order to obtain a special module with $(k, m, n) \in T$, we must have at least one component $\omega_1^{(*)} \otimes 1$ or $1 \otimes \omega_1^{(*)}$.

The following lemma 4.14 from Kimura et al. [17] is our starting point.

Lemma 13.32 *Assume that for* $s_1 + s_2 > 0$ *or* $t_1 + t_2 > 0$*, the module*

$$\Big(\mathrm{GL}_1^{k+s_1+s_2+t_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,$$
$$(\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \;\oplus\; (\omega_1 \otimes \omega_1)^{\oplus k} \;\oplus\; (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\Big),$$

with $km > n > m \geq 2$ and $k \geq 2$, is prehomogeneous. Then one of the following cases holds:

1. (a) $s_2 = t_1 = 0, t_2 \geq 1$.
   (b) $s_2 = t_2 = 0$.
   (c) $s_1 = t_2 = 0, s_2 \geq 1$.
2. (a) $s_2 = 0, t_1 \geq 1, t_2 = 1$.
   (b) $s_1 = 1, s_2 \geq 1, t_2 = 0$.
3. (a) $s_2 = 0, t_1 = 1, t_2 \geq 0$.
   (b) $s_1 \geq 2, s_2 = 1, t_2 = 0$.
4. $s_2 = t_2 = 1$.

Remark 13.33 As $km > n$, at least one castling transformation must be applied to obtain m_0, n_0 with $km_0 \leq n_0$, so we may assume $j = \nu(k, m, n) \geq 1$. If $t > 0$, we then have by lemmata 4.10 and 4.11 in [17][10] that

$$\left(\mathrm{GL}_m \times \mathrm{GL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t})\right)$$

is prehomogenous if and only if

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{GL}_m \times \mathrm{GL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t})\right)$$

is prehomogeneous. As $k + s + t \geq 3$, we can consider $\mathrm{GL}_m \times \mathrm{GL}_n$ a subgroup of $\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n$, and either of the above modules is prehomogeneous if and only if

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t})\right)$$

is prehomogeneous. In particular, the latter module is not special, because

$$\dim(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n) \geq n^2 + m^2 + 1 > n^2 + m^2 = \dim(\mathrm{GL}_m \times \mathrm{GL}_n) \geq \dim(V).$$

Now we determine when the cases in lemma 13.32 are special modules. Note that we often use castling equivalence in the sense of lemma 9.25.

Proposition 13.34 *From case 1 (a) in lemma 13.32 we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t})$,
  with $sa_{j+1} + ta_{j+2} = m_0$ and $n_0 = km_0$.

Let $p = km + t - n$ and $q = kp - m = k^2m + kt - kn - m$. From case 1 (b) we get the following special modules:

[10] Be warned though that the meaning of m and n changes in the course of section 4.2 in [17].

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp > m$, $s > 0$, $ta_{j+1} + sa_{j+2} = \tilde{m}_0$ *and* $\tilde{n}_0 = k\tilde{m}_0$ *for* $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1} m$ *and* $j = \nu(k, p, m)$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp > m$, $kq = p$, $2 \leq t$ *and* $kt = q$.

- $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$,
with $kp > m$, $k = q + 1$ *and* $p = q^2 + q$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$,
with $kp > m$, $k = q$ *and* $p = q^2$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp > m$, $kq > p$, $ta_{j+2} = \tilde{m}_0$ *and* $k\tilde{m}_0 = \tilde{n}_0$ *for* $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1} p$ *and* $j = \nu(k, q, p)$.

- $(\mathrm{GL}_1^{1+k+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp = m$ *and* $t + k = p + 1$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp = m$ *and* $t + k = p$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp = m$, $2 \leq s$ *and* $t + ks = p$.

Let $p = km + t - n$, $q = kp + s - m$ *and* $r = kq - p$. *From case 1 (c) we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$,
with $kp > m$, $kq > p$, $t > 0$, $sa_{j+1} + ta_{j+2} = \tilde{m}_0$ *and* $\tilde{n}_0 = k\tilde{m}_0$ *for* $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1} p$ *and* $j = \nu(k, q, p)$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$,
with $kp > m$, $kq > p$, $kr = q$, $2 \leq s$ *and* $ks = r$.

- $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$,
with $kp > m$, $kq > p$, $k = r + 1$ *and* $q = r^2 + r$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$,
with $kp > m$, $kq > p$, $k = r$ *and* $q = r^2$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$,
with $kp > m$, $kq > p$, $kr > q$, $sa_{j+2} = \tilde{m}_0$ *and* $k\tilde{m}_0 = \tilde{n}_0$ *for* $\tilde{m}_0 = a_{j+1}r - a_j q$, $\tilde{n}_0 = a_j r - a_{j-1} q$ *and* $j = \nu(k, r, q)$.

