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Holonomy Groups of Flat Pseudo-Riemannian Homogeneous Manifolds

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Notation and Conventions

Throughout this text, we use the following notation:

- The pseudo-Euclidean space $\mathbb{R}^n$ endowed with an indefinite inner product of signature r, s (where $n = r + s$) is denoted by $\mathbb{R}^{r,s}$. In particular, $\mathbb{R}^{n,0} \cong \mathbb{R}^{0,n}$ is the Euclidean space, and $\mathbb{R}^{n-1,1} \cong \mathbb{R}^{1,n-1}$ is the Minkowski space.

- Unless stated otherwise, the vectors $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{R}^n$.

- Elements of $\mathbb{R}^n$ are represented by column vectors. To save space, these columns will sometimes also be written as n-tuples without further remark.

- The kernel and the image of a linear map A are denoted by $\ker A$ and $\operatorname{im} A$, respectively.

- Groups will be denoted by boldface letters, $\boldsymbol{G}$, and Lie algebras will be denoted by German letters, $\mathfrak{g}$. The Lie algebra of a Lie group $\boldsymbol{G}$ is denoted by $\mathfrak{Lie}(\boldsymbol{G})$.

- The neutral element of an abstract group $\boldsymbol{G}$ is denoted by $1_{\boldsymbol{G}}$ or 1. For matrix groups, we also write I_n or I for the $n \times n$-identity matrix.

- The action of a group element g on elements x of some set is denoted by $g.x$.

- The one-dimensional additive and multiplicative groups are denoted by $\mathbf{G}_+$ and $\mathbf{G}^\times$, respectively.

- By a mild abuse of language, when we speak of the *Zariski closure* of a group $\boldsymbol{G} \subset \mathbf{Aff}(\mathbb{R}^n)$, we shall always mean the *real Zariski closure*, that is, the $\mathbb{R}$-points of its complex Zariski closure.

- The differential of a smooth map f is denoted by f_* or $\mathrm{d}f$.

- Different parts of a proposition or a theorem are labeled by (a), (b), etc., and the parts of the proof referring to these are labeled the same way. Different steps in the proof of one statemend are labeled by small Roman numerals (i), (ii), etc.

See also the appendices for some standard notations.

Contents

Introduction

In flatness is the preservation of the world. So seek the Wolf in thyself!
– Metallica

Pseudo-Riemannian manifolds are among the most important objects in geometry, and they are of particular importance for modern mathematical physics. When developing structure theories for certain classes of pseudo-Riemannian manifolds, one studies their symmetries. By this we mean the isometry group of a manifold, and its fundamental group in particular. In this thesis, we study the structure of flat pseudo-Riemannian homogeneous spaces via their affine holonomy groups. For complete manifolds they coincide with the fundamental group. Incomplete manifolds are harder to understand, and here the affine holonomy group can be a more tangible homomorphic image of the fundamental group.

Setting the Stage

Non-degenerate symmetric bilinear forms appear naturally in many branches of mathematics and physics. In geometry, the positive definite forms define Euclidean geometry in $\mathbb{R}^n$. The indefinite forms define more general geometries, of which Lorentz geometry for signature $(n-1,1)$ is the most prominent. In physics, Lorentz geometry appears as the geometry of special relativity.

Going from $\mathbb{R}^n$ to smooth manifolds, one studies pseudo-Riemannian manifolds M endowed with a pseudo-Riemanian metric $\langle \cdot, \cdot \rangle$, a field of non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle_p$ on the tangent spaces $\mathrm{T}_p M$. Unsurprisingly, the Riemannian case, where the metric is positive definite, is the most studied and best understood of these. In theoretical physics, Riemannian geometry is the language of classical mechanics. Modern theories like general relativity, gravitation and cosmology are built on pseudo-Riemannian geometry.

There are significant differences between positive definite and indefinite metrics: The famous theorem by Hopf and Rinow states that for a Riemannian manifold M, geodesic completeness is equivalent to metric completeness (meaning any Cauchy sequence converges). This is due to the fact that the geodesics are closely related to the metric structure on M induced by the Riemannian metric.

For an indefinite metric such a relation does not exist and the Hopf-Rinow

Theorem does not hold. So geodesically incomplete manifolds appear naturally[1)] in pseudo-Riemannian geometry. The Schwarzschild spacetimes describing black holes in general relativity and the de Sitter spacetimes describing a flat expanding universe in cosmology are examples of incomplete pseudo-Riemannian manifolds.[2)]

Flat Pseudo-Riemannian Manifolds

M is said to be of *constant curvature* if the sectional curvature is constant everywhere. If M is also connected and geodesically complete (meaning its geodesics can be extended indefinitely), then M is called a *space form*. Every space form can be written as a quotient $M = \tilde{M}/\mathbf{\Gamma}$, where the universal cover $\tilde{M}$ is one of the *model spaces* of constant curvature, and $\mathbf{\Gamma}$ is a group of isometries acting freely and properly discontinuously on $\tilde{M}$. In the Riemannian case, a famous theorem by Killing and Hopf states that the model spaces are up to scaling $\mathbf{S}^n$ for constant curvature $+1$, $\mathbf{H}^n$ for constant curvature -1, and $\mathbb{R}^n$ for flat manifolds.

The problem of finding all Riemannian space forms was first formulated by Killing in 1891. A complete classification is known only for constant positive curvature.[3)]

The Euclidean (flat) space forms are determined by subgroups $\mathbf{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^n)$ acting properly discontinuously, so the first step in a classification is to study these groups. Their structure theory is based on three famous theorems by Bieberbach from 1910 and 1911.[4)]

The first Bieberbach Theorem roughly states that *in every crystallographic subgroup* $\mathbf{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^n)$ *(that is, a discrete uniform subgroup), the subgroup of pure translations in* $\mathbf{\Gamma}$ *is a normal subgroup of finite index*. The second and third Bieberbach Theorem state that *two crystallographics groups are isomorphic if and only if they are conjugate by an affine transformation*, and that *in each dimension there are only finitely many isomorphism classes of crystallographic groups*.

On the manifold level, Bieberbach's theorems mean that *any compact connected flat Riemannian manifold is a quotient* $\mathbb{R}^n/\mathbf{\Gamma}$*, and that it is finitely covered by a flat torus* $\mathbb{R}^n/\mathbf{\Gamma}_0$*, where* $\mathbf{\Gamma}_0$ *is a lattice of translations*. Further-

[1)] Of course, this presumes one is willing to accept the appearance of black holes as a natural phenomenon.

[2)] See Sachs and Wu [40], section 1.4, and Besse [4], section 3.L.

[3)] See part III in Wolf's book [52].

[4)] See Charlap [9].

more, *in each dimension there are only finitely many affinity classes of flat compact connected Riemannian manifolds*. The 3-dimensional compact Euclidean space forms were classified up to affinity by Hantzsche and Wendt in 1934, and this was extended by Wolf[5)] to the non-compact case and up to isometry. In 1956, Calabi presented a procedure to classify the compact Euclidean space forms of dimension n given the classification in dimension $n-1$.[6)] By this method, Calabi and others arrived at a classification of compact Euclidean space forms of dimension 4 up to affinity.

The classification problem for indefinite metrics is much harder. There is no general structure theory for complete flat pseudo-Riemannian manifolds, let alone incomplete ones. A tentative analogue to Bieberbach's Theorems is the *Auslander conjecture* from 1964, which states that *in every crystallographic subgroup* $\mathbf{\Gamma} \subset \mathbf{Aff}(\mathbb{R}^n)$*, there exists a solvable subgroup of finite index*. This conjecture has been verified only in special cases, most notably for compact flat Lorentz manifolds. Carrière [8] proved that every compact flat Lorentz manifold is complete. Then the Auslander conjecture for this case follows from the work of Goldman and Kamishima [18] in 1984. There is a classification (up to commensurability of the fundamental groups) of compact flat Lorentz manifolds by Grunewald and Margulis [19] from 1989.

Flat Pseudo-Riemannian Homogeneous Manifolds

In this thesis, we are concerned with *homogeneous* manifolds. This is a very special class of manifolds, as they have a transitive group of symmetries. In particular, the orbits of a group action on a manifold are homogeneous spaces.

Homogeneity is a strong property: In the Riemannian case it implies completeness and the only complete flat homogeneous Riemannian manifolds are quotients $M = \mathbb{R}^n/\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a group of pure translations.[7)] Hence M is a product $\mathbb{R}^m \times \mathbf{T}^{n-m}$, where $\mathbf{T}^{n-m}$ is the flat torus.

If we consider indefinite metrics, things are more complicated than in the Riemannian case. Here, incomplete manifolds can appear even among the homogeneous spaces.

The theory for complete spaces was pioneered by Wolf [47, 48, 49] in

[5)]See Wolf [52], section 3.5.
[6)]This is explained in Wolf [52], section 3.6.
[7)]See section 6.1 in this thesis.

the 1960s. The complete flat pseudo-Riemannian manifolds are quotients $\mathbb{R}^{r,s}/\boldsymbol{\Gamma}$, where *the fundamental group* $\boldsymbol{\Gamma}$ *is an isometry group with transitive centraliser*. Wolf discovered some further properties of $\boldsymbol{\Gamma}$, most importantly that $\boldsymbol{\Gamma}$ *is a 2-step nilpotent group*, and he also derived a unipotent matrix representation for abelian $\boldsymbol{\Gamma}$. From this he concluded that *in the Riemannian case, the Lorentz case and for dimensions* ≤ 4, $\boldsymbol{\Gamma}$ *is a group of pure translations*. Wolf [48] also mistakenly claimed that $\boldsymbol{\Gamma}$ is always abelian, and that a compact flat pseudo-Riemannian homogeneous manifold is a quotient by a group of pure translations.

Only in 2008 a first counterexample to both assumptions was given by Baues [2], namely a compact manifold $M = \boldsymbol{N}/\boldsymbol{\Lambda}$, where $\boldsymbol{N}$ is a 2-step nilpotent Lie group of rather special type,[8)] and $\boldsymbol{\Lambda}$ a lattice in $\boldsymbol{N}$.

On complete flat affine manifolds M, the linear parts of the fundamental group map onto the *linear holonomy group* at any point $p \in M$. This is the group of linear maps that arise from parallel transport around the loops based at p. In Baues' example above, the holonomy is abelian even though $\boldsymbol{\Gamma}$ is not. One might be tempted to conjecture that if the fundamental group is not abelian, at least the linear holonomy always has to be abelian. But such hopes are in vain: In chapter 11 we give the first known example of a complete flat pseudo-Riemannian homogeneous manifold with non-abelian linear holonomy. It is a unipotent representation of the integral Heisenberg group acting on $\mathbb{R}^{7,7}$. In fact, one cannot find such an example in lower dimensions, as we show in Theorem 2.25:

Theorem *If* $M = \mathbb{R}^n/\boldsymbol{\Gamma}$ *is a complete flat pseudo-Riemannian homogeneous manifold and has non-abelian linear holonomy, then* $n \geq 14$.

Additionally, we develop a structure theory for complete flat pseudo-Riemannian manifolds. For this we study the action on $\mathbb{R}^n$ of the Zariski closure $\boldsymbol{G}$ of $\boldsymbol{\Gamma}$. From a classical theorem of Rosenlicht [39] it follows that the quotient space $\mathbb{R}^n/\boldsymbol{G}$ is algebraically isomorphic to $\mathbb{R}^{n-k}$. From this we conclude in Theorem 5.5 that *there exists an algebraic section* $\sigma : \mathbb{R}^n/\boldsymbol{G} \to \mathbb{R}^n$, *and so* $\mathbb{R}^n$ *is algebraically isomorphic to a trivial principal* $\boldsymbol{G}$*-bundle*

$$\boldsymbol{G} \to \mathbb{R}^n \to \mathbb{R}^{n-k}.$$

This gives us the following structure theorem:

Theorem *A complete flat pseudo-Riemannian homogeneous manifold* $\mathbb{R}^n/\boldsymbol{\Gamma}$ *is diffeomorphic to a trivial fibre bundle*

$$\boldsymbol{G}/\boldsymbol{\Gamma} \to \mathbb{R}^n \to \mathbb{R}^{n-k}.$$

8) This is the example in chapter 9.

We also find that the affine connection on the orbits of $\boldsymbol{G}$ pulls back to a bi-invariant flat affine connection given by $\nabla_X Y = \frac{1}{2}[X, Y]$ for $X, Y \in \mathfrak{g}$, so the affine structure is an invariant for $\boldsymbol{G}$.

Special Cases

In low dimensions and for certain signatures, the fundamental groups of flat pseudo-Riemannian homogeneous manifolds can be precisely characterised. In the complete case, Wolf [48] already stated that the fundamental group consists only of pure translations if the manifold is Riemannian, Lorentz or of dimensions ≤ 4. In Wolf [52] he proved that $\boldsymbol{\Gamma}$ is free abelian whenever the *Witt index* is ≤ 2 (this is $\min\{r, s\}$ if the signature is (r, s)).

We refine this characterisation by giving necessary and sufficient conditions on the generators of $\boldsymbol{\Gamma}$ in order for $\mathbb{R}^{n-2,2}/\boldsymbol{\Gamma}$ to be a homogeneous manifold. With the help of this characterisation, we are able to give structure theorems for the abelian fundamental groups of complete flat pseudo-Riemannian spaces in dimensions ≤ 6, and we also give a structure theorem for the non-abelian fundamental groups arising in dimension 6.

Some special signatures for incomplete spaces were studied by Duncan and Ihrig. In [11], they classify the incomplete flat Lorentz homogeneous spaces, and in [12] they gave a classification of those incomplete flat homogeneous manifold where the signature is $(n-2, 2)$ and some further conditions on the manifold are assumed. See the following paragraph for a discussion.

Incomplete Flat Pseudo-Riemannian Homogeneous Spaces

As noted before, the incomplete manifolds are an important class of manifolds with indefinite metric. Even in the flat homogeneous case, they are hard to understand.

An important class of incomplete flat pseudo-Riemannian homogeneous spaces are quotients $M = D/\boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is the affine holonomy group, and $D \subset \mathbb{R}^{r,s}$ is an open orbit of $Z_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{\Gamma})$. D is a *homogeneous domain*, meaning D is finitely covered by the universal pseudo-Riemannian cover $\tilde{M}$ of M. In this thesis, we consider only the incomplete spaces of this type. It is not clear whether all incomplete spaces are of this type.

Duncan and Ihrig [11, 12, 13] systematically studied incomplete flat pseudo-Riemannian homogeneous manifolds. The homogeneous domains D they studied are *translationally isotropic*, meaning that the set T of translations leaving D invariant contains $T^\perp$. In [11] they showed that *any flat homogeneous Lorentz space is of the form $D/\mathbf{\Gamma}$, where D is a translationally isotropic half-plane*.

Duncan and Ihrig [12] showed that Wolf's theory of fundamental groups for complete manifolds carries over to incomplete manifolds essentially without change, except for the fact that $\mathbf{\Gamma}$ need not act freely on $\mathbb{R}^{r,s}$ (only on D), as its centraliser is no longer required to act transitively on all of $\mathbb{R}^{r,s}$ (only on D).

They classified the translationally isotropic domains in $\mathbb{R}^{n-2,2}$ and used this in [13] to give a classification of those flat homogeneous spaces of type $D/\mathbf{\Gamma}$ with metric signature $(n-2, 2)$ and translationally isotropic domain D. In Theorem 3.17, we show

Theorem *If $\mathbf{\Gamma}$ has abelian linear holonomy, then the domain D is translationally isotropic.*

So the requirement in Duncan and Ihrig's classification that D be translationally isotropic turns out to be no restriction at all. Thus their article [13] contains the *classification of all incomplete flat pseudo-Riemannian homogeneous spaces $D/\mathbf{\Gamma}$ of signature $(n-2, 2)$, where D is the open orbit of the centraliser of $\mathbf{\Gamma}$*.

Again, the question arises when $\mathbf{\Gamma}$ can have a non-abelian linear holonomy group. We give the answer in Theorem 2.24:

Theorem *If $M = \mathbb{R}^n/\mathbf{\Gamma}$ is a flat pseudo-Riemannian manifold, not necessarily complete, and has non-abelian linear holonomy, then $n \geq 8$.*

More precisely, the Witt index is ≥ 4. Main Example 10 shows that this dimension bound is sharp. It is the first known example of an incomplete flat pseudo-Riemannian homogeneous space with non-abelian fundamental group.

Overview of the Thesis and the Results

Some background material on pseudo-scalar products, affine differential geometry, and algebraic groups is included in the appendices as a reference for the reader. It covers topics that might not be considered standard topics for differential geometry lectures. Also, some standard notations are introduced there (which can also be looked up in the index).

Part I deals with the general theory of flat pseudo-Riemannian homogeneous spaces and contains our general results: In **chapter 1** we summarise the state of the art theory of flat pseudo-Riemannian homogeneous spaces.

We introduce the term *Wolf group* to describe a class of certain isometry groups. This class includes the affine holonomy groups of flat pseudo-Riemannian spaces.

In **chapter 2** we generalise Wolf's structure theory for Wolf groups, by including those with non-abelian linear holonomy group. This theory is helpful for constructing examples, and also leads to two dimension bounds for flat homogeneous pseudo-Riemannian spaces with non-abelian holonomy groups, namely that a Wolf group with non-abelian linear holonomy must act on a space dimension ≥ 8 or even ≥ 14 if the action is free.

As the centralisers of Wolf groups are of particular importance, we collect some of their general properties in **chapter 3**. Provided the Wolf group in question has abelian linear holonomy, we also show that the centraliser's open orbit is translationally isotropic.

Chapter 4 contains the structure theory for compact flat pseudo-Riemannian homogeneous spaces as developed by Baues [2]. It is included here to give a more complete presentation of the whole subject.

In **chapter 5** we study the action of a Zariski closed Wolf group $\mathbf{G}$ on $\mathbb{R}^n$. The orbits of this action are found to be affine subspaces of $\mathbb{R}^n$. Furthermore, we conclude that $\mathbb{R}^n$ is algebraically isomorphic to a trivial principal bundle $\mathbf{G} \to \mathbb{R}^n \to \mathbb{R}^{n-k}$ where $k = \dim \mathbf{G}$. An immediate consequence is that every complete flat pseudo-Riemannian manifold $M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$ is a fibre bundle $\mathbf{G}/\mathbf{\Gamma} \to M \to \mathbb{R}^{n-k}$. Also, we investigate the affine and metric structures on the orbits of $\mathbf{G}$ and find that the induced affine connection on the orbits of $\mathbf{G}$ is an invariant for $\mathbf{G}$. In the special case that the induced metric on the orbits is non-degenerate, we further find that $\mathbf{G}$ has to contain a subgroup of a certain type and consequently $\dim \mathbf{G} \geq 6$ holds.

Special cases are examined in **chapter 6**. We recall the results on Riemannian and Lorentz flat homogeneous spaces by Wolf, Duncan and Ihrig, and then give a characterisation of the fundamental groups of complete flat homogeneous spaces with metric signature $(n-2, 2)$. We then obtain structure theorems for the fundamental groups in dimensions 5 and 6. Combined with a theorem by Wolf, this gives a rough classification in the abelian case.