- $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$,
with $kp > m$, $kq = p$ *and* $s + k = q + 1$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1),$ *with* $kp > m$, $kq = p$ *and* $s + k = q$.
- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t}),$ *with* $kp > m$, $kq = p$, $2 \le t$ *and* $s + kt = q$.
- $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k}),$ *with* $k = p$ *and* $kp + s - 1 = m$.
- $(\mathrm{GL}_1^{s+k} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k}),$ *with* $k = p + 1$ *and* $kp + s - 1 = m$.
- $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)),$ *with* $k = p$ *and* $kp + s - 1 = m$.
- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k}),$ *with* $ks = p$ *and* $kp = m$.

Proof: We consider the subcases of case 1 from lemma 13.32 one by one.

(a) As stated in remark 13.33, the module can be special only if we replace $\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n$ by $\mathrm{GL}_m \times \mathrm{GL}_n$, and in this case it is special if and only if $sa_{j+1} + ta_{j+2} = m_0$ and $n_0 = km_0$ by proposition 13.30.

(b) The module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})\right)$$

is castling-equivalent to

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_p \times \mathrm{SL}_m,\ (\omega_1^{\oplus t} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus s})\right)$$

with $p = km + t - n$. In particular, one of them is special if and only if the other one is. By the proof of theorem 4.15 in [17], we have $p < m$.

- If $kp > m$ and $s > 0$, this is the case 1 (a). The module is special if and only if we replace the group by $\mathrm{GL}_p \times \mathrm{GL}_m$ and if $ta_{j+1} + sa_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$, with $j = \nu(k, p, m)$, $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1} m$ holds.
- If $kp > m$ and $s = 0$, then $t > 0$ and the module is castling-equivalent to

$$\left(\mathrm{GL}_1^{k+t} \times \mathrm{SL}_q \times \mathrm{SL}_p,\ (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t})\right)$$

with $q = kp - m < p$. Let $\tilde{n} = p$, $\tilde{m} = q$, $\tilde{s}_1 = \tilde{s}_2 = 0$, $\tilde{t}_1 = 0$, $\tilde{t}_2 = t$.
If $kq \le p$ and $t \ge 2$, this is the case KII IV-16-ii (a) or 16-iv (b) with parameters $k \ge 2$, $\tilde{n}$, $\tilde{m}$, $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{t}_1$ and $\tilde{t}_2$. We get a special module from IV-16-ii (a) (see proposition 13.21) by replacing the group by $\mathrm{GL}_q \times \mathrm{GL}_p$ and with $\tilde{s}_1 + k\tilde{t}_2 = \tilde{m}$, i.e. $kt = k^2m + kt - kn - m = q$. We do not get special modules from IV-16-iv (b) (see proposition 13.23).

If $kq \leq p$ and $t = 1$, it is the second case in remark 11.6. By proposition 13.10 (and by looking up the simple special modules in proposition 13.3), there are two candidates for special modules. The first one must satisfy $k = \tilde{m} + 1$ (i.e. $k = q + 1 = k^2m + k - kn - m + 1$) and $\tilde{n} = k\tilde{m}$ (i.e. $p = km+1-n = \tilde{m}^2+\tilde{m} = (k^2m+k-kn-m+1)^2+k^2m+k-kn-m+1$). The second one is obtained by replacing $\mathrm{GL}_1^{k+1} \times \mathrm{SL}_q \times \mathrm{SL}_p$ by $\mathrm{GL}_q \times \mathrm{GL}_p$ with parameters satisfying $k = \tilde{m} = q = k^2m + k - kn - m$ and $\tilde{n} = p = km + 1 - n = k\tilde{m} = (k^2m + k - kn - m)^2$.

If $kq > p$, this is the case 1 (a). The module is special if and only if we replace the group by $\mathrm{GL}_q \times \mathrm{GL}_p$ and if $ta_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$, with $j = \nu(k, q, p)$, $\tilde{m}_0 = a_{j+1}q - a_jp$, $\tilde{n}_0 = a_jq - a_{j-1}p$ holds.

- If $kp \leq m$ and $s = 0$, it is the third case in remark 11.6. By proposition 13.8, it is special if $kp = m$ and $(\mathrm{GL}_1^l \times \mathrm{SL}_p, \omega_1^{\oplus t})$ is a special module (from proposition 13.3) for some $l \leq t$. We obtain two candidates. The first candidate must satisfy $l = t$ and $t = p+1 = km+t-n+1$, i.e. $km+1 = n$. For the second candidate, replace the group by $\mathrm{GL}_p \times \mathrm{GL}_m$ (i.e. $l = 1$) with t satisfying $t = p = km + t - n$, i.e. $km - n = 0$. In either case, we have a contradiction to $km > n$, so we do not get a special module.
- If $kp \leq m$ and $s = 1$, it is the second case in remark 11.6. By proposition 13.10, it is special if and only if $kp = m$ and $(\mathrm{GL}_1^l \times \mathrm{SL}_p, \omega_1^{\oplus t+k})$ is a special module (from proposition 13.3) for some $l \leq t+k$. We obtain two special modules. The first module satisfies $l = t+k$ and $t+k = p+1 = km+t-n+1$. For the second module, replace the group by $\mathrm{GL}_p \times \mathrm{GL}_m$ (i.e. $l = 1$) with $t + k = p = km + t - n$.
- If $kp \leq m$ and $2 \leq s$, it is the case KII IV-16-ii (a) or KII IV-16-iv (b) with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t$, $\tilde{s}_2 = \tilde{t}_1 = 0$, $\tilde{t}_2 = s$. For KII IV-16-ii (a), i.e. $kp = m$, we get a special module by replacing the group by $\mathrm{GL}_p \times \mathrm{GL}_m$ with $\tilde{s}_1 + k\tilde{t}_2 = \tilde{m}$, i.e. $t + ks = p$, see proposition 13.21. There are no special modules for IV-16-iv (b) by proposition 13.23.

(c) The module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1^{*\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})\right)$$

is castling-equivalent to

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_p \times \mathrm{SL}_m,\ (\omega_1^{\oplus t} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus s})\right)$$

with $p = km + t - n$. By the proof of theorem 4.15 in [17], we have $p < m$.

- If $kp > m$, this is the case 1 (b) with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{t} = s \geq 1$ and $\tilde{s} = t$. We get the special modules from the previous case 1 (b) by plugging in these parameters.
- If $kp + s \leq m$, this is the third case in remark 11.6. By proposition 13.8 it is special if and only if $kp + s = m$, if we replace GL_1^{k+s} by GL_1, and if $(\mathrm{GL}_1^l \times \mathrm{SL}_p, \omega_1^{\oplus t})$ is a special module for some $l \leq t$. Thus, we get two

candidates for special modules. The first candidate has parameters $t = p$ and $l = 1$, i.e. $km - n = 0$. For the second candidate, replace the group by $\mathrm{GL}_p \times \mathrm{GL}_m$ with parameters $t = p + 1$ and $l = p + 1$, i.e. $km - n + 1 = 0$. In either case, we have a contradiction to $km > n$, so we do not obtain any special modules.

- If $kp + s - 1 = m$ and $t = 0$ or $t = 1$, then this is the fourth case in remark 11.6. By proposition 13.9 it is special if and only if $(\mathrm{GL}_1^l \times \mathrm{SL}_p, \omega_1^{\oplus k} \oplus \omega_1^{*\oplus t})$ is a simple special module for some $l \leq k + t$. Checking with proposition 13.3, we get three special modules. The first module has parameters $t = 0$, $l = 1$ and $k = p$, and we replace GL_1^k by GL_1. The second module has parameters $t = 0$, $l = p + 1$ and $k = p + 1$. The third module has parameters $t = 1$, $l = p + 1$ and $k = p$.
- If $kp + 1 = m$, $s \geq 3$ and $t = 0$, this is the case KII IV-16-iii. According to proposition 13.22, there are no special modules for $k > 1$.
- If $kp = m$, $2 \leq s$ and $t = 0$, this is the case KII IV-16-ii (b) with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t = 0$, $\tilde{s}_2 = 0$, $\tilde{t}_1 = s$ and $\tilde{t}_2 = 0$. By proposition 13.21, it is special if and only if we replace the group by $\mathrm{GL}_p \times \mathrm{GL}_m$, and the parameters satisfy $\tilde{s}_2 + k\tilde{t}_1 = \tilde{m}$, i.e. $ks = p$. ■

Proposition 13.35 *Let $p = km + t - n - 1$. From case 2 (a) in lemma 13.32 we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$,
  with $kp > m$, $(t-2)a_{j+1} + (k+s)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1} m$ and $j = \nu(k, p, m)$.
- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$,
  with $kp = m$ and $t - 2 + k(k + s) = p$.

Let $p = km + t - n$ and $q = kp + t - m - 1$. From case 2 (b) in lemma 13.32 we get the following special modules:

- $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$,
  with $kp > m$, $kq > p$, $(s-2)a_{j+1} + (k+t)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1} p$ and $j = \nu(k, q, p)$.
- $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$,
  with $kp > m$, $kq = p$ and $s - 2 + k(k + t) = q$.
- $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$,
  with $m = s - 1 + kp$ and $k + t = p + 1$.
- $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$,
  with $m = s - 1 + kp$ and $k + t = p$.

Proof: We consider the subcases of case 2 from lemma 13.32 one by one.

(a) By lemma 13.6, the module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))\right)$$

is special if and only if

$$\left(\mathrm{GL}_1^{k+s} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1},\ (\mu \otimes \omega_1^{\oplus k+s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t-2})\right)$$

is special, which is castling equivalent (in the sense of lemma 9.25) to

$$\left(\mathrm{GL}_1^{k+s} \times \mathrm{GL}_p \times \mathrm{SL}_m,\ (\omega_1^{\oplus t-2} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (\mu \otimes 1 \otimes \omega_1^{*\oplus k+s})\right)$$

with $p = km + (t-2) - (n-1) = km + t - n - 1$.