Part II contains our main examples. Beginning with **chapter 7**, we collect some facts which are useful for the construction of new examples. Most importantly, we discuss a criterion for a free group action to be proper. This guarantees the action of a discrete Wolf group to be properly discontinuous on the open orbit of its centraliser.

Chapters 8, 9, 10, **11** and **12** contain our main examples of flat pseudo-Riemannian homogeneous spaces. We include a compact example due to Baues, which was the first known example of a non-abelian Wolf group (chapter 9). We then construct the first known examples with non-abelian holonomy group: One being incomplete (chapter 10) in dimension 8 and one being complete (chapter 11) in dimension 14. These examples show in particular that the dimension bounds found in chapter 2 are sharp. Finally, we give an example of a non-compact flat pseudo-Riemannian homogeneous manifold such that induced metrics on the orbits of the Zariski closure of G are non-degenerate (chapter 12).

Literature

The main sources for this thesis are the article by Baues [2] and the latest edition of Wolf's book [52] on spaces of constant curvature: Baues [2] approaches the subject from a much more general point of view than we do here. Many results are given for affinely flat homogeneous spaces and then specialised to the (pseudo-)metric or symplectic case. Wolf [52] studies spaces of constant curvature, focusing on the flat and the spherical case. The structure theory in chapter 2 is a direct continuation of Wolf's work.

The work by Duncan and Ihrig in [11, 12, 13] initiated a theory of incomplete flat pseudo-Riemannian homogeneous spaces. Their results can also be found in a generalised form in Baues [2].

Other articles related to the topics of this thesis include those on the theory of affine spaces. Carrière [7] gives a short survey of the subject (see also the references therein). The theory of affine homogeneous domains goes back to the study of Siegel domains (Vey [44]) and convex homogeneous domains (Koszul [26], Vinberg [45]). See also Goldmann and Hirsch [17] and Jo and Kim [24] for more recent results. Yagi [53] studied compact affinely flat homogeneous manifolds. In doing so, he explored the relationship between left-invariant affine connections on homogeneous spaces and associative products on the Lie algebra of affine vector fields.

The standard text book on pseudo-Riemannian geometry is O'Neill's book [30]. It discusses many of the differences between Riemannian manifolds and manifolds with indefinite metrics, in particular the question of completeness. Furthermore, it contains many applications to special and general relativity. Charlap's book [9] is an introduction to Bieberbach groups and compact flat Riemannian manifolds.

Open Questions

In this thesis we have investigated many diverse questions concerning the structure of flat pseudo-Riemannian homogeneous manifolds and their affine holonomy groups. Yet the theory is complicated and a complete structure theory or classification does not seem to be immediately at hand. We discuss some questions that arise immediately from this thesis:

- In chapter 2 we developed a representation theory for Wolf groups. Can this theory be refined to a point where one can say precisely which 2-step nilpotent groups admit representations as Wolf groups given the dimension n of $\mathbb{R}^n$? In particular, which constraints does the dimension n put on the dimension of $\boldsymbol{G}$?

- The centraliser of a Wolf group $\boldsymbol{G}$ is of fundamental importance, as knowledge of the centraliser provides informations on the geometry of $\mathbb{R}^{r,s}/\boldsymbol{G}$ and perhaps on how $\mathbb{R}^{r,s}/\boldsymbol{G}$ embeds into $\mathbb{R}^{r,s}$. Yet, as the examples confirm, the centraliser can have a very complicated structure. A sharper look at Main Examples 10 and 11 might lead to some results on the centralisers in general.

- Turning the question for the centraliser on its head, one can ask which Wolf groups $\boldsymbol{W}(\boldsymbol{L})$ associated to a given isometry group $\boldsymbol{L}$ with open orbit can appear. The discrete subgroups of $\boldsymbol{W}(\boldsymbol{L})$ would then yield all examples of flat pseudo-Riemannian homogeneous manifolds for $\boldsymbol{L}$. One approach to the classification problem can be the determination of all possible pairs $(\boldsymbol{L}, \boldsymbol{W}(\boldsymbol{L}))$.

- The question whether the open orbit of the centraliser is always translationally isotropic is important for incomplete manifolds. A deeper understanding of the centraliser might provide the existence of certain subgroup that guarantees this property (see the remarks following Theorem 3.17), or it might lead to the construction of counterexamples.

- We have seen in chapter 4 that the compact cases are essentially determined by a Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus_\omega \mathfrak{a}^*$ endowed with an invariant bilinar form. In the lowest possible dimension 6 this algebra is a butterfly algebra. Can the strong conditions that the invariant form imposes on $\mathfrak{g}$ be used to determine the possible algebras in higher dimensions? More generally, this would determine those Wolf groups whose orbits can have non-degenerate induced metric.

- Further structure and classification theorems on special cases might give clues for developing a general classification. For complete manifolds of dimension 7 or signature $(n-3,3)$ a structure theory might be built on the structure theory for dimension 6 from chapter 6.
- Which affine homogeneous domains D can arise for incomplete manifolds, and what are their topological properties? For example, the domain D in Main Example 10 is not simply connected and homotopy equivalent to a circle.

Part I

Flat Pseudo-Riemannian Homogeneous Spaces

1 Isometries of Flat Pseudo-Riemannian Homogeneous Spaces

This chapter presents the fundamentals of the theory of flat pseudo-Riemannian homogeneous spaces. Many of these results were developed by Wolf [48, 52]. Our exposition follows chapter 3 in Wolf [52] and a recent survey article by Baues [2].

Some additional background and notation can be found in appendices A, B and C. In particular, $\mathbb{R}^{r,s}$ denotes the Euclidean space $\mathbb{R}^{r+s}$, endowed with a pseudo-scalar product $\langle \cdot, \cdot \rangle$ of signature (r, s). Its isometry group is denoted by $\mathbf{Iso}(\mathbb{R}^{r,s})$.

1.1 Flat Pseudo-Riemannian Manifolds

The Hopf-Killing Theorem states that every complete flat affine manifold is a quotient $M = \mathbb{R}^n/\mathbf{\Gamma}$, where $\mathbf{\Gamma} \subset \mathbf{Aff}(\mathbb{R}^n)$ is the fundamental group of M. In particular, this holds for homogeneous Riemannian manifolds M, as they are always complete. But for a pseudo-Riemannian (or more generally affine) manifold M, homogeneity does not automatically imply completeness.

If M is a complete flat pseudo-Riemannian manifold, then the Hopf-Killing Theorem still holds, so that

$$M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$$

with fundamental group $\mathbf{\Gamma} \in \mathbf{Iso}(\mathbb{R}^{r,s})$. If M is not complete, the model spaces are harder to describe (but see Wolf [52], Theorem 2.4.9).

If M is flat, it is possible to model M on $\mathbb{R}^{r,s}$ at least locally in the following sense:

Definition 1.1 Let X be a homogeneous space for a Lie group $\boldsymbol{G}$. A manifold M is **locally modelled** on $(X, \boldsymbol{G})$ if M can be covered by coordinate charts $\chi_i : U_i \to X$ such that the coordinate changes $g_{ij} = \chi_i \circ \chi_j^{-1}$ are elements of $\boldsymbol{G}$. The manifold M together with a maximal atlas of such charts is called a **(X, G)-manifold**.

If M is locally modelled on $(X, \boldsymbol{G})$, we obtain the following correspondence between the universal cover $\tilde{M}$ and X:

Definition 1.2 Let M be a $(X, \boldsymbol{G})$-manifold. Fix a base point $p_0 \in M$. The **development map**

$$\text{dev} : \tilde{M} \to X$$

is the local diffeomorphism that agrees with the analytic continuation of a chart around p_0 along each path in a neighbourhood of the path's endpoint.

The development map encodes the notion of "unrolling" a piece of $\tilde{M}$ on X. For a hands-on description of the development map, see section 3.4 in Thurston [43].

Proposition 1.3 *Let M be a $(X, \boldsymbol{G})$-manifold, and let $\varphi \in \mathbf{Diff}(\tilde{M})$ that looks like the action of an element of $\boldsymbol{G}$ in local charts. Then there exists an element $\delta(\varphi) \in \boldsymbol{G}$ such that*

$$\text{dev} \circ \varphi = \delta(\varphi) \circ \text{dev}. \tag{1.1}$$

This holds for the elements of the fundamental group $\boldsymbol{\Gamma} = \pi_1(M, p_0)$, so that δ induces a homomorphism

$$\delta : \boldsymbol{\Gamma} \to \boldsymbol{G}. \tag{1.2}$$

See Thurston [43], section 3.4 for a proof.

Theorem 1.4 *A flat pseudo-Riemannian manifold M with metric of signature (r, s) is a $(\mathbb{R}^{r,s}, \mathbf{Iso}(\mathbb{R}^{r,s}))$-manifold.*

For a proof, see Baues [2], Theorem 2.3.

Definition 1.5 In the flat case, the homomorphism $\delta : \boldsymbol{\Gamma} \to \mathbf{Iso}(\mathbb{R}^{r,s})$ from (1.2) is called the **affine holonomy homomorphism**. Its images are affine transformations (A, v) of $\mathbb{R}^{r+s}$, and if they are composed with the projection L on the linear part (that is, $\text{L}(A, v) = A$), we obtain the **linear holonomy homomorphism**

$$\text{hol} = \text{L} \circ \delta : \boldsymbol{\Gamma} \to \mathbf{O}_{r,s}. \tag{1.3}$$

Accordingly, $\delta(\boldsymbol{\Gamma})$ and $\text{hol}(\boldsymbol{\Gamma})$ are called the **affine holonomy group** and **linear holonomy group**, respectively.

This naming is well justified: For fixed base point $p_0 \in \mathbb{R}^{r,s}$, the homomorphism hol maps $\mathbf{\Gamma}$ onto the holonomy group $\mathbf{Hol}(M, p_0)$ of M at the point p_0 (see Wolf [52], Theorem 3.4.2). $\mathbf{Hol}(M, p_0)$ is defined as the group of linear maps $\tau_\gamma : \mathrm{T}_{p_0}M \to \mathrm{T}_{p_0}M$ such that τ_γ is the parallel transport along a closed loop γ based at p_0. The image of γ under this homomorphism is in fact the linear part of $\delta(\gamma)$ (see Wolf [52], Lemma 3.4.4).

1.2 Killing Fields and the Development Representation

Let M be a flat pseudo-Riemannian manifold of signature (r, s), $\tilde{M}$ its universal pseudo-Riemannian cover and $\tilde{\mathbf{\Gamma}} \subset \mathbf{Iso}(\tilde{M})$ the group of deck transformations. Set

$$\mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}}) = \{g \in \mathbf{Iso}(\tilde{M}) \mid g\tilde{\mathbf{\Gamma}}g^{-1} = \tilde{\mathbf{\Gamma}}\},$$

the normaliser of $\tilde{\mathbf{\Gamma}}$ in $\mathbf{Iso}(\tilde{M})$. Then $\mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}})$ is a covering group of $\mathbf{Iso}(M)$ (see Proposition C.1).

Lemma 1.6 *The map (1.1) induces a homomorphism of Lie groups*

$$\delta : \mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}}) \to \mathbf{Iso}(\mathbb{R}^{r,s}). \tag{1.4}$$

Then $\delta(\mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}}))$ *normalises* $\mathbf{\Gamma} = \delta(\tilde{\mathbf{\Gamma}})$*, and* $\delta(\mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}})^\circ)$ *centralises* $\mathbf{\Gamma}$*.*

This δ is called the **development representation** of $\mathbf{Iso}(\tilde{M}, \tilde{\mathbf{\Gamma}})$.

Proposition 1.7 *The differential of the development representation (1.4) induces an anti-isomorphism*

$$\delta' : \mathfrak{kill}(M) \to \mathfrak{iso}(\mathbb{R}^{r,s}).$$

δ' is also called **development representation**.

We may choose a point $p \in M$ as origin for (local) affine coordinates (see appendix B.1). Then T_pM is identified with $\mathbb{R}^{r+s}$, and $X \in \mathfrak{kill}(M)$ is represented by a matrix in $\mathfrak{aff}(\mathbb{R}^{r+s})$:

$$\delta'(X) = \begin{pmatrix} \nabla X & X_p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\mathsf{A}_X & X_p \\ 0 & 0 \end{pmatrix}, \tag{1.5}$$

compare Example B.4. Here, A_X is the tensor field $\mathscr{L}_X - \nabla_X$, which coincides with $-\nabla X$ for torsion free connections (see Proposition B.5).

For the quotient $M = \tilde{M}/\mathbf{\Gamma}$ to be homogeneous it is necessary and sufficient that the centraliser of $\mathbf{\Gamma}$ in $\mathbf{Iso}(\tilde{M})$ acts transitively on $\tilde{M}$ (Corollary C.2). Infinitesimally, this means the local flows of the Killing fields must commute with some subgroup $\boldsymbol{L} \subset \mathbf{Iso}(\tilde{M})$ which acts transitively on $\tilde{M}$.

We say that a Killing field X **commutes** with a group $\boldsymbol{L} \subset \mathbf{Iso}(M)$ (or **centralises** $\boldsymbol{L}$) if the local flow of X commutes with $\boldsymbol{L}$ at every point in M. The proofs of the following results are to be found in paragraph 3.4.1 of Baues [2].

Lemma 1.8 *Let $X, Y \in \mathfrak{kill}(M)$, where $[X, Y] = 0$. Then*

$$\langle \nabla_Y X, X \rangle = 0, \quad \langle \nabla_X X, Y \rangle = 0.$$

Proposition 1.9 *Let $X \in \mathfrak{kill}(M)$, such that X commutes with a group $\boldsymbol{L} \subset \mathbf{Iso}(M)$ which has an open orbit on M. Then:*

(a) $\nabla_X X = 0$.

(b) *If M is also flat,* $\mathsf{A}_X \mathsf{A}_X = 0$.

Proposition 1.10 *Let $X, Y, Z \in \mathfrak{kill}(M)$, such that X, Y, Z commute with a group $\boldsymbol{L} \subset \mathbf{Iso}(M)$ which has an open orbit on M. Then:*

(a) $[X, Y] = -2\mathsf{A}_Y X = 2\mathsf{A}_X Y$.

If M is also flat, then:

(b) $\mathsf{A}_X \mathsf{A}_Y Z = \mathsf{A}_Y \mathsf{A}_X Z$.

(c) $[[X, Y], Z] = 0$.

(d) $\mathsf{A}_{[X,Y]} = [\mathsf{A}_X, \mathsf{A}_Y] = 2\mathsf{A}_X \mathsf{A}_Y$.

Corollary 1.11 *Let M be a flat pseudo-Riemannian manifold M, and $\boldsymbol{L}$ a subgroup of $\mathbf{Iso}(M)$ which has an open orbit on M. Then the Lie algebra $\mathfrak{kill}(M)^{\boldsymbol{L}}$ of $\boldsymbol{L}$-invariant vector fields is 2-step nilpotent.*

1.3 Wolf Groups

The theory of complete flat pseudo-Riemannian homogeneous spaces and their fundamental groups was pioneered by Wolf [47, 48]. He realised that the essential property of a fundamental group $\mathbf{\Gamma}$ of a flat pseudo-Riemannian homogeneous space is that it is an isometry group whose centraliser $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\mathbf{\Gamma})$ acts transitively on $\mathbb{R}^{r,s}$ (Corollary C.2). To include also incomplete manifolds, one requires $\boldsymbol{L}$ to act with an open orbit in $\mathbb{R}^{r,s}$ which is stabilised by $\mathbf{\Gamma}$.

Thus, the following definition seems appropriate:

Definition 1.12 A **Wolf group** $\boldsymbol{G}$ is a subgroup of $\mathbf{Iso}(\mathbb{R}^{r,s})$ such that the centraliser $\mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$ has an open orbit in $\mathbb{R}^{r,s}$ which is invariant under $\boldsymbol{G}$.

Definition 1.13 Let $\boldsymbol{L}$ be an algebraic subgroup of $\mathbf{Iso}(\mathbb{R}^{r,s})$ acting with open orbit on $\mathbb{R}^{r,s}$. The group $\boldsymbol{W}(\boldsymbol{L}) = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{L})$ is called the **Wolf group associated to** $\boldsymbol{L}$.

Remark 1.14 Clearly $\boldsymbol{G} \subset \boldsymbol{W}(\boldsymbol{L})$ for any Wolf group with centraliser $\boldsymbol{L}$. The Lie algebra of $\boldsymbol{W}(\boldsymbol{L})$ consists of the matrices in $\mathfrak{iso}(\mathbb{R}^{r,s})$ commuting with $\mathfrak{Lie}(\boldsymbol{L})$.

In the following, let $\boldsymbol{G}$ be a Wolf group and $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$.

Remark 1.15 A Wolf group $\boldsymbol{G}$ acts freely on the open orbit D of its centraliser: For all $p, q \in D$, there is $l \in \boldsymbol{L}$ such $l.p = q$. So if $g \in \boldsymbol{G}$ fixes p, then $g.(l.p) = l.(g.p) = l.p$. So g acts trivially on the open orbit D. Hence $g = \mathrm{id}$.

The action of $g \in \boldsymbol{G}$ induces a Killing field $X^+ \in \mathfrak{kill}(\mathbb{R}^{r,s})^{\boldsymbol{L}}$. By Corollary 1.11, these Killing fields are contained in a 2-step nilpotent Lie subalgebra of $\mathfrak{kill}(\mathbb{R}^{r,s})$. So:

Theorem 1.16 *A Wolf group is 2-step nilpotent.*

Lemma 1.17 *The centraliser* $\boldsymbol{L}$ *of* $\boldsymbol{G}$ *is an algebraic subgroup of* $\mathbf{Iso}(\mathbb{R}^{r,s})$.

Proof: $\mathbf{Iso}(\mathbb{R}^{r,s})$ is algebraic, and its subgroup $\boldsymbol{L}$ is defined by polynomial equations. ∎

Lemma 1.18 *If* $\boldsymbol{G}$ *is a Wolf group, its Zariski closure* $\overline{\boldsymbol{G}}$ *in* $\mathbf{Iso}(\mathbb{R}^{r,s})$ *is also a Wolf group. In particular,* $\overline{\boldsymbol{G}}$ *acts freely on the open orbit of* $\boldsymbol{L}$.

Proof: For all $g \in \boldsymbol{G}$, the conjugation map $\mathrm{c}_g(h) = ghg^{-1}$ is the identity when restricted to $\boldsymbol{L}$. By continuity in g, $\mathrm{c}_g|_L = \mathrm{id}_L$ for all $g \in \overline{\boldsymbol{G}}$. ∎

Corollary 1.19 *The Wolf group* $\boldsymbol{W}(\boldsymbol{L})$ *associated to* $\boldsymbol{L}$ *is Zariski-closed.*

Proposition 1.20 *Let $\boldsymbol{G}$ be a Wolf group.*

(a) *$\boldsymbol{G}$ is unipotent.*

(b) *Every $g \in \boldsymbol{G}$ is an affine transformation of the form $g = (I + A, v)$, where $A^2 = 0$, $Av = 0$. In particular, $\log(g) = (A, v)$.*

(c) *$v \perp \operatorname{im} A$ and $\operatorname{im} A$ is totally isotropic.*

Proof: Let $\boldsymbol{L}$ be the centraliser of $\boldsymbol{G}$, $\mathfrak{g} = \mathfrak{Lie}(\boldsymbol{G})$, $\mathfrak{l} = \mathfrak{Lie}(\boldsymbol{L})$ and $\mathfrak{w} = \mathfrak{Lie}(\boldsymbol{W}(\boldsymbol{L}))$.

(a) By Corollary 6.14 in Baues [2], every element in the affine centraliser $\mathrm{Z}_{\mathbf{Aff}(\mathbb{R}^{r+s})}(\boldsymbol{L})$ is unipotent. In particular, this holds for all $g \in \boldsymbol{W}(\boldsymbol{L})$. So $\boldsymbol{W}(\boldsymbol{L})$ is a unipotent algebraic group and hence $\exp(\mathfrak{w}) = \boldsymbol{W}(\boldsymbol{L})$. As $\boldsymbol{G} \subseteq \boldsymbol{W}(\boldsymbol{L})$, it is unipotent as well.

(b) From (1.5) we know that every $X = (A, v) \in \mathfrak{w}$ is of the form

$$X = \begin{pmatrix} -\mathsf{A}_{X^+} & X_p^+ \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{n+1}(\mathbb{R})$$

for the Killing field X^+ induced by the action of the one-parameter subgroup $\exp(tX)$. So $A = -\mathsf{A}_{X^+}$ and $v = X_p^+$. The centraliser of X^+ has an open orbit, so it follows from Proposition 1.10 that $A^2 = 0$ and $Av = 0$. If $g = \exp(X)$, it follows that $g = (I + A, v)$. By part (a), every $g \in \boldsymbol{G}$ is of this form.

(c) Let $g = (I + A, v) \in \boldsymbol{G}$. By Proposition C.5, A_{X^+} and hence A is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$. So for all $x \in \mathbb{R}^{r,s}$,

$$\langle Ax, v \rangle = -\langle x, Av \rangle = -\langle x, 0 \rangle = 0,$$

and $v \perp \operatorname{im} A$ follows. Also, for all $x, y \in \mathbb{R}^{r,s}$,

$$\langle Ax, Ay \rangle = -\langle x, A^2 y \rangle = -\langle x, 0 \rangle = 0.$$

It follows that $\operatorname{im} A$ is totally isotropic. ∎

Corollary 1.21 *If $g = (I + A, v) \in \boldsymbol{G}$, then*

$$\ker A = (\operatorname{im} A)^{\perp}, \quad \operatorname{im} A = (\ker A)^{\perp}.$$

Now the properties of Killing fields in Proposition 1.10 translate directly into properties of Wolf groups:

Proposition 1.22 *Let $\boldsymbol{G}$ be a Wolf group. If $g_i = (I + A_i, v_i) \in \boldsymbol{G}$, $i = 1, 2, 3$, then:*

(a) $A_1 A_2 = -A_2 A_1$.

(b) $A_1 v_2 = -A_2 v_1$.

(c) $A_1 A_2 v_3 = 0$.

(d) $[(A_1, v_1), (A_2, v_2)] = (2A_1 A_2, 2A_1 v_2)$. *More concisely, if we write $X_i = (A_i, v_i)$:*

$$X_1 X_2 = \frac{1}{2}[X_1, X_2]. \tag{1.6}$$

From this, Wolf derived some further properties for Wolf groups with abelian linear holonomy, see section 3.7 in Wolf [52]. To find a convenient representation, he introduced the subspace

$$U_{\boldsymbol{G}} = \sum_{(I+A,v)\in\boldsymbol{G}} \operatorname{im} A. \tag{1.7}$$

From Corollary 1.21 it follows that

$$U_{\boldsymbol{G}}^{\perp} = \bigcap_{(I+A,v)\in\boldsymbol{G}} \ker A. \tag{1.8}$$

These spaces play an important role in the characterisation of Wolf groups with abelian linear holonomy:

Proposition 1.23 *Let $\boldsymbol{G}$ be a Wolf group. Then the following are equivalent:*

(a) *The linear holonomy $\mathrm{L}(\boldsymbol{G})$ is abelian.*

(b) *If $g_1, g_2 \in \boldsymbol{G}$, then $A_1 A_2 = 0$.*

(c) *The space $U_{\boldsymbol{G}}$ is totally isotropic.*

If these conditions hold, choose a Witt basis with respect to the Witt decomposition for the totally isotropic subspace $U_{\boldsymbol{G}}$. In this Witt basis, the linear part of $g \in \boldsymbol{G}$ is

$$\mathrm{L}(g) = \begin{pmatrix} I & 0 & C \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \tag{1.9}$$

where C is a skew-symmetric matrix, and the translation part $\mathrm{T}(g)$ is an element of $U_{\boldsymbol{G}}^{\perp} \backslash U_{\boldsymbol{G}}$.

For a proof, see Wolf [52], Proposition 3.7.9.

Corollary 1.24 *Let $\mathbf{G}$ be a Wolf group with abelian linear holonomy $\mathrm{L}(\mathbf{G})$. Then:*

(a) *The commutator subgroup $[\mathbf{G}, \mathbf{G}]$ consists of pure translations only, and any pure translation in $\mathbf{G}$ is central.*

(b) *The following are equivalent:*

  (i) $\mathbf{G}$ *is abelian.*

  (ii) $A_1 v_2 = 0$ *for all* $(I + A_i, v_i) \in \mathbf{G}$.

  (iii) $v \in U_G^{\perp}$ *for all* $(I + A, v) \in \mathbf{G}$.

For a proof, see Corollary 3.7.11 in Wolf [52]. Some additional results due to Wolf on spaces of low dimension or with special signatures will be discussed in chapter 6.

2 Representations of Wolf Groups

We study representations of Wolf groups $\boldsymbol{G}$, in particular of those with non-abelian linear holonomy. Examples for this case are the Main Examples 10 and 11. For now, we do not assume $\boldsymbol{G}$ to act freely (that is, its centraliser $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$ has an open orbit which is not necessarily all of $\mathbb{R}^{r,s}$).

Consider the Lie algebra $\mathfrak{g}$ of (the Zariski closure of) $\boldsymbol{G}$. Recall from (1.5) that the linear part of an element $X \in \mathfrak{g}$ is given by the operator $-\mathsf{A}_X$ as defined in (B.2), and its translation part is X_p, where we consider X as a Killing vector field and $p \in \mathbb{R}^{r,s}$ as the origin of our coordinate system. The algebra $\mathrm{L}(\mathfrak{g})$ consisting of the linear parts of $\mathfrak{g}$ is also 2-step nilpotent, and it is abelian if and only if the linear holonomy group of $\boldsymbol{G}$ is abelian.

For $X \in \mathfrak{g}$, we use the notation $X = (A_X, v_X)$ with $A_X = \mathrm{L}(X)$, $v_X = \mathrm{T}(X)$.

2.1 Some Bookkeeping

Lemma 2.1 *If G has non-abelian linear holonomy, there exist* $X, Y \in \mathfrak{g}$ *such that* $A_X A_Y \neq 0$.

PROOF: Recall $2A_X A_Y = A_{[X,Y]}$ from Proposition 1.22. As $\mathrm{L}(\mathfrak{g})$ is not abelian, there exist $A_X, A_Y \in \mathrm{L}(\mathfrak{g})$ such that $0 \neq [A_X, A_Y] = A_{[X,Y]}$. ■

The centre $\mathrm{Z}(\boldsymbol{G})$ is an abelian Wolf group. The Lie algebra $\mathfrak{z}(\mathfrak{g}) = \mathfrak{Lie}(\mathrm{Z}(\boldsymbol{G}))$ is the centre of $\mathfrak{g}$, and $\mathfrak{z}(\mathrm{L}(\mathfrak{g})) = \mathrm{L}(\mathfrak{z}(\mathfrak{g}))$ is the centre of $\mathrm{L}(\mathfrak{g})$. The space

$$U_{\mathrm{Z}(\boldsymbol{G})} = \sum_{Z \in \mathfrak{z}(\mathfrak{g})} \operatorname{im} A_Z$$

is a totally isotropic space contained in $U_{\boldsymbol{G}} = \sum_{X \in \mathfrak{g}} \operatorname{im} A_X$. The latter is *not* totally isotropic if $\mathrm{L}(\boldsymbol{G})$ is not abelian (see Proposition 1.23).

In this chapter, the role of the totally isotropic subspace $U_{\mathrm{Z}(\boldsymbol{G})}$ from the abelian case will be played by the possibly larger totally isotropic subspace

$$U_0 = U_{\boldsymbol{G}} \cap U_{\boldsymbol{G}}^{\perp} = \sum_{X \in \mathfrak{g}} \operatorname{im} A_X \cap \bigcap_{X \in \mathfrak{g}} \ker A_X. \tag{2.1}$$

The equality of the two sides follows from (1.8). For clarity, we shall sometimes write $U_0(\boldsymbol{G})$ for U_0.

Lemma 2.2

(a) $U_{\mathrm{Z}(\boldsymbol{G})} \subseteq U_{\boldsymbol{G}}$.

(b) $U_{Z(G)} \perp U_G$.

(c) U_0 *is a totally isotropic subspace and* $U_{Z(G)} \subseteq U_0$.

(d) $U_{Z(G)}^{\perp} \supseteq U_0^{\perp} \supseteq U_G^{\perp} \supseteq U_0 \supseteq U_{Z(G)}$.

Proof: (a) holds by definition, (c) follows from (a) and (b), and (d) follows from (b) and (c).

For (b), let $A_Z v \in U_{Z(G)}$ for $Z \in \mathfrak{z}(\mathfrak{g})$ and $v \in \mathbb{R}^{r,s}$. As Z is central in $\mathfrak{g}$, it follows that $A_X A_Z v = 0$ for all $v \in \mathbb{R}^{r,s}$. Let $w \in \mathbb{R}^{r,s}$. It follows from the skew symmetry of A_X that

$$\langle \underbrace{A_X w}_{\in U_G}, \underbrace{A_Z v}_{\in U_{Z(G)}} \rangle = -\langle w, A_X A_Z v \rangle = 0.$$

So $U_G \perp U_{Z(G)}$. ■

Lemma 2.3 *Fix a dual space* $U_{Z(G)}^*$ *for* $U_{Z(G)}$. *Let* $X \in \mathfrak{g}$ *such that* A_X *is not central in* $\mathfrak{l}(\mathfrak{g})$. *This means there exists* $Y \in \mathfrak{g}$ *such that* $A_X A_Y \neq 0$ *and* $v \in \mathbb{R}^{r,s}$ *such that* $A_X A_Y v \neq 0$. *Then:*

(a) *If* $w \in \mathbb{R}^{r,s}$ *is dual to* $A_X A_Y v$, *then* $-A_X w$ *is dual to* $A_Y v$, *and* $A_X w, A_Y v$ *are linearly independent.*

(b) $\operatorname{im} A_X A_Y \subseteq U_{Z(G)}$.

(c) $A_Y v \in U_G \backslash U_0$.

(d) $v \notin U_{Z(G)}^{\perp}$.

Proof:

(a) As $A_X A_Y v \neq 0$ and $\langle \cdot, \cdot \rangle$ is non-degenerate, there exists $w \in \mathbb{R}^{r,s}$ such that
$$1 = \langle w, A_X A_Y v \rangle = -\langle A_X w, A_Y v \rangle = \langle A_Y A_X w, v \rangle.$$
As $A_X v$, $A_Y w$ are isotropic, they cannot be linearly dependent.

(b) $A_Y A_X = \frac{1}{2}[A_Y, A_X] = -A_X A_Y$ is central because $\mathfrak{l}(\mathfrak{g})$ is 2-step nilpotent. So $\operatorname{im} A_X A_Y \subseteq U_{Z(G)}$.

(c) $A_X(A_Y v) \neq 0$, and U_0 is contained in $\ker A_X$. So $A_Y v \notin U_0$.

(d) Let $v = v^* + v_0$, where $v^* \in U^*_{Z(G)}$ and $v_0 \in U^\perp_{Z(G)}$. Then

$$\langle A_Y A_X w, v\rangle = \langle A_Y A_X w, v^* + v_0\rangle = \langle A_Y A_X w, v^*\rangle = 1.$$

So $v^* \neq 0$. ■

Corollary 2.4 *Let $A_X w$, $A_Y v$ as in Lemma 2.3. Then $A_X w$, $A_Y v$ span a Minkowski plane in U_G and thus are not contained in the totally isotropic space U_0.*

Lemma 2.5 *For every $v \in U_0^* \backslash \{0\}$, there exists an $X \in \mathfrak{g}$ such that $A_X v \neq 0$. If $v \in U^*_{Z(G)}$, then X can be chosen as a central element.*

Proof: By definition of U_0^*, there exists $A_X w$ in U_0 dual to v. By the skew-symmetry of A_X,

$$0 \neq \langle v, A_X w\rangle = -\langle A_X v, w\rangle.$$

So $A_X v \neq 0$. ■

Consider a Witt decomposition

$$\mathbb{R}^{r,s} = U_{Z(G)} \oplus W \oplus U^*_{Z(G)}.$$

Here, $U^\perp_{Z(G)} = U_{Z(G)} \oplus W$. We have seen that $A_X U^*_{Z(G)} \subseteq U_{Z(G)}$ for all $X \in \mathfrak{g}$. We shall now study how A_X acts on the other subspaces in this decomposition.

Lemma 2.6 $A_X U^\perp_{Z(G)} \subseteq U_0$ *for all* $X \in \mathfrak{g}$.

Proof: Let $u \in U^\perp_{Z(G)}$. For all $v \in \mathbb{R}^{r,s}$ and $X, Y \in \mathfrak{g}$, because A_X is skew and $A_X A_Y$ is central, we get

$$\langle A_Y v, A_X u\rangle = -\langle A_X A_Y v, u\rangle = 0.$$

Hence $A_X u \perp U_G$, that is $A_X u \in U_G \cap U_G^\perp = U_0$. ■

Lemma 2.7 $A_X U_0 = \{0\}$ *for all* $X \in \mathfrak{g}$.

Proof: $U_0 \subset U_G \subset \ker A_X$. ■

The following proposition sums up the above:

Proposition 2.8 *The chain of subspaces*

$$\mathbb{R}^{r,s} \supset U^\perp_{Z(G)} \supset U_0 \supset \{0\} \tag{2.2}$$

is stabilised by $\mathfrak{L}(\mathfrak{g})$ *such that each subspace is mapped to the next one in the chain.*

2.2 The Matrix Representation

The bookkeeping from the previous section allows us to establish some rules for the matrix representation of the linear parts A_X of elements $X \in \mathfrak{g}$. We fix a Witt basis with respect to the Witt decomposition

$$\mathbb{R}^{r,s} = U_0 \oplus W_0 \oplus U_0^*. \tag{2.3}$$

Here, W_0 is a subspace complementary to U_0 in $U_0^\perp$, so that $\langle\cdot,\cdot\rangle$ is non-degenerate on W_0. By $\tilde{I}$ we denote the signature matrix of $\langle\cdot,\cdot\rangle$ on W_0.

Theorem 2.9 *Let $X \in \mathfrak{g}$. Then the matrix representation of $A_X = \mathrm{L}(X)$ for the given Witt basis is*

$$A_X = \begin{pmatrix} 0 & -B_X^\top \tilde{I} & C_X \\ 0 & 0 & B_X \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

with $B_X \in \mathbb{R}^{(n-2k)\times k}$ and $C_X \in \mathfrak{so}_k$ (where $k = \dim U_0$). The columns of B_X are isotropic and mutually orthogonal with respect to $\tilde{I}$.

Proof: With respect to the given Witt basis, A_X is represented by a matrix

$$\begin{pmatrix} A & -B^\top \tilde{I} & C \\ D & E & B \\ F & -D^\top \tilde{I} & -A^\top \end{pmatrix}.$$

By Lemma 2.7, $U_0 \subseteq \ker A_X$. So $A = 0$, $D = 0$ and $F = 0$. Lemma 2.6 states that $U_{Z(G)}^\perp$ gets mapped to U_0 by A_X, and the same holds for $U_0^\perp \subset U_{Z(G)}^\perp$ (see Lemma 2.2). It follows that $E = 0$.

The condition $A_X^2 = 0$ translates to

$$A_X A_X = \begin{pmatrix} 0 & 0 & -B_X^\top \tilde{I} B_X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

so all columns of B_X are isotropic and mutually orthogonal with respect to $\tilde{I}$. Alternatively, this follows from the fact that the image of A_X is totally isotropic. ∎

The above proposition shows that U_0 is a good choice of totally isotropic subspace; for example, had we chosen a Witt basis with respect to $U_{Z(G)}$ instead, then we would not have been able to conclude $E = 0$, and the matrix representation would become more complicated.

Remark 2.10 If $\mathrm{L}(\mathfrak{g})$ is not abelian, then there exist $X, Y \in \mathfrak{g}$ such that

$$0 \neq \frac{1}{2}[A_X, A_Y] = \begin{pmatrix} 0 & -B_X^\top \tilde{I} & C_X \\ 0 & 0 & B_X \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -B_Y^\top \tilde{I} & C_Y \\ 0 & 0 & B_Y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -B_X^\top \tilde{I} B_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the submatrices B_X and B_Y are not all 0 and $-B_X^\top \tilde{I} B_Y \neq 0$. This corresponds to the fact that the images of non-commuting A_X and A_Y contain vectors $v \in \operatorname{im} A_X$, $w \in \operatorname{im} A_Y$ that are dual to each other (recall that $v, w \in U_0^\perp$, so their inner product is determined by their respective W_0-components, which is a linear combination of the columns of B_X and B_Y, respectively).

Given A_X, in the following B_X and C_X refer to the representation (2.4). The columns of B_X represent the non-zero W_0-components of the image of A_X.

The restriction of $\langle \cdot, \cdot \rangle$ to W_0 is non-degenerate and represented by the matrix $\tilde{I}$. If $v = u + w + u^*$ is the Witt decomposition of $v \in \mathbb{R}^{r,s}$, then

$$\langle A_X v, A_X v \rangle = \langle B_X u^*, B_X u^* \rangle.$$

This is because U_0 is totally isotropic and W_0 is orthogonal to U_0.

If A_X is not central, let A_Y denote an operator such that $A_X A_Y \neq 0 \neq [A_X, A_Y]$.

Remark 2.11 Write $v = u + w + u^*$ for the Witt decomposition of $v \in \mathbb{R}^{r,s}$. In the corresponding matrix representation,

$$A_X v = \begin{pmatrix} 0 & -B_X^\top \tilde{I} & C_X \\ 0 & 0 & B_X \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ w \\ u^* \end{pmatrix} = \begin{pmatrix} -B_X^\top \tilde{I} w + C_X u^* \\ B_X u^* \\ 0 \end{pmatrix}. \tag{2.5}$$

We derive some necessary conditions which have to be satisfied by the representation matrices (2.4) a Wolf group. Each rule will be given a catchy name for ease of reference. These rules are particularly helpful for constructing new examples.