- If $kp > m$, this is the case 1 (a) of lemma 13.32, with the group replaced by $\mathrm{GL}_1^{k+s} \times \mathrm{GL}_p \times \mathrm{SL}_m$ and parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t-2$, $\tilde{s}_2 = \tilde{t}_1 = 0$ and $\tilde{t}_2 = k+s$. By the proof of theorem 4.16 in [17], we have $m > p$. Argueing as in remark 13.33, we still have to replace $\mathrm{GL}_1^{k+s} \times \mathrm{GL}_p \times \mathrm{SL}_m$ by $\mathrm{GL}_p \times \mathrm{GL}_m$ in order to obtain special modules. Then we obtain them by plugging in the parameters in proposition 13.34.
- If $kp \leq m$, as $k+s \geq 2$, this is the case KII IV-16-ii (a) or KII IV-16-iv (b) with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t-2$, $\tilde{s}_2 = \tilde{t}_1 = 0$ and $\tilde{t}_2 = k+s$. For IV-16-ii (a), i.e. $kp = m$, we get a special module by replacing the group by $\mathrm{GL}_p \times \mathrm{GL}_m$, see proposition 13.21. There are no special modules for IV-16-iv (b) by proposition 13.23.

(b) The module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})\right)$$

with $2 \leq s$ is castling equivalent to

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_p \times \mathrm{SL}_m,\ (\omega_1^{\oplus t} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus s-1}))\right)$$

with $p = km + t - n$ and $p < m$ (cf. theorem 4.5 in [17]).

- If $kp > m$, this is the case 2 (a) with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t$, $\tilde{s}_2 = 0$, $\tilde{t}_1 = s-1$ and $\tilde{t}_2 = 1$. We get the special modules from the previous case 2 (a) by plugging in these parameters.
- If $kp \leq m$, this is the second case from remark 11.6. By proposition 13.10, we obtain two special modules. The first module satisfies $m = s - 1 + kp$ and $k + t = p + 1$. For the second module, replace $\mathrm{GL}_1^{k+t+s} \times \mathrm{SL}_p \times \mathrm{SL}_m$ by $\mathrm{GL}_1^{1+s} \times \mathrm{SL}_p \times \mathrm{SL}_m$ with parameters $m = s - 1 + kp$ and $k + t = p$. ∎

Proposition 13.36 *From case 3 (a) in lemma 13.32 we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, *with* $(k+s)a_{j+1} + (t-2)a_{j+2} = \tilde{m}_0$ *and* $\tilde{n}_0 = k\tilde{m}_0$*, for* $\tilde{m}_0 = a_{j+1}m - a_j(n-1)$, $\tilde{n}_0 = a_j m - a_{j-1}(n-1)$ *and* $j = \nu(k, m, n-1)$.

Let $p = km + t - n$*. From case 3 (b) in lemma 13.32 we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, *with* $kp > m$, $(k+t)a_{j+1} + (s-2)a_{j+2} = \tilde{m}_0$ *and* $\tilde{n}_0 = k\tilde{m}_0$*, for* $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ *and* $j = \nu(k, p, m-1)$.

Proof: We consider the subcases of case 3 from lemma 13.32 one by one.

(a) By lemma 13.6, the module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ (\omega_1^{\oplus s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))\right)$$

is special if and only if

$$\left(\mathrm{GL}_1^{k+s} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1},\ (\mu \otimes \omega_1^{\oplus k+s} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus t-2})\right)$$

is special. Prehomogeneity implies $m < n-1$, see theorem 4.5 in [17]. As $km > n-1$, this is the case 1 (a) in lemma 13.32 with the group replaced by $\mathrm{GL}_1^{k+s} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1}$ and parameters $\tilde{n} = n-1$, $\tilde{m} = m$, $\tilde{s}_1 = k+s$, $\tilde{s}_2 = \tilde{t}_1 = 0$ and $\tilde{t}_2 = t-2$. Argueing as in remark 13.33, we still have to replace $\mathrm{GL}_1^{k+s} \times \mathrm{SL}_m \times \mathrm{GL}_{n-1}$ by $\mathrm{GL}_m \times \mathrm{GL}_{n-1}$ in order to obtain special modules. Then we obtain them by plugging in the parameters in proposition 13.34.

(b) The module

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})\right)$$

is castling-equivalent to

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_p \times \mathrm{SL}_m,\ (\omega_1^{\oplus t} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus s-1}))\right)$$

with $p = km + t - n$. By theorem 4.5 in [17] we have $p < n$.

- If $kp > m$, this is the case 3 (a) in lemma 13.32, with parameters $\tilde{n} = m$, $\tilde{m} = p$, $\tilde{s}_1 = t$, $\tilde{s}_2 = 0$, $\tilde{t}_1 = 1$ and $\tilde{t}_2 = s-1$. We get the special modules by plugging in these parameters in the previous case 3 (a).
- If $kp \leq m$, this is the case KII IV-16-iv (c). By proposition 13.23, there are no special modules for $k > 1$. ■

Proposition 13.37 *Let* $p = km + t - n - 1$*. From case 4 in lemma 13.32, we get the following special modules:*

- $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, *with* $kp > m-1$, $(k+t-2)a_{j+1} + (k+s-2)a_{j+2} = \tilde{m}_0$ *and* $\tilde{n}_0 = k\tilde{m}_0$ *for* $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ *and* $j = \nu(k, p, m-1)$.

- $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, *with* $m-1 = kp$ *and* $(k+t-2) + k(k+s-2) = p$.

Proof: By applying lemma 13.6 twice, we have that

$$\left(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n,\ ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*))\right)$$

is special if and only if

$$\left(\mathrm{GL}_{m-1} \times \mathrm{GL}_{n-1},\ (\omega_1^{\oplus k+s-2} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus k+t-2})\right)$$

is special. The latter module is castling equivalent to

$$\left(\mathrm{GL}_p \times \mathrm{GL}_{m-1},\ (\omega_1^{\oplus k+t-2} \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{*\oplus k+s-2})\right)$$

with $p = k(m-1) + (k+t-2) - (n-1) = km + t - n - 1$ and $p < m-1$ (cf. theorem 4.5 in [17]).

- If $kp > m-1$, this is the case 1 (a) from lemma 13.32 with parameters $\tilde{n} = m$, $\tilde{m} = n$, $\tilde{s}_1 = k+t-2$, $\tilde{s}_2 = \tilde{t}_1 = 0$ and $\tilde{t}_2 = k+s-2$. We get the special modules by plugging in these parameters in proposition 13.34.

- If $k = 2$, $kp \leq m-1$ and $s = 1$, this is the second case in remark 11.6. By proposition 13.10, it is special if and only if $m-1 = 2p$ and $(\mathrm{GL}_p, \omega_1^{\oplus t+2})$ is special, i.e. $t+2 = p$.

- If $kp \leq m-1$ and $s > 1$ or $k > 2$, this is the case KII IV-16-ii (a) or KII IV-16-iv (b). If $kp = m-1$, it is IV-16-ii (a) and the module is special if and only if $(k+t-2) + k(k+s-2) = p$ by proposition 13.21. For IV-16-iv (b), there are no special modules by proposition 13.23. Note that these conditions even include the previous case $k = 2$, $s = 1$. ■

14 Some Results and Observations

In this chapter, we present some results on special modules obtained by applying the results from the previous chapters of this part.

14.1 Special Modules for $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$

Consider the group $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$. If there were any irreducible special modules for this group, each of them would either appear in theorem 11.1, SK I (not in SK II or SK III by regularity), or be castling-equivalent to one of the modules there. By writing $\mathrm{GL}_2 = \mathrm{GL}_2 \times \mathrm{SL}_1$, we obtain the following theorem.

Theorem 14.1 *The only irreducible special modules for* $\mathrm{GL}_1 \times \mathrm{SL}_m \times \mathrm{SL}_n$ *are (up to equivalence and castling equivalence)*

- *SK I-4:* $(\mathrm{GL}_2 \times \mathrm{SL}_1, 3\omega_1 \otimes \omega_1, \mathrm{Sym}^3 \Bbbk^3)$.
- *SK I-8:* $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2 \Bbbk^3 \otimes \Bbbk^2)$.
- *SK I-11:* $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.

Note that $m \neq n$ in each of these cases. This holds even for non-irreducible modules, because every irreducible component would have to be non-regular by theorem 12.14, but by SK II and SK III in theorem 11.1, there are no non-regular modules for the case $m = n$.

Theorem 14.2 *There are no special modules for* $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$, $n \geq 2$.

Remark 14.3 If we admit a centre GL_1^2, then we trivially obtain special modules with semisimple part $\mathrm{SL}_n \times \mathrm{SL}_n$ as direct compositions (see definition 10.22) of special modules for $\mathrm{GL}_1 \times \mathrm{SL}_n$.

14.2 Special Modules with Semisimple Factors other than SL_n

One might be lead to believe that for every special module, the only simple factors of the group are the groups SL_n. But in fact, we found three special modules containing a simple factor Sp_2 (which is locally isomorphic to SO_5). These are

- KI I-16: $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- KI I-18: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-19: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.

Still, it remains an open question whether there are special modules with simple factors other than SL_n or Sp_2.

For a reductive group with one-dimensional torus it is not even known if any simple factor other than SL_n is possible. For the irreducible case it is known that this is not the case. For the non-irreducible case, theorem 12.14 tells us that the only reductive groups possible must be castling-equivalent to those labelled by SK III in theorem 11.1. We see that any such group may contain at most one simple factor $G \neq \mathrm{SL}_n$.