Lemma 2.12 (Isotropy rule) *The columns of B_X are isotropic and mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$.*

This was already shown in Theorem 2.9.

Lemma 2.13 (Duality rule) *Assume A_X is not central (that is $A_X A_Y \neq 0$). Then B_Y contains a column b_Y^i which is dual to a column b_X^j of B_X (that is $\langle b_X^j, b_Y^i \rangle \neq 0$). Further, $i \neq j$.*

PROOF: The first part is immediate from Remark 2.10.

Assume $i = j$. Then the ith column of $B_X + B_Y$ is a non-isotropic vector in W_0. But

$$\exp(A_X + A_Y) = \begin{pmatrix} I & -(B_X + B_Y)^\top \tilde{I} & C_X + C_Y \\ 0 & I & B_X + B_Y \\ 0 & 0 & I \end{pmatrix} \in \mathrm{L}(\boldsymbol{G}).$$

If $A_Z = \mathrm{L}(\exp(A_X + A_Y))$, then A_Z must have totally isotropic image, and in particular

$$B_Z = B_X + B_Y$$

must have isotropic columns. This contradiction implies $i \neq j$. ■

Lemma 2.14 (Crossover rule) *Given A_X and A_Y, let b_Y^i be column i of B_Y and b_X^k column k of B_X. Then*

$$\langle b_X^k, b_Y^i \rangle = -\langle b_X^i, b_Y^k \rangle.$$

If this expression is $\neq 0$, then $b_X^k, b_X^i, b_Y^k, b_Y^i$ are linearly independent.

PROOF: Recall the formula for the product $A_Z = A_X A_Y$ from Remark 2.10: The matrix block C_Z is

$$C_Z = -B_X^\top \tilde{I} B_Y,$$

so its entry in column k, row i, is the inner product

$$-b_X^{k\top} \tilde{I} b_Y^i = -\langle b_X^k, b_Y^i \rangle.$$

The skew-symmetry of C_Z implies

$$-\langle b_X^i, b_Y^k \rangle = \langle b_X^k, b_Y^i \rangle.$$

Now assume $\langle b_X^k, b_Y^i \rangle \neq 0$. As b_X^k and b_Y^i are both isotropic, it follows that the subspace S_{ki} spanned by b_X^k, b_Y^i is a 2-dimensional Minkowski plane. Similarly, the subspace S_{ik} spanned by b_X^i, b_Y^k is also a 2-dimensional Minkowski plane. By the isotropy rule, b_X^i is orthogonal to b_X^k. By the duality rule, b_X^i is also orthogonal to b_Y^i. So $b_X^i \in S_{ki}^\perp$, and similarly $b_Y^k \in S_{ki}^\perp$. So $S_{ik} \subset S_{ki}^\perp$. Now the non-degeneracy of $\langle \cdot, \cdot \rangle$ on the Minkowski planes implies

$$S_{ki} \cap S_{ik} = \{0\},$$

which means $S_{ki} \oplus S_{ik}$ is 4-dimensional with basis $b_X^k, b_X^i, b_Y^k, b_Y^i$. ■

Lemma 2.15 *Assume $A_X A_Y \neq 0$ and that the columns b^i_X in B_X and b^j_Y in B_Y satisfy $\langle b^i_X, b^j_Y\rangle \neq 0$. The subspace W_0 in (2.3) has a Witt decomposition*

$$W_0 = W_{ij} \oplus W' \oplus W^*_{ij}, \tag{2.6}$$

where $W_{ij} = \mathbb{R}b^i_X \oplus \mathbb{R}b^j_X$, $W^*_{ij} = \mathbb{R}b^i_Y \oplus \mathbb{R}b^j_Y$, $W' \perp W_{ij}$, $W' \perp W^*_{ij}$, and $\langle\cdot,\cdot\rangle$ is non-degenerate on W'. Furthermore,

$$\mathrm{wi}(W_0) \geq \mathrm{rk}\, B_X \geq 2 \quad \textit{and} \quad \dim W_0 \geq 2\,\mathrm{rk}\, B_X \geq 4. \tag{2.7}$$

Proof: $\mathbb{R}b^i_X \oplus \mathbb{R}b^j_X$ is totally isotropic because $\mathrm{im}\, B_X$ is. By the crossover rule, $\{b^j_Y, b^i_Y\}$ is a dual basis to $\{b^i_X, b^j_X\}$ (after scaling, if necessary).

W_0 contains $\mathrm{im}\, B_X$ as a totally isotropic subspace, hence it also contains a dual space. Hence $\mathrm{wi}(W_0) \geq \mathrm{rk}\, B_X \geq \dim W_{ij} \geq 2$ and $\dim W_0 \geq 2\,\mathrm{rk}\, B_X \geq 2\dim W_{ij} = 4$. ∎

2.3 Translation Parts

Lemma 2.16 *Let $X = (A_X, v_X), Y = (A_Y, v_Y) \in \mathfrak{g}$. Then $A_X v_X = 0 = A_Y v_Y$. If the $\boldsymbol{G}$-action is free and $[X, Y] \neq 0$, then*

$$A_X v_Y \neq 0, \quad A_Y v_X \neq 0,$$

and in particular, v_X and v_Y are linearly independent.

Proof: By now it is well-known that $A_X v_X = 0$.

Assume $\boldsymbol{G}$ acts freely. If $[X, Y] \neq 0$, then $2A_X v_Y = -2A_Y v_X$ is the translation part of $[X, Y]$. If this translation part was 0, then $\exp([X, Y])$ has 0 as a fixed point, but this contradicts the free action of $\boldsymbol{G}$. ∎

Lemma 2.17 *Let $Z = \frac{1}{2}[X, Y]$. With respect to the Witt decomposition (2.3), its translation part is*

$$\tau(Z) = \begin{pmatrix} u_Z \\ 0 \\ 0 \end{pmatrix},$$

with $u_Z \in U_0$.

Proof: $\tau(Z) = \tau([X, Y]) = A_X v_Y \in \mathrm{im}\, A_X \subseteq U_{\boldsymbol{G}}$. Also, since Z is central, $\tau(Z) \in \bigcap_{X\in\mathfrak{g}} \ker A_X = U^{\perp}_{\boldsymbol{G}}$. So $\tau(Z) \in U_0 = U_{\boldsymbol{G}} \cap U^{\perp}_{\boldsymbol{G}}$. ∎

Lemma 2.18 (Translation rule) *Let* $X = (A_X, v_X), Y = (A_Y, v_Y) \in \mathfrak{g}$. *Then*

$$u_X^*, u_Y^* \in \ker B_X \cap \ker B_Y,$$

where u_X^*, u_Y^* *denote the respective* U_0^**-components of* v_X, v_Y.

Proof: Let $Z = \frac{1}{2}[X, Y]$ and let $v_Z = u_Z + w_Z + u_Z^*$ be the Witt decomposition of v_Z. From Lemma 2.17 it follows that $w_Z = 0$, $u_Z^* = 0$. But because $v_Z = A_X v_Y = -A_Y v_X$, it follows from (2.5) that $0 = w_Z = B_X u_Y^* = -B_Y u_X^*$. Further, $A_X v_X = 0 = A_Y v_Y$ implies $B_X u_X^* = 0 = B_Y u_Y^*$. ■

2.4 Criteria for Fixed Points

In this section we prove that in certain cases the central element Z always has a fixed point, so these spaces cannot have a free $\boldsymbol{G}$-action. The proofs are quite involved and rely heavily on the results of the previous sections, so we repeat the most important formulae for quick reference: We have a Witt decomposition with respect to the totally isotropic subspace U_0,

$$\mathbb{R}^{r,s} = U_0 \oplus W_0 \oplus U_0^*. \tag{2.8}$$

The Witt decomposition of $v \in \mathbb{R}^{r,s}$ is written

$$v = u + w + u^* \tag{2.9}$$

for $u \in U_0, w \in W_0, u^* \in U_0^*$. For a Wolf group $\boldsymbol{G}$ acting on $\mathbb{R}^{r,s}$, every element $g \in \boldsymbol{G}$ has the form

$$g = I + X = \exp(X) \tag{2.10}$$

for some $X \in \log(\boldsymbol{G})$, and

$$X = (A_X, v_X) \quad \text{with } A_X = \text{L}(X), v_X = \text{T}(X). \tag{2.11}$$

With respect to the Witt decomposition (2.8), X is represented by

$$X = (A_X, v_X) = \left(\begin{pmatrix} 0 & -B_X^\top \tilde{I} & C_X \\ 0 & 0 & B_X \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} u_X \\ w_X \\ u_X^* \end{pmatrix} \right), \tag{2.12}$$

where C_X is skew. The submatrices B_X, C_X can be read as linear maps,

$$B_X^\top \tilde{I} : W_0 \to U_0, \qquad B_X : U_0^* \to W_0, \qquad C_X : U_0^* \to U_0. \tag{2.13}$$

Every $X \in \log(\boldsymbol{G})$ satisfies $X^2 = 0$, which implies

$$A_X v_X = 0, \tag{2.14}$$

or with (2.12)

$$-B_X^\top \tilde{I} w_X + C_X u_X^* = 0, \tag{2.15}$$
$$B_X u_X^* = 0. \tag{2.16}$$

For $X, Y \in \log(\boldsymbol{G})$,

$$A_X A_Y = -A_Y A_X = \frac{1}{2}[A_X, A_Y], \qquad A_X v_Y = -A_Y v_X. \tag{2.17}$$

Then by (2.12) and Lemma 2.17,

$$Z = XY = (A_Z, v_Z) = (A_X A_Y, A_X v_Y) = \left(\begin{pmatrix} 0 & 0 & C_Z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} u_Z \\ 0 \\ 0 \end{pmatrix}\right), \tag{2.18}$$

where

$$C_Z = -B_X^\top \tilde{I} B_Y = B_Y^\top \tilde{I} B_X, \tag{2.19}$$
$$v_Z = u_Z = -B_X^\top \tilde{I} w_Y + C_X u_Y^* = B_Y^\top \tilde{I} w_X - C_Y u_X^*. \tag{2.20}$$

Choose X, Y such that $A_X A_Y = A_Z \neq 0$. Then

$$C_Z = -B_X^\top \tilde{I} B_Y \neq 0 \quad \text{and} \quad \operatorname{rk} C_Z \text{ is even because it is skew.} \tag{2.21}$$

Let b_X^i denote column i from B_X and b_Y^j column j from B_Y in (2.12). Then the isotropy rule says

$$\langle b_X^i, b_X^j \rangle = 0 \qquad \text{for all } i, j, \tag{2.22}$$

the duality rule says there exists i, j such that

$$\langle b_X^i, b_Y^j \rangle \neq 0, \tag{2.23}$$

and the crossover rule says

$$\langle b_X^i, b_Y^j \rangle = -\langle b_X^j, b_Y^i \rangle \tag{2.24}$$

and $b_X^i, b_X^j, b_Y^i, b_Y^j$ are linearly independent if this is $\neq 0$. The restriction of $\langle \cdot, \cdot \rangle$ to W_0 is represented by the matrix $\tilde{I}$, so

$$C_Z = -B_X^\top \tilde{I} B_Y = \left(\langle b_Y^j, b_X^i \rangle\right)_{ij}. \tag{2.25}$$

Lemma 2.19 (Fixed point rule) *If* $u_Z \in \operatorname{im} B_X^\top \tilde{I} B_Y$*, then* $\exp(Z)$ *has a fixed point.*

Proof: By (2.19), $C_Z = -B_X^\top \tilde{I} B_Y$, and by (2.20), $v_Z = A_X v_Y = u_Z$. If there exists u^* such that $u_Z = C_Z u^*$, then it follows from (2.18) that

$$Z.\begin{pmatrix} 0 \\ 0 \\ -u^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_Z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -u^* \end{pmatrix} + \begin{pmatrix} u_Z \\ 0 \\ 0 \end{pmatrix} = 0.$$

So $\exp(Z)$ has a fixed point. ■

Lemma 2.20 *If* $\operatorname{rk} B_X^\top \tilde{I} B_Y = \operatorname{rk} B_X$ *and the* $\boldsymbol{G}$*-action is free, then*

$$u_X^* \neq 0, \quad u_Y^* \neq 0.$$

Proof: From (2.20) we get

$$u_Z = -B_X^\top \tilde{I} w_Y + C_X u_Y^*.$$

Also, $\operatorname{im} B_X^\top \tilde{I} B_Y \subset \operatorname{im} B_X^\top$. But by our rank assumption, $\operatorname{im} B_X^\top \tilde{I} B_Y = \operatorname{im} B_X^\top$. So, if $u_Y^* = 0$, then $u_Z \in \operatorname{im} B_X^\top = \operatorname{im} B_X^\top \tilde{I} B_Y$, which implies the existence of a fixed point by the fixed point rule. So $u_Y^* \neq 0$ if the action is free. Using $v_Z = A_X v_Y = -A_Y v_X$, we can conclude $u_X^* \neq 0$ in a similar manner. ■

Corollary 2.21 *If* $\dim U_0 = 2$*, then* $\boldsymbol{G}$ *has a fixed point.*

Proof: By Lemma 2.15, $2 \leq \operatorname{rk} B_X \leq \dim U_0 = 2$, so B_X is of full rank. Now (2.16) implies $u_X^* = 0$, so by Lemma 2.20, the $\boldsymbol{G}$-action is not free. ■

Lemma 2.22 *If* $\dim U_0 = 3$ *and* $\dim(\operatorname{im} B_X + \operatorname{im} B_Y) \leq 5$*, then* $\exp(Z)$ *has a fixed point.*

Proof: By Lemma 2.15, $\operatorname{rk} B_X, \operatorname{rk} B_Y \geq 2$. We distinguish two cases:

(i) Assume $\operatorname{rk} B_X = 2$ (or $\operatorname{rk} B_Y = 2$).

Because $C_Z = -B_X^\top \tilde{I} B_Y$ is skew, it is also of rank 2. Then

$$\operatorname{im} B_X^\top \tilde{I} B_Y = \operatorname{im} B_X^\top.$$

$\ker B_X$ is a 1-dimensional subspace due to $\dim U_0 = \dim U_0^* = 3$. Because $u_X^*, u_Y^* \in \ker B_X$, we have $u_X^* = \lambda u_Y^*$ for some number $\lambda \neq 0$.

From (2.20) and (2.15), we get

$$\lambda u_Z = -B_X^\top \tilde{I} \lambda w_Y + C_X \lambda u_Y^* = -B_X^\top \tilde{I} \lambda w_Y + C_X u_X^*,$$
$$0 = -B_X^\top \tilde{I} w_X + C_X u_X^*.$$

So

$$\lambda u_Z = \lambda u_Z - 0 = B_X^\top \tilde{I}(w_X - \lambda w_Y).$$

In other words, $u_Z \in \operatorname{im} B_X^\top = \operatorname{im} B_X^\top \tilde{I} B_Y$, and $\exp(Z)$ has a fixed point by the fixed point rule.

(ii) Assume $\operatorname{rk} B_X = \operatorname{rk} B_Y = 3$.

As $[A_X, A_Y] \neq 0$, (2.23) and (2.24) imply the existence of a pair of columns b_X^i, b_X^j in B_X and a pair of columns b_Y^j, b_Y^i in B_Y such that $\alpha = \langle b_X^i, b_Y^j \rangle = -\langle b_X^j, b_Y^i \rangle \neq 0$. For simplicity say $i = 1$, $j = 2$. As $\operatorname{rk} B_X = 3$, the column b_X^3 is linearly independent of b_X^1, b_X^2, and these columns span the totally isotropic subspace $\operatorname{im} B_X$ of W_0.

- Assume $b_Y^3 \in \operatorname{im} B_X$ (or $b_X^3 \in \operatorname{im} B_Y$).

  Then b_Y^3 is a multiple of b_X^3: In fact, let $b_Y^3 = \lambda_1 b_X^1 + \lambda_2 b_X^2 + \lambda_3 b_X^3$. Then $\langle b_Y^3, b_X^i \rangle = 0$ because $\operatorname{im} B_X$ is totally isotropic. Since $\operatorname{im} B_Y$ is totally isotropic and by (2.24),

  $$0 = \langle b_Y^3, b_Y^1 \rangle = \lambda_1 \langle b_X^1, b_Y^1 \rangle + \lambda_2 \langle b_X^2, b_Y^1 \rangle + \lambda_3 \langle b_X^3, b_Y^1 \rangle$$
  $$= \lambda_2 \alpha - \lambda_3 \langle b_Y^3, b_X^1 \rangle = \lambda_2 \alpha.$$

  Because $\alpha \neq 0$, this implies $\lambda_2 = 0$ and in the same way $\lambda_1 = 0$. So $b_Y^3 = \lambda_3 b_X^3$.

  Now $b_X^3 \perp b_X^i, b_Y^j$ for all i, j. We have $u_Y^* = 0$ because $B_Y u_Y^* = 0$ and B_Y is of maximal rank. Then $\langle b_X^3, w_Y \rangle = \langle b_Y^3, w_Y \rangle = 0$, because $0 = B_Y^\top \tilde{I} w_Y + C_Y u_Y^* = B_Y^\top \tilde{I} w_Y$. Hence (2.19) and (2.20) take the form

  $$C_Z = -B_X^\top \tilde{I} B_Y = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u_Z = -B_X^\top \tilde{I} w_Y = \begin{pmatrix} -\langle b_X^1, w_Y \rangle \\ -\langle b_X^2, w_Y \rangle \\ 0 \end{pmatrix}.$$

  It follows that $u_Z \in \operatorname{im} C_Z$, so in this case $\exp(Z)$ has a fixed point by the fixed point rule.

- Assume $b_Y^3 \notin \operatorname{im} B_X$ and $b_X^3 \notin \operatorname{im} B_Y$.

  This means b_Y^3 and b_X^3 are linearly independent. If $b_Y^3 \perp \operatorname{im} B_X$, then $b_X^3 \perp \operatorname{im} B_Y$ by the crossover rule. With respect to the Witt decomposition $W_0 = W_{12} \oplus W' \oplus W_{12}^*$ (Lemma 2.15), this means b_X^3, b_Y^3 span a 2-dimensional subspace of $(W_{12} \oplus W_{12}^*)^\perp = W'$. But then $\dim(\operatorname{im} B_X + \operatorname{im} B_Y) = 6$, contradicting the lemma's assumption that this dimension should be ≤ 5.

So $b_Y^3 \not\perp \operatorname{im} B_X$ and $b_X^3 \not\perp \operatorname{im} B_Y$ hold. Because further $b_X^3 \perp \operatorname{im} B_X$, $b_Y^3 \perp \operatorname{im} B_Y$ and $\dim(\operatorname{im} B_X + \operatorname{im} B_Y) \leq 5$, there exists a $b \in W'$ (with W' from the Witt decomposition above) such that

$$\begin{aligned} b_X^3 &= \lambda_1 b_X^1 + \lambda_2 b_X^2 + \lambda_3 b, \\ b_Y^3 &= \mu_1 b_Y^1 + \mu_2 b_Y^2 + \mu_3 b. \end{aligned}$$

Because B_X, B_Y are of maximal rank, we have $u_X^* = 0 = u_Y^*$ by (2.16). Then

$$0 = B_Y^\top \tilde{I} w_Y = \begin{pmatrix} \langle b_Y^1, w_Y \rangle \\ \langle b_Y^2, w_Y \rangle \\ \langle b_Y^3, w_Y \rangle \end{pmatrix},$$

and this implies $\langle b, w_Y \rangle = 0$. Put $\xi = \langle b_X^1, w_Y \rangle$, $\eta = \langle b_X^2, w_Y \rangle$. Then

$$u_Z = -B_X^\top \tilde{I} w_Y = -\begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix}$$

and (recall $\alpha = \langle b_X^1, b_Y^2 \rangle = -\langle b_X^2, b_Y^1 \rangle$)

$$C_Z = B_Y^\top \tilde{I} B_X = \begin{pmatrix} 0 & -\alpha & -\lambda_2 \alpha \\ \alpha & 0 & \lambda_1 \alpha \\ \lambda_2 \alpha & -\lambda_1 \alpha & 0 \end{pmatrix}.$$

So

$$C_Z \cdot \frac{1}{\alpha} \begin{pmatrix} -\eta \\ \xi \\ 0 \end{pmatrix} = -\begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix} = u_Z.$$

By the fixed point rule, $\exp(Z)$ has a fixed point. ■

Lemma 2.23 *If* $\dim U_0 = 4$ *and* $\operatorname{rk} B_X^\top \tilde{I} B_Y = \operatorname{rk} B_X = \operatorname{rk} B_Y$*, then* $\exp(Z)$ *has a fixed point.*

PROOF: By assumption,

$$\operatorname{im} B_X^\top \tilde{I} B_Y = \operatorname{im} B_X^\top = \operatorname{im} B_Y^\top.$$

(i) First, assume $u_X^* = \lambda u_Y^*$ for some number $\lambda \neq 0$. Writing out $A_X v_Y = v_Z$ and $A_X v_X = 0$, we get from (2.20) and (2.15)

$$\begin{aligned}\lambda u_Z &= -B_X^\top \tilde{I} \lambda w_Y + C_X \lambda u_Y^* = -B_X^\top \tilde{I} \lambda w_Y + C_X u_X^*,\\ 0 &= -B_X^\top \tilde{I} w_X + C_X u_X^*.\end{aligned}$$

So

$$\lambda u_Z = \lambda u_Z - 0 = B_X^\top \tilde{I}(w_X - \lambda w_Y).$$

In other words, $u_Z \in \operatorname{im} B_X^\top = \operatorname{im} B_X^\top \tilde{I} B_Y$, and $\exp(Z)$ has a fixed point by the fixed point rule.

(ii) Now, assume u_X^* and u_Y^* are linearly independent. The translation rule (Lemma 2.18) can be reformulated as

$$\operatorname{im} B_X^\top = \operatorname{im} B_Y^\top \subseteq \ker u_X^{*\top} \cap \ker u_Y^{*\top}.$$

$\ker u_X^{*\top}$, $\ker u_Y^{*\top}$ are 3-dimensional subspaces of the 4-dimensional space U_0^*, and their intersection is of dimension 2 (because u_X^*, u_Y^* are linearly independent). By Lemma 2.15, $\operatorname{rk} B_X \geq 2$, so it follows that

$$\operatorname{im} B_X^\top = \operatorname{im} B_Y^\top = \ker u_X^{*\top} \cap \ker u_Y^{*\top}.$$

With (2.15) we conclude $C_X u_X^* = b$ for some $b \in \operatorname{im} B_X^\top$. Thus, by the skew-symmetry of C_X,

$$0 = (u_Y^{*\top} C_X u_X^*)^\top = -u_X^{*\top} C_X u_Y^*.$$

So $C_X u_Y^* \in \ker u_X^{*\top}$. In the same way $C_Y u_X^* \in \ker u_Y^{*\top}$. But $u_Z = C_X u_Y^* + b_1 = -C_Y u_X^* + b_2$ for some $b_1, b_2 \in \operatorname{im} B_X^\top$. Hence

$$\begin{aligned}u_X^{*\top} u_Z &= \underbrace{u_X^{*\top} C_X u_Y^*}_{=0} + \underbrace{u_X^{*\top} b_1}_{=0} = 0,\\ u_Y^{*\top} u_Z &= -\underbrace{u_Y^{*\top} C_Y u_X^*}_{=0} + \underbrace{u_Y^{*\top} b_2}_{=0} = 0.\end{aligned}$$

So $u_Z \in \ker u_X^{*\top} \cap \ker u_Y^{*\top} = \operatorname{im} B_X^\top = \operatorname{im} B_X^\top \tilde{I} B_Y$. With the fixed point rule we conclude that there exists a fixed point for $\exp(Z)$. ∎

2.5 Dimension Bounds

In this section, we prove a lower bound for the dimension of an affine space admitting an action of a Wolf group $\boldsymbol{G}$ whose linear holonomy group $\mathrm{L}(\boldsymbol{G})$ is non-abelian (Theorem 2.24). For this result, we do not assume $\boldsymbol{G}$ to act freely, so it holds in particular for incomplete manifolds. We then sharpen this lower bound under the assumption that $\boldsymbol{G}$ has transitive centraliser (Theorem 2.25). Examples show that both lower bounds are sharp.

Throughout the section, let X, Y be any elements of $\log(\boldsymbol{G})$ with non-commuting linear part, $[A_X, A_Y] = 2A_X A_Y \neq 0$. Let $Z = \frac{1}{2}[X, Y]$. Then $Z = (A_Z, v_Z) = (A_X A_Y, A_X v_Y)$.

Recall that the Witt decomposition is $\mathbb{R}^{r,s} = U_0 \oplus W_0 \oplus U_0^*$, and write $v = u + w + u^*$ for the Witt decomposition of any $v \in \mathbb{R}^{r,s}$. Also, recall (2.5):

$$A_X v = \begin{pmatrix} 0 & -B_X^\top \tilde{I} & C_X \\ 0 & 0 & B_X \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ w \\ u^* \end{pmatrix} = \begin{pmatrix} -B_X^\top \tilde{I} w + C_X u^* \\ B_X u^* \\ 0 \end{pmatrix}.$$

Theorem 2.24 *Let $\boldsymbol{G}$ be a Wolf group acting on $\mathbb{R}^{r,s}$, $n = r + s$, with non-abelian holonomy group. Then*

$$\mathrm{wi}(\mathbb{R}^{r,s}) \geq 4, \quad n \geq 8.$$

As Main Example 10 shows, this is a sharp lower bound.

Proof: If $\mathrm{L}(\boldsymbol{G})$ is not abelian, there exist $g_X = (I + A_X, v_X), g_Y = (I + A_Y, v_Y)$ such that $A_X A_Y \neq 0$.

By the duality rule, there are columns in B_X, B_Y which are dual to one another. Then, by the crossover rule, B_X and B_Y together contain at least four linearly independent columns. This implies

$$\dim W_0 \geq 4.$$

Further, $B_X^\top \tilde{I} B_Y \neq 0$. So if $Z = [X, Y]$, this means the skew-symmetric matrix C_Z is non-zero. Hence C_Z must have at least two columns, that is

$$\dim U_0 \geq 2.$$

By Lemma 2.15, $\mathrm{wi}(W_0) \geq 2$, so now it follows that $\mathbb{R}^{r,s}$ contains a totally isotropic subspace of dimension ≥ 4. Hence

$$\mathrm{wi}(\mathbb{R}^{r,s}) \geq 4.$$

Further,

$$n \geq 2\mathrm{wi}(\mathbb{R}^{r,s}) \geq 8,$$

follows. ■

If the centraliser of $\boldsymbol{G}$ acts transitively, then $\boldsymbol{G}$ must not have a fixed point. With the help of the fixed point citeria from section 2.4, we obtain the following theorem:

Theorem 2.25 *Let $\boldsymbol{G}$ be a Wolf group acting on $\mathbb{R}^{r,s}$, $n = r+s$, with transitive centraliser. If $\boldsymbol{G}$ has a non-abelian holonomy group, then*

$$\mathrm{wi}(\mathbb{R}^{r,s}) \geq 7, \quad n \geq 14.$$

As Main Example 11 shows, this is a sharp lower bound.

Proof: We only need to show $\mathrm{wi}(\mathbb{R}^{r,s}) \geq 7$, then it follows immediately that

$$n \geq 2\mathrm{wi}(\mathbb{R}^{r,s}) \geq 14.$$

If the centraliser is transitive, then $\boldsymbol{G}$ acts freely. From Corollary 2.21 we know that $\dim U_0 \geq 3$. By Lemma 2.15, $\mathrm{wi}(W_0) \geq 2$, and if $\dim U_0 \geq 5$, then

$$\mathrm{wi}(\mathbb{R}^{r,s}) = \dim U_0 + \mathrm{wi}(W_0) \geq 5 + 2 = 7,$$

and we are done. So let $2 < \dim U_0 < 5$.

(i) First, let $\dim U_0 = 4$. Assume $\mathrm{rk}\, B_X = \mathrm{rk}\, B_Y = 2$. Because $C_Z = -B_X^\top \tilde{I} B_Y \neq 0$ is skew, it is of rank 2. So $\mathrm{rk}\, B_X = \mathrm{rk}\, B_Y = 2 = \mathrm{rk}\, B_X^\top \tilde{I} B_Y$. By Lemma 2.23, the action of $\boldsymbol{G}$ is not free.

Now assume $\mathrm{rk}\, B_X \geq 3$. It follows from Lemma 2.15 that $\mathrm{wi}(W_0) \geq 3$ and $\dim W_0 \geq 6$, so once more

$$\mathrm{wi}(\mathbb{R}^{r,s}) = \dim U_0 + \mathrm{wi}(W_0) \geq 4 + 3 = 7.$$

So the theorem holds for $\dim U_0 = 4$.

(ii) Let $\dim U_0 = 3$. If $\dim(\mathrm{im}\, B_X + \mathrm{im}\, B_Y) \leq 5$, there exists a fixed point by Lemma 2.22, so $\boldsymbol{G}$ does not act freely. So let $\dim(\mathrm{im}\, B_X + \mathrm{im}\, B_Y) = 6$: As $[A_X, A_Y] \neq 0$, the crossover rule (Lemma 2.14) implies the existence of a pair of columns b_X^i, b_X^j in B_X and a pair of columns b_Y^j, b_Y^i in B_Y such that $\alpha = \langle b_X^i, b_Y^j \rangle = -\langle b_X^j, b_Y^i \rangle \neq 0$. For simplicity say $i = 1$, $j = 2$. The columns b_X^1, b_X^2, b_X^3 span the totally isotropic subspace $\mathrm{im}\, B_X$ of

W_0, and b_Y^1, b_Y^2, b_Y^3 span $\operatorname{im} B_Y$. We have a Witt decomposition with respect to $W_{12} = \mathbb{R}b_X^1 \oplus \mathbb{R}b_X^2$ (Lemma 2.15),

$$W_0 = W_{12} \oplus W' \oplus W_{12}^*,$$

where $W_{12}^* = \mathbb{R}b_Y^1 \oplus \mathbb{R}b_Y^2$. Because $b_X^3 \perp \operatorname{im} B_X$ and $b_Y^3 \perp \operatorname{im} B_Y$,

$$b_X^3 = \lambda_1 b_X^1 + \lambda_2 b_X^2 + b', \qquad b_Y^3 = \mu_1 b_Y^1 + \mu_2 b_Y^2 + b'',$$

where $b', b'' \in W'$ are linearly independent because $\dim(\operatorname{im} B_X + \operatorname{im} B_Y) = 6$. From $0 = \langle b_X^3, b_X^3 \rangle$ it follows that $\langle b', b' \rangle = 0$, and similarly $\langle b'', b'' \rangle = 0$. The crossover rule then implies

$$\lambda_1 \langle b_Y^2, b_X^1 \rangle = \langle b_Y^2, b_X^3 \rangle = -\langle b_Y^3, b_X^2 \rangle = -\mu_1 \langle b_Y^1, b_X^2 \rangle = \mu_1 \langle b_Y^2, b_X^1 \rangle,$$
$$\lambda_2 \langle b_Y^1, b_X^2 \rangle = \langle b_Y^1, b_X^3 \rangle = -\langle b_Y^3, b_X^1 \rangle = -\mu_2 \langle b_Y^2, b_X^1 \rangle = \mu_2 \langle b_Y^1, b_X^2 \rangle.$$

As the inner products are $\neq 0$, it follows that $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$. Then, by the duality rule,

$$0 = \langle b_X^3, b_Y^3 \rangle = \underbrace{(\lambda_1 \mu_2 - \lambda_2 \mu_1)}_{=0} \langle b_Y^2, b_X^1 \rangle + \langle b', b'' \rangle = \langle b', b'' \rangle.$$

So b' and b'' span a 2-dimensional totally isotropic subspace in the non-degenerate space W', so this subspace has a 2-dimensional dual and $\dim W' \geq 4$, $\operatorname{wi}(W') \geq 2$, follows. Hence

$$\operatorname{wi}(W_0) = \dim W_{12} + \operatorname{wi}(W') \geq 2 + 2 = 4,$$

and again

$$\operatorname{wi}(\mathbb{R}^{r,s}) = \dim U_0 + \operatorname{wi}(W_0) \geq 3 + 4 = 7,$$

and the theorem is proved. ■

Corollary 2.26 *If M is a flat homogeneous pseudo-Riemannian manifold such that its fundamental group has non-abelian linear holonomy group, then*

$$\dim M \geq 8$$

and the signature (r, s) of M satisfies $r \geq s \geq 4$. Moreover, if M is complete,

$$\dim M \geq 14$$

and the signature satisfies $r \geq s \geq 7$.

3 The Centraliser

In this chapter we will study some properties of the centraliser $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$ of a Wolf group $\boldsymbol{G}$.

3.1 Algebraic Properties

Let $\boldsymbol{G}$ be a Wolf group, let $\boldsymbol{L}$ denote the centraliser of $\boldsymbol{G}$ in $\mathbf{Iso}(\mathbb{R}^{r,s})$ acting with open orbit on $\mathbb{R}^{r,s}$. The unipotent radical of $\boldsymbol{L}$ is denoted by $\boldsymbol{U}$.

Recall that the centraliser $\boldsymbol{L}$ is an algebraic subgroup of $\mathbf{Iso}(\mathbb{R}^{r,s})$.

Proposition 3.1 *The centraliser $\boldsymbol{L}$ acts transitively on $\mathbb{R}^n$ if and only if its unipotent radical $\boldsymbol{U} \subset \boldsymbol{L}$ acts transitively.*

Proof: $\boldsymbol{L}$ can be written as $\boldsymbol{L} = \boldsymbol{H} \cdot \boldsymbol{U}$ for some reductive group $\boldsymbol{H}$ by Theorem G.9. As an affine action of reductive group $\boldsymbol{H}$ has a fixed point on $\mathbb{R}^n$ (Baues [2], Lemma 2.2), $\boldsymbol{U}$ must act transitively on $\mathbb{R}^n$ if $\boldsymbol{L}$ does. ■

Remark 3.2 If $\boldsymbol{L}$ does not act transitively, then its orbit is a proper open subset of $\mathbb{R}^n$. This implies that $\boldsymbol{L}$ is not unipotent, as orbits of unipotent groups are closed (Proposition G.7).

Proposition 3.3 *Assume $\boldsymbol{U}$ acts transitively. As a set, $\mathbb{R}^n$ can be identified with $\boldsymbol{U}/\boldsymbol{U}_p$ for some $p \in \mathbb{R}^n$. Further:*

(a) *The manifolds $\mathbb{R}^n$ and $\boldsymbol{U}/\boldsymbol{U}_p$ are diffeomorphic.*

(b) *The quotient $\boldsymbol{U}/\boldsymbol{U}_p$ exists as an affine algebraic variety, and as such it is isomorphic to $\mathbb{R}^n$.*

Proof:

(a) The diffeomorphism is well-known from the theory of homogeneous spaces (Helgason [21], chapter II, Theorem 3.2 and Proposition 4.3).

(b) That the geometric quotient $\boldsymbol{U}/\boldsymbol{U}_p$ exists as a quasi-projective variety is due to the fact that $\boldsymbol{U}$ is an algebraic group and the stabiliser $\boldsymbol{U}_p$ is a Zariski closed subgroup (Proposition E.16). Because $\boldsymbol{U}$ is unipotent, it is even an affine variety (Proposition G.24). As $\mathbb{R}^n$ is also a geometric quotient for the action of $\boldsymbol{U}_p$ on $\boldsymbol{U}$ (Proposition E.7), it is isomorphic to $\boldsymbol{U}/\boldsymbol{U}_p$ as an affine variety. ■

Lemma 3.4 *The orbit map $\theta_p : \boldsymbol{G} \to \boldsymbol{G}.p$, $g \mapsto g.p$ is an isomorphism of affine algebraic varieties.*

Proof: The map θ_p is a bijective morphism since $\boldsymbol{G}$ acts freely. Because $G.p$ is smooth, we can use the corollary in AG 18.4 in Borel [5] to conclude that θ_p is open, which means its inverse map is a morphism as well. ∎

It is convenient to write the quotient $\mathbb{R}^n/\boldsymbol{G}$ as a homogeneous space of certain unipotent groups, as these are particularly well-behaved objects and many properties can be deduced rather easily. To this end, we make the following definition: Let F_p denote the orbit of $\boldsymbol{G}$ through p.[9)] Set

$$\boldsymbol{U}_{F_p} = \{u \in \boldsymbol{U} \mid u.F_p \subseteq F_p\}. \tag{3.1}$$

The following properties of $\boldsymbol{U}_{F_p}$ rely on the fact that $\boldsymbol{U}$ commutes with $\boldsymbol{G}$.

Proposition 3.5 *$\boldsymbol{U}_{F_p}$ is an algebraic subgroup of $\boldsymbol{U}$, and its action on F_p is transitive.*

Proof:

(i) $\boldsymbol{U}_{F_p}$ is a subgroup of $\boldsymbol{U}$:

- Clearly $I \in \boldsymbol{U}_{F_p}$.
- Let $u_1, u_2 \in \boldsymbol{U}_{F_p}$. Given $g_1 \in \boldsymbol{G}$, we have $u_2.(g_1.p) = g_2.p$ and $u_1.(g_2.p) = g_3.p$ for some $g_2, g_3 \in \boldsymbol{G}$. So $(u_1u_2).(g.p) \in F_p$ for any $g \in \boldsymbol{G}$, hence $u_1u_2 \in \boldsymbol{U}_{F_p}$.
- For $u \in \boldsymbol{U}_{F_p}$, we have

$$p = u^{-1}u.p = u^{-1}g.p = gu^{-1}.p$$

for a certain $g \in \boldsymbol{G}$. Then $u^{-1}.p = g^{-1}.p$, and hence for arbitrary $g' \in \boldsymbol{G}$,

$$u^{-1}.(g'.p) = g'.(u^{-1}.p) = g'g.p \in F_p.$$

So $u^{-1} \in \boldsymbol{U}_{F_p}$.