Part V

Appendix

A Tables of Groups and Lie Algebras

A.1 The Classical Groups and their Lie Algebras

G	$\mathfrak{g} = \mathfrak{Lie}(G)$	$\dim_{\mathbb{k}}(\mathfrak{g})$
general linear group $\mathrm{GL}_n = \{A \in \mathrm{Mat}_n \mid \det(A) \neq 0\}$	$\mathfrak{gl}_n = \mathrm{Mat}_n$	n^2
special linear group $\mathrm{SL}_n = \{A \in \mathrm{Mat}_n \mid \det(A) = 1\}$	$\mathfrak{sl}_n = \{X \in \mathrm{Mat}_n \mid \mathrm{tr}(X) = 0\}$	$n^2 - 1$
orthogonal group $\mathrm{O}_n = \{A \in \mathrm{GL}_n \mid AA^\top = I_n\}$	$\mathfrak{o}_n = \{X \in \mathrm{Mat}_n \mid X^\top = -X\}$	$\frac{1}{2}n(n-1)$
special orthogonal group $\mathrm{SO}_n = \{A \in \mathrm{O}_n \mid \det(A) = 1\}$	$\mathfrak{so}_n = \mathfrak{o}_n$	$\frac{1}{2}n(n-1)$
unitary group $\mathrm{U}_n = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid AA^* = I_n\}$	$\mathfrak{u}_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \overline{X} = -X^\top\}$	n^2
special unitary group $\mathrm{SU}_n = \{A \in \mathrm{U}_n \mid \det(A) = 1\}$	$\mathfrak{su}_n = \{X \in \mathfrak{u}_n \mid \mathrm{tr}(X) = 0\}$	$n^2 - 1$
symplectic group $\mathrm{Sp}_n = \{A \in \mathrm{GL}_{2n} \mid A^\top J A = J\}$	$\mathfrak{sp}_n = \{X \in \mathrm{Mat}_{2n} \mid X^\top J + JX = 0\}$	$n(2n+1)$

Note that $\mathfrak{u}_n$ and $\mathfrak{su}_n$ are Lie algebras over $\mathbb{k} = \mathbb{R}$, but not over $\mathbb{C}$.

A.2 Complex Simple Lie-Algebras

Type	$\mathfrak{g}$		$\dim(\mathfrak{g})$
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n \geq 1$	$n^2 + 2n$
B_n	$\mathfrak{o}_{2n+1}(\mathbb{C})$	$n \geq 2$	$2n^2 + n$
C_n	$\mathfrak{sp}_n(\mathbb{C})$	$n \geq 3$	$2n^2 + n$
D_n	$\mathfrak{o}_{2n}(\mathbb{C})$	$n \geq 4$	$2n^2 - n$
G_2		-	14
F_4		-	52
E_6		-	72
E_7		-	133
E_8		-	248

Further we have $A_1 = B_1 = C_1$, $B_2 = C_2$ and $A_3 = D_3$.

A.3 Some Isomorphisms of Classical Lie Algebras

For an algebraically closed field $\mathbb{k}$ of characteristic 0, we have the following isomorphisms of Lie algebras:

$$\mathfrak{so}_2(\mathbb{k}) \cong \mathbb{k} \cong \mathfrak{gl}_1(\mathbb{k})$$
$$\mathfrak{sp}_1(\mathbb{k}) \cong \mathfrak{sl}_2(\mathbb{k})$$
$$\mathfrak{so}_3(\mathbb{k}) \cong \mathfrak{sl}_2(\mathbb{k})$$
$$\mathfrak{so}_4(\mathbb{k}) \cong \mathfrak{sl}_2(\mathbb{k}) \oplus \mathfrak{sl}_2(\mathbb{k})$$
$$\mathfrak{so}_5(\mathbb{k}) \cong \mathfrak{sp}_2(\mathbb{k})$$
$$\mathfrak{so}_6(\mathbb{k}) \cong \mathfrak{sl}_4(\mathbb{k})$$

Not all of these isomorphisms hold over the real numbers. We have

$$\mathfrak{sp}_1(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R})$$
$$\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{su}_2$$
$$\mathfrak{so}_4(\mathbb{R}) \cong \mathfrak{su}_2 \oplus \mathfrak{su}_2$$
$$\mathfrak{so}_6(\mathbb{R}) \cong \mathfrak{su}_4$$

Recall that two groups with isomorphic Lie algebras are locally isomorphic.

B Tables of Special Modules

In this appendix, we present the special modules found in this thesis. Note that this list is not claimed to be a complete classification of the respective cases.

B.1 Special Modules with Torus GL_1

B.1.1 1-Simple Special Modules with Torus GL_1

- SK I-4: $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{k}^2)$.
- Ks I-2: $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{k}^n)^{\oplus n})$.

B.1.2 2-Simple Special Modules with Torus GL_1

- SK I-8: $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\mathbb{k}^3 \otimes \mathbb{k}^2)$.
- SK I-11: $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{k}^5 \otimes \mathbb{k}^4)$.
- KII I-2: $(\mathrm{GL}_1 \times G \times \mathrm{SL}_n, (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \oplus (\varrho_{k+1}^* \otimes 1) \oplus \ldots \oplus (\varrho_l^* \otimes 1))$, with $n = -1 + \sum_{i=1}^k \dim(\varrho_i)$ and $(\mathrm{GL}_1 \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ a special module for a simple group G.
- KII IV-16-iv (a): $(\mathrm{GL}_m \times \mathrm{SL}_{m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m+1})$, $m \geq 2$.