(ii) Next, we show transitivity: As $\boldsymbol{U}$ acts transitively on $\mathbb{R}^n$, for every $g.p \in F_p$ we find an element $u \in \boldsymbol{U}$ such that $u.p = g.p$. But then, for any other $g' \in \boldsymbol{G}$ we have

$$u.(g'.p) = g'.(u.p) = g'g.p \in F_p,$$

so $u \in \boldsymbol{U}_{F_p}$, and hence $\boldsymbol{U}_{F_p}$ acts transitively.

9) We use this notation because the $\boldsymbol{G}$-orbits will appear as fibres later on.

(iii) It remains to prove that $\boldsymbol{G}$ is algebraic: The previous argument also shows that every $u \in \boldsymbol{U}$ with $u.p \in F_p$ is contained in $\boldsymbol{U}_{F_p}$. This means that $\boldsymbol{U}_{F_p}$ is the preimage of F_p under the orbit map $\theta_p : \boldsymbol{U} \to \mathbb{R}^n$, $u \mapsto u.p$. But $\boldsymbol{G}$ is a unipotent group, so its orbit F_p is Zariski closed (Proposition G.7). Further, θ_p is a morphism, so $\theta_p^{-1}(F_p) = \boldsymbol{U}_{F_p}$ is also Zariski closed. ■

Lemma 3.6 *The stabiliser $\boldsymbol{U}_p$ is a normal subgroup of $\boldsymbol{U}_{F_p}$.*

Proof: Let $u_p \in \boldsymbol{U}_p$, $u \in \boldsymbol{U}_{F_p}$ and let $g \in \boldsymbol{G}$ such that $u^{-1}.p = g.p$. Then

$$uu_pu^{-1}.p = uu_pg.p = ug.(u_p.p) = ug.p = uu^{-1}.p = p.$$

So $uu_pu^{-1} \in \boldsymbol{U}_p$, hence $\boldsymbol{U}_p$ is normalised by $\boldsymbol{U}_{F_p}$. ■

Theorem 3.7 *Fix $p \in \mathbb{R}^n$. For $u\boldsymbol{U}_p \in \boldsymbol{U}_{F_p}/\boldsymbol{U}_p$ let g_u denote the element in $\boldsymbol{G}$ satisfying $u.p = g_u.p$ (as $\boldsymbol{G}$ acts freely, g_u is unique). The map*

$$\Phi : \boldsymbol{U}_{F_p}/\boldsymbol{U}_p \to \boldsymbol{G}, \quad u\boldsymbol{U}_p \mapsto g_u^{-1}$$

is an isomorphism of algebraic groups.

Proof: First, we prove that Φ is an isomorphism of algebraic varieties, then we check that is a homomorphism of groups as well:

(i) By Proposition 3.3 we can identify the elements of $\mathbb{R}^n$ and $\boldsymbol{U}/\boldsymbol{U}_p$. The element $\overline{p} = \boldsymbol{U}_p$ in $\boldsymbol{U}/\boldsymbol{U}_p$ corresponds to p in $\mathbb{R}^n$.

By Lemma 3.6, $\boldsymbol{U}_p$ is a normal closed subgroup of $\boldsymbol{U}_{F_p}$, which itself is a closed subgroup of $\boldsymbol{U}$. By Proposition E.18, the orbit map

$$\varrho_p : \boldsymbol{U}_{F_p}/\boldsymbol{U}_p \to \overline{p}.(\boldsymbol{U}_{F_p}/\boldsymbol{U}_p), \quad u\boldsymbol{U}_p \mapsto \overline{p}.u$$

for the right-action of $\boldsymbol{U}_{F_p}$ on $\boldsymbol{U}/\boldsymbol{U}_p \cong \mathbb{R}^n$ is an algebraic isomorphism. But $\overline{p}.(\boldsymbol{U}_{F_p}/\boldsymbol{U}_p) = F_p$ under the isomorphism from Proposition 3.3, where $\overline{p}.u$ on the left hand side of the equations is identified with $u.p$ on the right hand side of the equation.

As $\boldsymbol{G}$ acts freely, we also have $F_p \cong \boldsymbol{G}$ via the orbit map $\theta_p(g) = g.p$ (Lemma 3.4) with inverse morphism

$$\theta_p^{-1} : F_p \to \boldsymbol{G}, \quad q \mapsto g_q$$

where g_q is the unique element in $\boldsymbol{G}$ with $g_q.p = q$. If $q = u.p$ for some $u \in \boldsymbol{U}_{F_p}$, then $g_q = g_u$ by definition.

Let ι denote the inversion morphism on $\boldsymbol{G}$. Set

$$\Phi = \iota \circ \theta_p^{-1} \circ \varrho_p.$$

Then Φ is an isomorphism of affine varieties, and indeed

$$\Phi(u\boldsymbol{U}_p) = \iota(\theta_p^{-1}(\varrho_p(u\boldsymbol{U}_p))) = \iota(\theta_p^{-1}(\overline{p}.u)) = \iota(\theta_p^{-1}(u.p)) = \iota(g_u) = g_u^{-1}.$$

(ii) Φ is a group homomorphism: For any $u_1, u_2 \in \boldsymbol{U}_{F_p}$

$$g_{u_1u_2}.p = u_1u_2.p = u_1g_{u_2}.p = g_{u_2}u_1.p = g_{u_2}g_{u_1}.p.$$

Then $g_{u_1u_2} = g_{u_2}g_{u_1}$ by the freeness of the $\boldsymbol{G}$-action, that is

$$\Phi(u_1\boldsymbol{U}_pu_2\boldsymbol{U}_p) = g_{u_1u_2}^{-1} = (g_{u_2}g_{u_1})^{-1} = g_{u_1}^{-1}g_{u_2}^{-1} = \Phi(u_1\boldsymbol{U}_p)\Phi(u_2\boldsymbol{U}_p).$$

So Φ is a homomorphism of groups by step (ii), and together with step (i) an isomorphism of algebraic groups. ■

3.2 Matrix Representation of the Centraliser

Let $\boldsymbol{G}, \boldsymbol{L}$ as before, and $\mathfrak{l} = \mathfrak{Lie}(\boldsymbol{L})$. The structure theory for $\boldsymbol{G}$ from chapter 2 allows us to make some statements on the matrix representation of the centraliser $\boldsymbol{L}$.

Recall the Witt decomposition (2.3),

$$\mathbb{R}^{r,s} = U_0 \oplus W_0 \oplus U_0^*,$$

where U_0 is totally isotropic and $\langle \cdot, \cdot \rangle$ is non-degenerate on W_0. Recall also from (2.1) that $\boldsymbol{G}$ acts trivially on U_0.

Proposition 3.8 *Let $S \in \mathfrak{l}$. Then the matrix representation of the linear part of S for the given Witt basis is*

$$\mathrm{L}(S) = \begin{pmatrix} S_1 & -S_2^\top \tilde{I} & S_3 \\ 0 & S_4 & S_2 \\ 0 & 0 & -S_1^\top \end{pmatrix}, \tag{3.2}$$

with $S_2 \in \mathbb{R}^{(n-2k)\times k}$ and $S_3 \in \mathfrak{so}_k$ (where $k = \dim U_0$).

Proof: Recall that $U_G^\perp = \bigcap \ker A_X$ is the subspace of $\mathbb{R}^{r,s}$ on which all of $\mathrm{L}(\boldsymbol{G})$ acts trivially. If $p \in U_G^\perp$ and $l \in \boldsymbol{L}$, then

$$\mathrm{L}(l)p = \mathrm{L}(l)\mathrm{L}(g)p = \mathrm{L}(g)\mathrm{L}(l)p.$$

So $\mathrm{L}(\boldsymbol{L})$ leaves $U_G^\perp$ invariant. But $\mathrm{L}(\boldsymbol{L})$ also leaves $U_{\boldsymbol{G}} = \sum \operatorname{im} A_X$ invariant. So $\mathrm{L}(\boldsymbol{L})$ leaves $U_0 = U_{\boldsymbol{G}} \cap U_G^\perp$ invariant. For $S \in \mathfrak{l}$, it follows that the matrix blocks D and F are 0 in the matrix representation (A.2) of $\mathrm{L}(S)$. ∎

Remark 3.9 In general, the matrix blocks S_1, S_2, S_3, S_4 in (3.2) can not be assumed to be 0. See (10.1) in Main Example 10.

Proposition 3.10 *Given $X \in \log(\boldsymbol{G})$, the linear part A_X of X is uniquely determined by the translation part v_X of X.*

Proof: We may assume that 0 is in the open orbit D of the centraliser $\boldsymbol{L}$ (otherwise conjugate with a translation moving some point $p \in D$ to 0). Then the translation parts of the elements of $\mathfrak{l}$ must span all of $\mathbb{R}^{r,s}$, that is, for every unit vector e_i there exists $S_i \in \mathfrak{l}$ with $\mathrm{T}(S_i) = e_i$. The fact that S_i commutes with $X \in \log(\boldsymbol{G})$ implies

$$A_X e_i = \mathrm{L}(S_i) v_X,$$

where the left-hand side is the ith column of A_X. So all columns of A_X are determined by v_X. ∎

Remark 3.11 The converse of Proposition 3.10 clearly is not true in general, as a Wolf group $\boldsymbol{G}$ consisting only of pure translations shows.

3.3 Remarks on Translationally Isotropic Domains

Definition 3.12 Let D be an open domain in $\mathbb{R}^{r,s}$, and let $T \subset \mathbf{Iso}(\mathbb{R}^{r,s})$ be the set of pure translations leaving D invariant (meaning $T.D = T + D \subset D$). If $T^\perp \subset T$, then D is called **translationally isotropic**.

Throughout this section, $\boldsymbol{\Gamma}$ is a discrete Wolf group and D is an open orbit of the centraliser $\boldsymbol{L} = Z_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{\Gamma})$. In all known examples of this type, the domain D is translationally isotropic.

We shall prove that if $\boldsymbol{\Gamma}$ has abelian holonomy, then D must be translationally isotropic.

Remark 3.13 D being translationally isotropic is equivalent to the condition that $v + D \not\subset D$ implies $v \not\perp T$.

Lemma 3.14 *Let $\mathbf{\Gamma}$ be a Wolf group and $D \subseteq \mathbb{R}^{r,s}$ an open orbit of the centraliser $\mathbf{L}$. Let U be a totally isotropic subspace in $\mathbb{R}^{r,s}$. If $U^\perp \subseteq T$, then D is translationally isotropic.*

Proof: Assume $U^\perp \subseteq T$. If a vector v satisfies $v + D \not\subset D$, then $v \notin U^\perp$. But then $v \not\perp U \subset T$. By Remark 3.13, D is translationally isotropic. ■

Recall once more the Witt decomposition (2.3),

$$\mathbb{R}^{r,s} = U_0 \oplus W_0 \oplus U_0^*,$$

where U_0 is totally isotropic and $\langle\cdot,\cdot\rangle$ is non-degenerate on W_0. Recall also from (2.1) that $\mathbf{\Gamma}$ acts trivially on U_0.

So a consequence of Lemma 3.14 is that D is translationally isotropic if $\mathbf{L}$ contains a subgroup $\mathbf{H}$ such that $\mathbf{H}.p = p + U_0^\perp$ for all $p \in D$.

Lemma 3.15 *Let $\mathbf{\Gamma}$ be a Wolf group acting on $\mathbb{R}^{r,s}$, and identify U_0 with the group of translations by vectors in U_0. Then $U_0 \subset \mathbf{L} \cap T$.*

Proof: A translation by $u \in U_0$ is represented by (I, u). If $(I+A, v) \in \mathbf{\Gamma}$, then

$$(I + A, v)(I, u) = (I + A, u + Au + v) = (I + A, u + v) = (I, u)(I + A, v),$$

where we used the fact that $U_0 \subset \ker A$, see (2.2). So $(I, u) \in \mathbf{L}$ and thus (I, u) is a translation leaving $\mathbf{L}$-orbits invariant, meaning $(I, u) \in \mathbf{L} \cap T$. ■

Lemma 3.16 $\mathrm{hol}(\mathbf{\Gamma})$ *is abelian if and only if $U_0^\perp \subseteq \mathbf{L}$. If this holds, then D is translationally isotropic.*

Proof: Let $u \in U_0^\perp$. Then

$$(I + A, v)(I, u) = (I + A, u + Au + v) = (I + A, u + v) = (I, u)(I + A, v)$$

for all $(I + A, v) \in \mathbf{\Gamma}$ if and only if $Au = 0$ for all $(I + A, v) \in \mathbf{\Gamma}$. But this is equivalent to $u \in \bigcap_A \ker A = U_{\mathbf{\Gamma}}^\perp \subset U_0^\perp$, which again is equivalent to the linear holonomy of $\mathbf{\Gamma}$ being abelian by Proposition 1.23.

In this case, D is translationally isotropic by Lemma 3.14. ■

The previous lemma immediately implies:

Theorem 3.17 *Let $M = D/\mathbf{\Gamma}$ be a flat pseudo-Riemannian homogeneous manifold, where $D \subseteq \mathbb{R}^{r,s}$ is an open orbit of the centraliser $\mathbf{L}$ of $\mathbf{\Gamma}$ in* $\mathbf{Iso}(\mathbb{R}^{r,s})$*. If $\mathbf{\Gamma}$ has abelian linear holonomy, then D is a translationally isotropic domain.*

So for certain classes of examples one can check if there exists a subgroup $\boldsymbol{H} \subset \boldsymbol{L}$ containing $p + U_0^{\perp}$ in its orbit for all $p \in D$. This is the case in all known examples. Even in Main Example 10, where $\boldsymbol{\Gamma}$ has non-abelian linear holonomy, $\boldsymbol{H}$ is the group generated by the exponentials of elements S in (10.1), where $z = 0$ and $a = d = 0$.

4 Compact Flat Homogeneous Spaces

The results in this chapter are mostly due to Baues [2], chapters 4 and 5. All proofs which are omitted here can be found there. We diverge from our usual notation in order to match that of Baues [2]. Main Example 9 is an example of the class of spaces discussed here. It illustrates some of their properties in more detail than we do in this section.

4.1 Compact Flat Pseudo-Riemannian Homogeneous Spaces

A theorem due to Marsden states that every compact homogeneous pseudo-Riemannian manifold is complete (O'Neill [30], chapter 9, Proposition 39). So every flat compact pseudo-Riemannian homogeneous space M is a quotient $M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$ for some group $\mathbf{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{r,s})$ which acts freely and properly discontinuously on $\mathbb{R}^{r,s}$.

Theorem 4.1 *Let M be a compact flat pseudo-Riemannian homogeneous manifold. Then M is isometric to a quotient of a flat pseudo-Riemannian Lie group $\mathbf{N}$ with bi-invariant metric.*

Here, $\mathbf{N}$ acts simply transitively on $\mathbb{R}^{r,s}$. If $M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$, then $\mathbf{N}$ centralises $\mathbf{\Gamma}$. If δ denotes the development representation of the right-multiplication of $\mathbf{N}$, then $\mathbf{\Gamma} = \delta(\mathbf{\Lambda})$ for some lattice $\mathbf{\Lambda} \subset \mathbf{N}$. Fix a base point p in $\mathbb{R}^{r,s}$. Then the orbit map $\theta : \mathbf{N} \to \mathbb{R}^{r,s}$, $n \mapsto \delta(n).p$ is an isometry, and for all $\lambda \in \mathbf{\Lambda}$, $n \in \mathbf{N}$,

$$\theta(n\lambda) = \delta(\lambda).\theta(n).$$

So θ induces an isometry $\mathbf{N}/\mathbf{\Lambda} \to \mathbb{R}^{r,s}/\mathbf{\Gamma}$.

Remark 4.2 The converse to Theorem 4.1 also holds: If $\mathbf{N}$ is endowed with a bi-invariant flat pseudo-Riemannian metric and $\mathbf{\Lambda} \subset \mathbf{N}$ a discrete subgroup, then $\mathbf{N}/\mathbf{\Lambda}$ is a flat pseudo-Riemannian homogeneous manifold.

Lemma 4.3 *Let $M = \mathbf{N}/\mathbf{\Lambda}$ be a compact flat pseudo-Riemannian homogeneous manifold. Then $\mathbf{N}$ is a 2-step nilpotent Lie group.*

Proof: Since the metric is bi-invariant, it follows from Corollary 10, chapter 11 in O'Neill [30] that the curvature tensor of $\mathbf{N}$ is of the form

$$\mathsf{R}(X,Y)Z = \frac{1}{4}[X,[Y,Z]]$$

for $X, Y, Z \in \mathfrak{n} = \mathfrak{Lie}(N)$. As M is flat, it follows that $[[X, Y], Z]] = 0$ for all $X, Y, Z \in \mathfrak{n}$. ∎

The following theorem was first proved in Baues and Globke [3]:

Theorem 4.4 *Let $\mathbf{N}$ be a Lie group endowed with a bi-invariant flat pseudo-Riemannian metric, and let $\mathbf{\Lambda}$ be a lattice in $\mathbf{N}$. Then the compact flat pseudo-Riemannian homogeneous manifold $\mathbf{N}/\mathbf{\Lambda}$ has abelian linear holonomy.*

Proof: Let $\mathfrak{n}$ denote the Lie algebra of $\mathbf{N}$, and δ the development representation of $\mathbf{N}$ at a fixed base point $p \in \mathbb{R}^{r,s}$.

The development representation δ' of $\mathfrak{n}$ is the differential of δ at the identity. By (1.5) and Proposition 1.22, δ' is equivalent to the affine representation

$$X \mapsto \Big(\frac{1}{2}\mathrm{ad}(X), X_p\Big),$$

on the vector space $\mathfrak{n} \cong \mathbb{R}^{r,s}$. In particular, the linear part of δ' is equivalent to the adjoint representation ad of $\mathfrak{n}$. Since $\mathfrak{n}$ is 2-step nilpotent, the adjoint representation ad has abelian image. It follows that the linear part $\mathrm{L}(\delta(\mathbf{N}))$ is abelian. Since $\mathbf{\Gamma} \subset \delta(\mathbf{N})$, this implies that $\mathrm{L}(\mathbf{\Gamma})$ is abelian. ∎

Remark 4.5 We identify $\mathfrak{n} \cong \mathbb{R}^{r,s}$ via the differential of the orbit map θ. Then $\gamma \in \mathbf{\Gamma} = \delta(\mathbf{\Lambda})$ has the form

$$\gamma = (I + A, v) = \Big(I + \frac{1}{2}\mathrm{ad}(X), X_p\Big),$$

for some $X \in \mathfrak{n}$ with $\delta(\exp(X)) = \gamma$. The space $U_{\mathbf{\Gamma}}$ from (1.7) is

$$U_{\mathbf{\Gamma}} = \sum_{(I+A,v)\in\mathbf{\Gamma}} \mathrm{im}\, A = \sum_{X\in\log(\mathbf{\Lambda})} \mathrm{im}\,\mathrm{ad}(X) = [\mathfrak{n}, \mathfrak{n}],$$

the commutator subalgebra (taken as a linear subspace) of $\mathfrak{n}$. Using bi-invariance and 2-step nilpotency, one can show $U_{\mathbf{\Gamma}}$ is totally isotropic. By Proposition 1.23, this shows again that $\mathrm{L}(\mathbf{\Gamma})$ is abelian.

From Theorems 4.1 and 4.4, we obtain the following corollary:

Corollary 4.6 *Let $M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$ be a compact flat pseudo-Riemannian homogeneous space. Then* $\mathrm{hol}(\mathbf{\Gamma})$ *is abelian.*

4.2 Lie Algebras with Bi-Invariant Metric

A bi-invariant pseudo-Riemannian metric on a Lie group N is uniquely determined by a bi-invariant pseudo-scalar product $\langle\cdot,\cdot\rangle$ on its Lie algebra $\mathfrak{n}$. The metric is flat if and only if $\mathfrak{n}$ is 2-step nilpotent.

In this section we state a structure theorem for these Lie algebras due to Baues [2] (Theorem 5.15).

We start by giving a construction method for 2-step nilpotent Lie algebras with bi-invariant metric: Let $\mathfrak{a}$ be an abelian Lie algebra and $\mathfrak{a}^*$ its dual vector space. For any alternating bilinear map $\omega : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$, we obtain a Lie product on the space $\mathfrak{a} \oplus \mathfrak{a}^*$ by setting

$$[(x, x^*), (y, y^*)] = (0, \omega(x, y)) \tag{4.1}$$

for all $x, y \in \mathfrak{a}$ and $x^*, y^* \in \mathfrak{a}^*$. We denote the corresponding Lie algebra by

$$\mathfrak{n} = \mathfrak{a} \oplus_\omega \mathfrak{a}^*.$$

Clearly $\mathfrak{n}$ is 2-step nilpotent.

Let $m = \dim \mathfrak{a}$. An inner product of signature (m, m) on $\mathfrak{n}$ is given by

$$\langle (x, x^*), (y, y^*) \rangle = x^*(y) + y^*(x) \tag{4.2}$$

for $x, y \in \mathfrak{a}$ and $x^*, y^* \in \mathfrak{a}^*$.

Define a trilinear form $\tau_\omega : \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a} \to \mathbb{R}$ by

$$\tau_\omega(x, y, z) = \omega(x, y)(z). \tag{4.3}$$

A direct computation shows:

Lemma 4.7 *The inner product (4.2) is bi-invariant if and only if the trilinear form τ_ω is alternating.*

Theorem 4.8 (Baues) *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with bi-invariant inner product $\langle\cdot,\cdot\rangle$. Then there exists an abelian Lie algebra $\mathfrak{a}$, an alternating trilinear form τ_ω on $\mathfrak{a}$ and an abelian Lie algebra $\mathfrak{z}_0$ such that $\mathfrak{n}$ can be written as a direct product of metric Lie algebras*

$$\mathfrak{n} = (\mathfrak{a} \oplus_\omega \mathfrak{a}^*) \oplus \mathfrak{z}_0. \tag{4.4}$$

Proof: For all $X = [X_1, X_2], Y = [Y_1, Y_2] \in [\mathfrak{n}, \mathfrak{n}]$,

$$\langle X, Y \rangle = \langle [X_1, X_2], [Y_1, Y_2] \rangle = \langle [[X_1, X_2], Y_1], Y_2 \rangle = \langle 0, Y_2 \rangle = 0,$$

the second equality follows from the bi-invariance, and the third follows from the 2-step nilpotency. So $[\mathfrak{n}, \mathfrak{n}]$ is a totally isotropic subspace of $\mathfrak{n}$.

Bi-invariance shows that its orthogonal complement $[\mathfrak{n}, \mathfrak{n}]^\perp$ is the centre $\mathfrak{z}(\mathfrak{n})$. Let $\mathfrak{a}$ denote a totally isotropic subspace dual to $[\mathfrak{n}, \mathfrak{n}]$ in $\mathfrak{n}$ (then $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{a}^*$). Finally, let $\mathfrak{z}_0$ be a vector space complement of $\mathfrak{a}^*$ in $\mathfrak{z}(\mathfrak{n})$, that is

$$\mathfrak{z}(\mathfrak{n}) = \mathfrak{a}^* \oplus \mathfrak{z}_0.$$

Then $\mathfrak{z}_0$ commutes with and is orthogonal to $\mathfrak{a}$ and $\mathfrak{a}^*$. So

$$\mathfrak{n} = (\mathfrak{a} \oplus_\omega \mathfrak{a}^*) \oplus \mathfrak{z}_0$$

for some alternating bilinear map $\omega : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$. ■

5 Orbits of Wolf Groups

Throughout this chapter, let Γ be a discrete Wolf group with Zariski closure $\boldsymbol{G}$, let $\boldsymbol{L}$ denote the centraliser of $\boldsymbol{G}$ in $\mathbf{Iso}(\mathbb{R}^{r,s})$, and let $\mathfrak{g} = \mathfrak{Lie}(\boldsymbol{G})$.

5.1 The Orbits

Let F_p denote the orbit $\boldsymbol{G}.p$ for $p \in \mathbb{R}^n$.

Proposition 5.1 *Let $X_1, \ldots, X_k$ be a Malcev basis of $\mathfrak{g}$, with $X_i = (A_i, v_i)$. For every $p \in \mathbb{R}^n$, set $b_i(p) = A_i p + v_i$. Then the orbit F_p is the affine subspace*

$$F_p = p + \operatorname{span}\{b_1(p), \ldots, b_k(p)\} \tag{5.1}$$

of dimension $\dim F_p = \dim \boldsymbol{G} - \dim \boldsymbol{G}_p$. In particular, if p lies in an open orbit of the centraliser $\boldsymbol{L}$ of $\boldsymbol{G}$, then $\dim F_p = \dim \boldsymbol{G}$.

Proof: In exponential coordinates, every element $g \in \boldsymbol{G}$ is written as

$$g = g(t_1, \ldots, t_k) = \exp(t_1 X_1 + \ldots + t_k X_k)$$

for unique parameters t_i. As $\boldsymbol{G}$ is a Wolf group, $\exp(A, v) = (I + A, v)$ for all $(A, v) \in \mathfrak{g}$. So

$$g(t_1, \ldots, t_k) = (I + t_1 A_1 + \ldots + t_k A_k,\ t_1 v_1 + \ldots + t_k v_k).$$

Then

$$\begin{aligned} g(t_1, \ldots, t_k).p &= p + t_1(A_1 p + v_1) + \ldots + t_k(A_k p + v_k) \\ &= p + t_1 b_1(p) + \ldots + t_k b_k(p) \end{aligned}$$

is affine with respect to the t_i. Varying the t_i through all values of $\mathbb{R}$ shows that the orbit through p is an affine subspace.

The assertions on the dimensions are standard results, taking into account that $\boldsymbol{G}$ acts freely on an open orbit of $\boldsymbol{L}$. ∎

5.2 An Algebraic Principal Bundle

In this section, let $\dim \boldsymbol{G} = k$, and let $\boldsymbol{U}$ denote the unipotent radical of the centraliser $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$, and let F_p denote the orbit $\boldsymbol{G}.p$.

Assume the centraliser $\boldsymbol{L}$ acts transitively on $\mathbb{R}^n$ (then so does $\boldsymbol{U}$), which implies $\boldsymbol{G}$ acts freely on $\mathbb{R}^n$. We will show that $\mathbb{R}^n$ is a trivial principal fibre bundle with structure group $\boldsymbol{G}$.

Recall from Proposition 3.3 that there exists an isomorphism of affine varieties

$$\Psi : \boldsymbol{U}/\boldsymbol{U}_p \to \mathbb{R}^n,$$

and from Theorem 3.7 that there exists an isomorphism of algebraic groups

$$\Phi : \boldsymbol{U}_{F_p}/\boldsymbol{U}_p \to \boldsymbol{G}.$$

We write $\tilde{\boldsymbol{U}} = \boldsymbol{U}_{F_p}/\boldsymbol{U}_p$.

Lemma 5.2 Ψ *induces a bijection from the orbits of the right-action of* $\tilde{\boldsymbol{U}}$ *on* $\boldsymbol{U}/\boldsymbol{U}_p$ *to the orbits of* $\boldsymbol{G}$ *on* $\mathbb{R}^n$.

Proof: Let $\tilde{u} \in \tilde{\boldsymbol{U}}$ and $g = \Phi(\tilde{u})$. For any $u \in \boldsymbol{U}$, $\Psi(u\boldsymbol{U}_p) = u.p \in \mathbb{R}^n$, and $u\boldsymbol{U}_p.\tilde{u} = u\tilde{u}\boldsymbol{U}_p$ maps to $\Psi(u\tilde{u}\boldsymbol{U}_p) = u\tilde{u}.p$.

By definition of Φ (Theorem 3.7) and because $\boldsymbol{G}$ and $\boldsymbol{U}$ commute,

$$u\tilde{u}.p = ug^{-1}.p = g^{-1}u.p.$$

So Ψ maps the orbit $u\boldsymbol{U}_p.\tilde{\boldsymbol{U}}$ to the orbit $\boldsymbol{G}.(u.p)$.

If $u_1\boldsymbol{U}_p.\tilde{\boldsymbol{U}}$ and $u_2\boldsymbol{U}_p.\tilde{\boldsymbol{U}}$ are disjoint $\tilde{\boldsymbol{U}}$-orbits, then the bijectivity of Ψ and the above calculation show that $\boldsymbol{G}.(u_1.p)$ and $\boldsymbol{G}.(u_2.p)$ are also disjoint. ■

Lemma 5.3 $\Psi^* : \mathcal{O}(\mathbb{R}^n) \to \mathcal{O}(\boldsymbol{U}/\boldsymbol{U}_p)$ *induces an isomorphism from the* $\boldsymbol{G}$-*invariant regular functions on* $\mathbb{R}^n$ *to the* $\tilde{\boldsymbol{U}}$-*invariant regular functions on* $\boldsymbol{U}/\boldsymbol{U}_p$.

Proof: Let $\mathcal{O}(\mathbb{R}^n)^{\boldsymbol{G}}$ and $\mathcal{O}(\boldsymbol{U}/\boldsymbol{U}_p)^{\tilde{\boldsymbol{U}}}$ denote the respective subrings of invariants.

(i) Let $f \in \mathcal{O}(\mathbb{R}^n)^{\boldsymbol{G}}$. For any $u \in \boldsymbol{U}$, $\tilde{u} \in \tilde{\boldsymbol{U}}$ with $g = \Phi(\tilde{u}) \in \boldsymbol{G}$ we have

$$\begin{aligned}(\Psi^* f)(u\boldsymbol{U}_p.\tilde{u}) &= f(\Psi(u\tilde{u}\boldsymbol{U}_p)) \\ &= f(u\tilde{u}.p) = f(ug^{-1}.p) \\ &= f(g^{-1}.(u.p)) = f(u.p) = (\Psi^* f)(u\boldsymbol{U}_p).\end{aligned}$$

So $\Psi^* f \in \mathcal{O}(\boldsymbol{U}/\boldsymbol{U}_p)^{\tilde{\boldsymbol{U}}}$.

(ii) Conversely, let $h \in \mathcal{O}(\boldsymbol{U}/\boldsymbol{U}_p)^{\tilde{\boldsymbol{U}}}$. As Ψ^* is an isomorphism, there is $f \in \mathcal{O}(\mathbb{R}^n)$ such that $\Psi^* f = h$. Let $g \in \boldsymbol{G}$, $\tilde{u} = \Phi^{-1}(g)$ and $q = u.p \in \mathbb{R}^n$ for some $u \in \boldsymbol{U}$. Then, by assumption on h,

$$\begin{aligned} f(q) = f(u.p) = (\Psi^* f)(u\boldsymbol{U}_p) = h(u\boldsymbol{U}_p) &= h(u\boldsymbol{U}_p.\tilde{u}) \\ &= (\Psi^* f)(u\boldsymbol{U}_p.\tilde{u}) = f(u\tilde{u}.\boldsymbol{U}_p) \\ &= f(g^{-1}u.\boldsymbol{U}_p) = f(g^{-1}.q). \end{aligned}$$

As g and q were arbitrary, it follows that $f \in \mathcal{O}(\mathbb{R}^n)^{\boldsymbol{G}}$.

That the correspondence is an isomorphism follows from the fact that its a restriction of the isomorphism Ψ^* to subrings. ∎

Lemma 5.4 *$\boldsymbol{U}/\boldsymbol{U}_{F_p}$ is a geometric quotient for the action of $\boldsymbol{G}$ on $\mathbb{R}^n$.*

Proof: $\boldsymbol{U}/\boldsymbol{U}_{F_p}$ is an algebraic homogeneous space for a unipotent group. By Rosenlicht's Theorem G.27, $\boldsymbol{U}/\boldsymbol{U}_{F_p}$ is algebraically isomorphic to an affine space. Further, $\boldsymbol{U}/\boldsymbol{U}_{F_p} = (\boldsymbol{U}/\boldsymbol{U}_p)/(\boldsymbol{U}_{F_p}/\boldsymbol{U}_p) = (\boldsymbol{U}/\boldsymbol{U}_p)/\tilde{\boldsymbol{U}}$, so $\dim \boldsymbol{U}/\boldsymbol{U}_{F_p} = \dim \boldsymbol{U}/\boldsymbol{U}_P - \dim \tilde{\boldsymbol{U}} = \dim \mathbb{R}^n - \dim \boldsymbol{G} = n - k$. Let $\pi_0 : \boldsymbol{U}/\boldsymbol{U}_p \to \boldsymbol{U}/\boldsymbol{U}_{F_p}$ denote the quotient map. So we have morphisms

$$\begin{array}{ccc} \boldsymbol{U}/\boldsymbol{U}_p & \xleftarrow{\Psi^{-1}} & \mathbb{R}^n \\ \downarrow{\scriptstyle \pi_0} & \swarrow{\scriptstyle \pi} & \\ \boldsymbol{U}/\boldsymbol{U}_{F_p} & & \\ \| & & \\ \mathbb{R}^{n-k} & & \end{array}$$

where we define $\pi = \pi_0 \circ \Psi^{-1}$. This is a quotient map:

- Since Ψ is an isomorphism and π_0 a quotient map, the map π is a surjective and open morphism.
- Let $V \subset \mathbb{R}^n$ be a Zariski-open subset, and let $W = \Psi^{-1}(V)$. By Lemma G.10, every rational $\boldsymbol{G}$-invariant function on V is a quotient of two $\boldsymbol{G}$-invariant polynomials, and the analogous statement holds for $\tilde{\boldsymbol{U}}$-invariant functions on W. Then it follows from Lemma 5.3 that $\mathbb{C}[W]^{\tilde{\boldsymbol{U}}} \cong \mathbb{C}[V]^{\boldsymbol{G}}$. Now

$$\mathbb{C}[V]^{\boldsymbol{G}} \cong \mathbb{C}[W]^{\tilde{\boldsymbol{U}}} \cong \mathbb{C}[\pi_0(W)] = \mathbb{C}[\pi(V)],$$

  where the isomorphism in the middle comes from π_0 being a quotient map.

- Let $\overline{q} = u\boldsymbol{U}_{F_p} \in \boldsymbol{U}/\boldsymbol{U}_{F_p}$, where $q = u.p$. Then the fibre of π over $\overline{q}$ is
$$\pi^{-1}(\overline{q}) = \Psi(\pi_0^{-1}(\overline{q})) = \Psi(u\boldsymbol{U}_p.\boldsymbol{U}_{F_p}) = \boldsymbol{G}.(u.p) = F_q,$$
the orbit of $\boldsymbol{G}$ through q. We used Lemma 5.2 for the third equality, and the second equality holds because π_0 is a quotient map.

So π satisfies the properties of Definition E.3, hence $\boldsymbol{U}/\boldsymbol{U}_{F_p}$ is a geometric quotient for the $\boldsymbol{G}$-action. ■

We will write $\mathbb{R}^n/\boldsymbol{G}$ for $\boldsymbol{U}/\boldsymbol{U}_{F_p}$.

Theorem 5.5 *Assume that* $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{r,s})}(\boldsymbol{G})$ *acts transitively on* $\mathbb{R}^n$. *The orbit space* $\mathbb{R}^n/\boldsymbol{G}$ *is isomorphic to* $\mathbb{R}^{n-k}$ *as an affine algebraic variety, and there exists an algebraic cross section* $\sigma : \mathbb{R}^n/\boldsymbol{G} \to \mathbb{R}^n$. *In particular,* $\mathbb{R}^n$ *is algebraically isomorphic to the trivial principal bundle*
$$\boldsymbol{G} \to \mathbb{R}^n \to \mathbb{R}^{n-k}. \tag{5.2}$$

Proof: $\mathbb{R}^n/\boldsymbol{G}$ is algebraically isomorphic to $\mathbb{R}^{n-k}$ by Lemma 5.4. We show $\pi : \mathbb{R}^n \to \mathbb{R}^n/\boldsymbol{G} \cong \mathbb{R}^{n-k}$ is a principal bundle for $\boldsymbol{G}$: $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ are smooth (hence normal) varieties. By a theorem of Rosenlicht (see Corollary G.26), $\mathbb{R}^{n-k}$ can be covered by open sets W such that on each W there exists a local cross section $\sigma_W : W \to \mathbb{R}^n$, and π is a locally trivial fibration.

The $\boldsymbol{G}$-action is principal (Definition E.2), so for any $p \in \mathbb{R}^n$ and $g \in \boldsymbol{G}$, the map $\beta(g.p, p) = g$ is a morphism. Thus the bundle's coordinate changes are morphisms and the bundle is algebraic.

Now the claim follows from Theorem G.31. ■

As a consequence of (5.2) we immediately obtain the following theorem:

Theorem 5.6 *Let* $M = \mathbb{R}^{r,s}/\boldsymbol{\Gamma}$ *be a complete flat pseudo-Riemanninan homogeneous manifold, and let* $n = r + s$, $k = \operatorname{rk}\boldsymbol{\Gamma}$. *Then* M *is diffeomorphic to a trivial fibre bundle*
$$\boldsymbol{G}/\boldsymbol{\Gamma} \to M \to \mathbb{R}^{n-k}. \tag{5.3}$$

5.3 The Affine and Metric Structure on the Orbits

Let $\boldsymbol{G} \subset \mathbf{Iso}(\mathbb{R}^{r,s})$ be a Zariski closed Wolf group and $\mathfrak{g}$ its Lie algebra. We study the affine structure on the orbits $F_p = \boldsymbol{G}.p$ of the $\boldsymbol{G}$-action on $\mathbb{R}^{r,s}$, where p is contained in the open orbit D of the centraliser of $\boldsymbol{G}$. As usual, let $n = r + s$, $k = \dim \boldsymbol{G}$ and $\mathfrak{g} = \mathfrak{Lie}(\boldsymbol{G})$.