B.2 Special Modules with Sp_m

- KI I-16: $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- KI I-18: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-19: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.

B.3 All Special Modules from Chapter 13

B.3.1 1-Simple Special Modules

- SK I-4: $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\Bbbk^2)$.
- Ks I-2: $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$.
- Ks I-3: $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\Bbbk^n)^{\oplus n+1})$.
- Ks I-4: $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*})$.
- Ks I-11 for $n = 2$: $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\Bbbk^2 \otimes \Bbbk^2)$.

B.3.2 2-Simple Special Modules

- SK I-8: $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2)$.
- SK I-11: $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.
- KI I-1: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus (\Bbbk^4 \otimes \Bbbk^2))$.
- KI I-2: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-6: $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \Bbbk^5 \otimes \Bbbk^2) \oplus \Bbbk^{5*} \oplus \Bbbk^{5(*)})$.
- KI I-16: $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- KI I-18: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-19: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.
- KII I-1: $(\mathrm{GL}_1^j \times G \times \mathrm{GL}_n, ((\sigma_1 \oplus \ldots \oplus \sigma_s) \otimes \omega_1) \oplus ((\varrho_1 \oplus \ldots \oplus \varrho_l) \otimes 1))$, with $n = \sum_{i=1}^s \dim(\varrho_i)$ and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ a special module for a simple group G, $1 \le j \le l$.
- KII I-2: $(\mathrm{GL}_1^{j+t} \times G \times \mathrm{SL}_n, ((\varrho_1 \oplus \ldots \oplus \varrho_k) \otimes \omega_1) \oplus ((\varrho_{k+1}^* \oplus \ldots \oplus \varrho_l^*) \otimes 1) \oplus (1 \otimes \omega_1^{\oplus t}))$, with $n = t - 1 + \sum_{i=1}^k \dim(\varrho_i)$, $1 \le j \le l$, and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ a special module for a simple group G.

- KII I-3: $(\mathrm{GL}_1^{j+t}\times G\times \mathrm{SL}_n, ((\varrho_1\oplus\ldots\oplus\varrho_k)\otimes\omega_1)\oplus((\varrho_{k+1}\oplus\ldots\oplus\varrho_l)\otimes 1)\oplus(1\otimes(\omega_1^{\oplus t-1}\oplus\omega_1^*)))$, with $n = t-1+\sum_{i=1}^k \dim(\varrho_i)$, $1 \le j \le l$, and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ a special module for a simple group G.
- KII II-4-i (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)}) \oplus (\omega_1^{(*)} \otimes 1))$.
- KII II-4-ii (a): $(\mathrm{GL}_1^5\times\mathrm{SL}_3\times\mathrm{SL}_4, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_2^{(*)})\oplus((\omega_1\oplus\omega_1)^{(*)}\otimes 1)\oplus(1\otimes\omega_1^{(*)}))$.
- KII II-4-iii (d): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_2)\oplus((\omega_1\oplus\omega_1)^{(*)}\otimes 1)\oplus(1\otimes\omega_1))$.
- KII II-4-iii (f): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_2)\oplus(\omega_1\otimes 1)\oplus(\omega_1^*\otimes 1)\oplus(1\otimes\omega_1))$.
- KII II-4-iii (g): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (h): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (n): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_2^*)\oplus((\omega_1\oplus\omega_1)^{(*)}\otimes 1)\oplus(1\otimes\omega_1^{(*)}))$.
- KII II-4-iii (p): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (q): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_2^*)\oplus(\omega_1\otimes 1)\oplus(\omega_1^*\otimes 1)\oplus(1\otimes\omega_1^{(*)}))$.
- KII II-5-i (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1))$.
- KII II-5-ii (b): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
- KII II-5-iii (e): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- KII II-5-iv (a): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2^*))$.
- KII II-6 (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_3 \times \mathrm{SL}_5, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2))$.
- KII II-9 (a): $(\mathrm{GL}_1^4 \times \mathrm{SL}_6 \times \mathrm{SL}_7, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
- KII II-10: $(\mathrm{GL}_1^{r+1} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_k \otimes 1) \oplus (1 \otimes \varrho_{k+1}^*) \oplus \ldots \oplus (1 \otimes \varrho_r^*))$, where $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ is a special module.
- KII III-12: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
- KII III-13-i (d): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1^*)\oplus(1\otimes\omega_1^*)\oplus(\omega_2^{(*)}\otimes 1)\oplus(1\otimes\omega_1^*))$.
- KII III-13-i (e): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1^*)\oplus(1\otimes\omega_1^*)\oplus(\omega_2^*\otimes 1)\oplus(1\otimes\omega_1))$.
- KII III-13-i (g): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1^*)\oplus(1\otimes\omega_1^*)\oplus(\omega_2\otimes 1)\oplus(1\otimes\omega_1))$.
- KII III-13-ii (a): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(\omega_2\otimes 1))$.
- KII III-13-ii (b): $(\mathrm{GL}_1^5\times\mathrm{SL}_2\times\mathrm{SL}_3, (\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(1\otimes\omega_1)\oplus(\omega_2^*\otimes 1))$.
- KII IV-16-i: $(\mathrm{GL}_1^{n+2}\times\mathrm{SL}_n\times\mathrm{SL}_n, (\omega_1\otimes 1)^{\oplus s_1}\oplus(\omega_1^*\otimes 1)^{\oplus s_2}\oplus(\omega_1\otimes\omega_1)\oplus(1\otimes\omega_1)^{\oplus t_1}\oplus (1 \otimes \omega_1^*)^{\oplus t_2})$,
  with $s_1 + t_2 = n$ and $s_2 + t_1 = 1$.

- KII IV-16-i: $(\mathrm{GL}_1^{n+2} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, with $s_1 + t_2 = n + 1$.
- KII IV-16-i: $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, with $s_1 + t_2 = n$.
- KII IV-16-ii (a): $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$, with $s_1 + kt_2 = m$.
- KII IV-16-ii (b): $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1)^{\oplus t_1})$, with $s_2 + kt_1 = m$.
- KII IV-16-iii: $(\mathrm{GL}_1^{t_1-1} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1)^{\oplus t_1 - 1}$, with $s_2 + t_1 = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$, with $n = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m})$, with $n = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+2} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m+1})$, with $n = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$, with $n = m + 1$.
- KII IV-16-iv (c): $(\mathrm{GL}_1^{t_2} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus t_2})$, with $s_1 + t_2 = m$ and $n = m + 1$.
- KII IV-17 (1a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t})$, with $sa_{j+1} + ta_{j+2} = m_0$ and $n_0 = km_0$.

Let $p = km + t - n$ and $q = kp - m = k^2m + kt - kn - m$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $s > 0$, $ta_{j+1} + sa_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_jm$, $\tilde{n}_0 = a_jp - a_{j-1}m$ and $j = \nu(k, p, m)$.
- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq = p$, $2 \leq t$ and $kt = q$.
- KII IV-17 (1b): $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $k = q + 1$ and $p = q^2 + q$.
- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $k = q$ and $p = q^2$.
- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq > p$, $ta_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$ for $\tilde{m}_0 = a_{j+1}q - a_jp$, $\tilde{n}_0 = a_jq - a_{j-1}p$ and $j = \nu(k, q, p)$.

- KII IV-17 (1b): $(\mathrm{GL}_1^{1+k+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$ and $t + k = p + 1$.
- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$ and $t + k = p$.
- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$, $2 \le s$ and $t + ks = p$.

Let $p = km + t - n$, $q = kp + s - m$ and $r = kq - p$.

- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq > p$, $t > 0$, $sa_{j+1} + ta_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1}p$ and $j = \nu(k, q, p)$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $kr = q$, $2 \le s$ and $ks = r$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $k = r + 1$ and $q = r^2 + r$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $k = r$ and $q = r^2$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $kr > q$, $sa_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$ for $\tilde{m}_0 = a_{j+1}r - a_j q$, $\tilde{n}_0 = a_j r - a_{j-1}q$ and $j = \nu(k, r, q)$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $kq = p$ and $s + k = q + 1$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $kq = p$ and $s + k = q$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq = p$, $2 \le t$ and $s + kt = q$.
- KII IV-17 (1c): $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $k = p$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $k = p + 1$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $k = p$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $ks = p$ and $kp = m$.

Let $p = km + t - n - 1$.

- KII IV-17 (2a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$, with $kp > m$, $(t-2)a_{j+1} + (k+s)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1}m$ and $j = \nu(k, p, m)$.
- KII IV-17 (2a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$, with $kp = m$ and $t - 2 + k(k+s) = p$.

Let $p = km + t - n$ and $q = kp + t - m - 1$.

- KII IV-17 (2b): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $kp > m$, $kq > p$, $(s-2)a_{j+1} + (k+t)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1}p$ and $j = \nu(k, q, p)$.
- KII IV-17 (2b): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $kp > m$, $kq = p$ and $s - 2 + k(k+t) = q$.
- KII IV-17 (2b): $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $m = s - 1 + kp$ and $k + t = p + 1$.
- KII IV-17 (2b): $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $m = s - 1 + kp$ and $k + t = p$.
- KII IV-17 (3a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, with $(k+s)a_{j+1} + (t-2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$, for $\tilde{m}_0 = a_{j+1}m - a_j(n-1)$, $\tilde{n}_0 = a_j m - a_{j-1}(n-1)$ and $j = \nu(k, m, n-1)$.

Let $p = km + t - n$.

- KII IV-17 (3b) $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, with $kp > m$, $(k+t)a_{j+1} + (s-2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$, for $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ and $j = \nu(k, p, m-1)$.

Let $p = km + t - n - 1$.

- KII IV-17 (4): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, with $kp > m - 1$, $(k + t - 2)a_{j+1} + (k + s - 2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ and $j = \nu(k, p, m-1)$.
- KII IV-17 (4): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, with $m - 1 = kp$ and $(k + t - 2) + k(k + s - 2) = p$.

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