Recall that F_p is an affine subspace of $\mathbb{R}^{r,s}$(Proposition 5.1). Since $\boldsymbol{G}$ acts freely, the natural affine connection ∇ on F_p pulls back to a flat affine connection ∇ on $\boldsymbol{G}$ through the orbit map.

Because $X^2 = 0$ for all $X \in \mathfrak{g} \subset \mathrm{Mat}_{n+1}(\mathbb{R})$, $\exp(X) = I + X$. So $\boldsymbol{G} = I + \mathfrak{g}$ is an affine subspace of $\mathrm{Mat}_{n+1}(\mathbb{R})$ which therefore has a natural affine connection $\nabla^{\boldsymbol{G}}$. This connection is left-invariant because left-multiplication is linear on $\mathrm{Mat}_{n+1}(\mathbb{R})$. The orbit map $\theta : \boldsymbol{G} \to F_p$, $I + X \mapsto (I + X).p$ is an affine map (if one chooses $I \in \boldsymbol{G}$ and $p \in F_p$ as origins, the linear part of θ is X and the translation part is $+p$). It is also a diffeomorphism onto F_p because the action is free and exp is a diffeomorphism.

From the above we immediately obtain:

Corollary 5.7 $(\boldsymbol{G}, \nabla^{\boldsymbol{G}})$ *is affinely diffeomorphic to* $(\boldsymbol{G}, \nabla)$.

Corollary 5.8 *If two Wolf groups* $\boldsymbol{G}$ *and* $\boldsymbol{G}'$ *are isomorphic as Lie groups, then* $(\boldsymbol{G}, \nabla)$ *and* $(\boldsymbol{G}', \nabla')$ *are affinely isomorphic.*

Recall that if $\boldsymbol{G}$ is a Wolf group, then $XY = \frac{1}{2}[X, Y]$ for all $X, Y \in \mathfrak{g}$. So we have an associative product on $\mathfrak{g}$ satisfying $XY - YX = [X, Y]$. There exists a bi-invariant flat affine connection $\tilde{\nabla}$ on $\boldsymbol{G}$ given by

$$\tilde{\nabla}_X Y = \frac{1}{2}[X, Y], \tag{5.4}$$

where X, Y are left-invariant vector fields on $\boldsymbol{G}$.[10]. In fact, $\tilde{\nabla}$ is bi-invariant because $[X, Y]$ is $\mathrm{Ad}(g)$-invariant, and it is flat because $\mathfrak{g}$ is 2-step nilpotent.

Proposition 5.9 *The bi-invariant flat affine connection* $\tilde{\nabla}$ *on* $\boldsymbol{G}$ *coincides with the flat affine connection* ∇ *on* $\boldsymbol{G}$.

Proof: As both connections are left-invariant, is suffices to show that they coincide on left-invariant vector fields. Expressed in matrix terms, left-invariance for vector fields means $X_g = gX_I$ for all $X \in \mathfrak{g}$, $g \in \boldsymbol{G}$. So for all $X, Y \in \mathfrak{g}$,

$$\begin{aligned}(\nabla_X Y)_g &= \lim_{t\to 0} \frac{Y_{g\exp(tX)} - Y_g}{t} = \lim_{t\to 0} \frac{g(I + tX_I)Y_I - gY_I}{t} \\ &= \lim_{t\to 0} gX_I Y_I = gX_I Y_I = (XY)_g \\ &= (\tilde{\nabla}_X Y)_g,\end{aligned}$$

[10] The connection $\tilde{\nabla}$ is sometimes called the *(0)-connection*.

where the first and last equality hold by definition. ∎

The metric $\langle\cdot,\cdot\rangle$ on the fibre F_p pulls back to a field $(\cdot,\cdot)$ of (possibly degenerate) left-invariant symmetric bilinear forms on $\boldsymbol{G}$ which is parallel with respect to ∇. By abuse of language, we call $(\cdot,\cdot)$ the **fibre metric** on $\boldsymbol{G}$. Since all fibres are isometric, the pair $(\nabla,(\cdot,\cdot))$ does not depend on p and is an invariant of $\boldsymbol{G} \subset \mathbf{Iso}(\mathbb{R}^{r,s})$.

Let $X \in \mathfrak{g}$ and let X^+ denote the Killing field on $\mathbb{R}^{r,s}$ with flow $\exp(tX).p$ at $p \in \mathbb{R}^{r,s}$. The pulled back vector field on $\boldsymbol{G}$ is also denoted by X^+. It is a right-invariant vector field on $\boldsymbol{G}$.

Proposition 5.10 *The fibre metric* $(\cdot,\cdot)$ *is a bi-invariant metric on* $\boldsymbol{G}$*, that is*

$$([X,Y],Z) = -\,(Y,[X,Z])$$

for all left-invariant vector fields $X,Y,Z \in \mathfrak{g}$.

Proof: Fix $g \in \boldsymbol{G}$. Let $X,Y,Z \in \mathfrak{g}$ be a left-invariant vector fields on $\boldsymbol{G}$ and let X^+,Y^+,Z^+ the right-invariant vector fields on $\boldsymbol{G}$ such that $X^+_g = X_g$, $Y^+_g = Y_g$, $Z^+_g = Z_g$. Let ψ_t denote the flow of X^+ at g. Then $\psi_t(g) = \exp(tX).g$. Because Y is left-invariant,

$$(\mathscr{L}_{X^+}Y)_g = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}\psi_{-t} Y_{\psi_t(g)} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}L_{\exp(-tX)} Y_{\exp(tX).g} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} Y_g = 0.$$

This implies

$$(\nabla_{X^+}Y)_g = (\nabla_Y X^+)_g.$$

$\nabla_X Y$ is tensorial in X, so

$$(\nabla_X Y)_g = (\nabla_{X^+}Y)_g = (\nabla_Y X^+)_g = (\nabla_{Y^+}X^+)_g.$$

X^+,Y^+ are pullbacks of Killing fields on F_p. So Proposition 1.10 (a) gives $\nabla_{Y^+}X^+ = -\nabla_{X^+}Y^+$. Then

$$(\nabla_X Y)_g = -(\nabla_{X^+}Y^+)_g.$$

Now it follows from (5.4) and the computations above that

$$\begin{aligned}
(2[X,Y],Z)_g + (Y,2[X,Z])_g &= (\nabla_X Y, Z^+)_g + (Y^+,\nabla_X Z)_g \\
&= (-\nabla_Y X, Z^+)_g + (Y^+,-\nabla_Z X)_g \\
&= (\nabla_{Y^+}X^+, Z^+)_g + (Y^+,\nabla_{Z^+}X^+)_g \\
&= -(\mathsf{A}_{X^+}Y^+, Z^+)_g - (Y^+,\mathsf{A}_{X^+}Z^+)_g \\
&= 0
\end{aligned}$$

$(\cdot,\cdot)$ is a tensor, so we can replace Z by Z^+ and Y by Y^+. The last equality holds because the tensor A_{X^+} is skew-symmetric with respect to $(\cdot,\cdot)$.

As g was arbitrary, $([X,Y],Z) = -(Y,[X,Z])$ holds everywhere. So $(\cdot,\cdot)$ is bi-invariant. ∎

The fibre metric $(\cdot,\cdot)$ on $\boldsymbol{G}$ induces an invariant symmetric bilinear form on $\mathfrak{g}$ which we also denote by $(\cdot,\cdot)$.

Definition 5.11 The **radical** of $(\cdot,\cdot)$ in $\mathfrak{g}$ is the subspace

$$\mathfrak{r} = \{X \in \mathfrak{g} \mid (X,\mathfrak{g}) = \{0\}\}.$$

Remark 5.12 The radical $\mathfrak{r}$ is an ideal due to the invariance of $(\cdot,\cdot)$.

Lemma 5.13 *The commutator subalgebra* $[\mathfrak{g},\mathfrak{g}]$ *is a totally isotropic subspace of* $\mathfrak{g}$ *with respect to* $(\cdot,\cdot)$. *The centre* $\mathfrak{z}(\mathfrak{g})$ *is orthogonal to* $[\mathfrak{g},\mathfrak{g}]$.

Proof: $\mathfrak{g}$ is 2-step nilpotent. So $([X_1,X_2],[X_3,X_4]) = (X_2,[X_1,[X_3,X_4]]) = (X_2,0) = 0$ for all $X_i \in \mathfrak{g}$. If $Z \in \mathfrak{z}(\mathfrak{g})$, then $(Z,[X_1,X_2]) = -([X_1,Z],X_2) = 0$. ∎

Corollary 5.14 *Assume there exists* $Z \in [\mathfrak{g},\mathfrak{g}]$ *and* $Z^* \in \mathfrak{g}$ *such that* $(Z,Z^*) \neq 0$. *Then* $Z^* \notin \mathfrak{z}(\mathfrak{g})$.

Lemma 5.15 *If* $Z = [X,Y]$, *then* $Z \perp \operatorname{span}\{X,Y,Z\}$.

Proof: Use invariance and 2-step nilpotency. ∎

Theorem 5.16 *Let* $\boldsymbol{\Gamma} \cong \mathbf{H}_3(\mathbb{Z})$ *with Zariski closure* $\boldsymbol{G} \cong \mathbf{H}_3$. *Assume* $M = \mathbb{R}^{r,s}/\boldsymbol{\Gamma}$ *is a flat pseudo-Riemannian homogeneous manifold. The fibre metric induced on* $\mathbf{G}$ *is degenerate, and* $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{r}$. *The possible signatures are* $(0,0,3)$, $(1,0,2)$, $(1,1,1)$ *and* $(2,0,1)$. *The following Example 5.17 shows that all these cases can occur.*

Proof: Let X, Y denote the Lie algebra generators of $\mathfrak{h}_3$ and $Z = [X,Y]$. By Lemma 5.15, $Z \in \mathfrak{r}$. So the positive definite case is excluded. ∎

Example 5.17 In Main Example 11 $\boldsymbol{\Gamma} \cong \mathbf{H}_3(\mathbb{Z})$. Consider the elements $\gamma_i = (I + A_i, v_i)$, $i = 1,2,3$, as given in section 12.1. The fibre $F_0 = \boldsymbol{G}.0$ is $F_0 = \operatorname{span}\{v_1,v_2,v_3\}$. This is a totally isotropic subspace, so the signature of $(\cdot,\cdot)$ is $(0,0,3)$. Recall from Lemma 3.14 that translations by elements of the subspace $U_0(\boldsymbol{\Gamma})$ are always contained in the centraliser. So we can modify the translation parts of $\boldsymbol{\Gamma}$ by vectors in $U_0(\boldsymbol{\Gamma})$ without changing the centraliser (meaning the modified group is still a Wolf group). To obtain the other possible signatures for $(\cdot,\cdot)$, modify as follows:

- $(1,0,2)$: Replace $\gamma_1 = (I + A_1, v_1)$ by $\gamma_1' = (I + A_1, v_1 - e_4)$. Then $(v_1', v_1') = 1$ and v_1' is orthogonal to the unmodified v_2, v_3.

- $(1,1,1)$: Replace $\gamma_1 = (I + A_1, v_1)$ by $\gamma_1' = (I + A_1, v_1 + e_4)$ and $\gamma_2 = (I + A_2, v_2)$ by $\gamma_2' = (I + A_2, v_2 + e_3)$. Then $(v_1', v_1') = -1$, $(v_2', v_2') = 1$ and $(v_1', v_2') = 0$.

- $(2,0,1)$: Replace $\gamma_1 = (I + A_1, v_1)$ by $\gamma_1' = (I + A_1, v_1 - e_4)$ and $\gamma_2 = (I + A_2, v_2)$ by $\gamma_2' = (I + A_2, v_2 + e_3)$. Then $(v_1', v_1') = 1$, $(v_2', v_2') = 1$ and $(v_1', v_2') = 0$.

If $\mathfrak{g}$ is not abelian and the fibre metric is non-degenerate then there are some strong constraints on the structure of $\mathfrak{g}$.

Proposition 5.18 *If the fibre metric on* $\boldsymbol{G}$ *is non-degenerate, then the linear holonomy group of* $\boldsymbol{G}$ *is abelian.*

Proof: The orbits F_p are affine subspaces of $\mathbb{R}^n$ and isometric to $\boldsymbol{G}$. So $F_p/\boldsymbol{\Gamma} = \boldsymbol{G}/\boldsymbol{\Gamma}$ is a compact flat pseudo-Riemannian homogeneous space. By Theorem 4.4, the linear holonomy of $\boldsymbol{G}$ is abelian. ■

Additionally, $\mathfrak{g}$ must contain a subalgebra of a certain type.

Definition 5.19 A **butterfly algebra** $\mathfrak{b}_6$ is a 2-step nilpotent Lie algebra of dimension 6 endowed with an invariant pseudo-scalar product $(\cdot,\cdot)$ such that there exists $Z \in [\mathfrak{b}_6, \mathfrak{b}_6]$ with $Z \notin \mathfrak{r}$. A **butterfly group** $\mathbf{B}_6$ is a Lie group with $\mathfrak{Lie}(\mathbf{B}_6) = \mathfrak{b}_6$.

The naming in Definition 5.19 will become clear (even inevitable) after the proof of the following proposition:

Proposition 5.20 *A butterfly algebra* $\mathfrak{b}_6$ *admits a vector space decomposition*

$$\mathfrak{b}_6 = \mathfrak{v} \oplus [\mathfrak{b}_6, \mathfrak{b}_6],$$

where the subspaces $\mathfrak{v}$ *and* $[\mathfrak{b}_6, \mathfrak{b}_6]$ *are totally isotropic and dual to each other. In particular,* $(\cdot,\cdot)$ *is non-degenerate of signature* $(3,3)$.

Proof: Let $X, Y \in \mathfrak{b}_6$ such that $Z = [X, Y] \neq 0$. By Lemma 5.13 $[\mathfrak{b}_6, \mathfrak{b}_6]$ is totally isotropic. By assumption there exists $Z^* \in \mathfrak{b}_6 \backslash [\mathfrak{b}_6, \mathfrak{b}_6]$ such that

$$(Z, Z^*) = 1.$$

As a consequence of Lemma 5.15, X, Y, Z^* are linearly independent, so they span a 3-dimensional subspace $\mathfrak{v}$. Since $(\cdot,\cdot)$ is invariant,

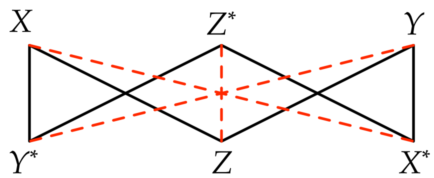
$$1 = ([X,Y],Z^*) = (Y,[X,Z^*]),$$
$$1 = ([X,Y],Z^*) = (X,[Z^*,Y]).$$

Set $X^* = [Z^*,Y]$ and $Y^* = [X,Z^*]$. Lemma 5.15 further implies that X^*,Y^*,Z are linearly independent, hence span a 3-dimensional subspace $\mathfrak{w}$ of $[\mathfrak{b}_6,\mathfrak{b}_6]$. Since $\mathfrak{b}_6$ is 2-step nilpotent $[\mathfrak{b}_6,\mathfrak{b}_6] \subset \mathfrak{z}(\mathfrak{b}_6)$. But $\mathfrak{v} \cap \mathfrak{z}(\mathfrak{b}_6) = \{0\}$, so it follows from dimension reasons that $\mathfrak{w} = [\mathfrak{b}_6,\mathfrak{b}_6] = \mathfrak{z}(\mathfrak{b}_6)$. By construction also

$$X \perp \operatorname{span}\{Y^*,Z\}, \quad Y \perp \operatorname{span}\{X^*,Z\}.$$

After a base change we may assume that $\mathfrak{v}$ is a dual space to $[\mathfrak{b}_6,\mathfrak{b}_6]$. ■

The bases $\{X,Y,Z^*\}$ and $\{X^*,Y^*,Z\}$ from the proof above are dual bases to each other. The following diagram describes the relations between these bases, where black lines from two elements indicate a commutator and dashed red lines indicate duality between the corresponding elements:



This explains the name. In particular, the following corollary justifies to speak of "the" butterfly algebra:[11)]

Corollary 5.21 *Any two butterfly algebras are isometric and isomorphic as Lie algebras.*

Proof: In every butterfly algebra one can find a basis as in the proof of Proposition 5.20. Mapping the elements of one butterfly algebra to the corresponding ones of another yields an isometry. ■

The butterfly algebra is of the type described in Theorem 4.8. The corresponding subspaces are $\mathfrak{a} = \mathfrak{v}$, $\mathfrak{a}^* = [\mathfrak{b}_6,\mathfrak{b}_6]$ and $\mathfrak{z}_0 = \{0\}$. Main Example 9 is a butterfly algebra. In particular:

Corollary 5.22 $\mathfrak{b}_6 = \mathfrak{h}_3 \oplus_{\mathrm{ad}^*} \mathfrak{h}_3^*$ *with* $(\cdot,\cdot)$ *as in (9.2).*

11)Some people misguidedly believe that the name *bat algebra* would be more apt.

Applied to the fibre metrics on Wolf groups, we may conclude:

Proposition 5.23 *If $\boldsymbol{G}$ is not abelian and the fibre metric on $\boldsymbol{G}$ is non-degenerate, then $\mathfrak{g}$ contains a butterfly subalgebra. In particular,* $\dim \boldsymbol{G} \geq 6$.

Proof: $\mathfrak{g}$ is 2-step nilpotent and contains a central element Z which has a dual element Z^*. As in the proof of Proposition 5.20 we find a basis of a butterfly subalgebra. ■

5.4 Pseudo-Riemannian Submersions

We continue to study the special case of a Wolf group $\boldsymbol{G}$ acting on $\mathbb{R}^{r,s}$ such that the induced metric on the orbits of $\boldsymbol{G}$ is non-degenerate. The quotient map $\pi : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}/\boldsymbol{G}$ is a pseudo-Riemannian submersion whose fibres are the $\boldsymbol{G}$-orbits F_p. The geometry of such submersions is encoded by two tensors T, S. These tensors might prove helpful in finding invariants for the classification of flat pseudo-Riemannian homogeneous manifolds. Their properties were studied by O'Neill [29]. In this section we follow his exposition.[12)]

Definition 5.24 Let M, B be pseudo-Riemannian manifolds. A **pseudo-Riemannian submersion** $\pi : M \to B$ is a smooth map of M onto B such that:

(a) The fibres $\pi^{-1}(b)$ are pseudo-Riemannian submanifolds of M for all $b \in B$.

(b) $\mathrm{d}\pi$ preserves the scalar products of vectors normal to fibres.

Definition 5.25 A vector field X is called **vertical** if X_p is tangent to the fibre through p for all $p \in M$, and X is called **horizontal** if X_p is orthogonal to the fibre through p. The projections of $w \in \mathrm{T}_pM$ on its vertical and horizontal components are denoted by $\mathrm{v}(w)$ and $\mathrm{H}(w)$, respectively.

Definition 5.26 For arbitrary vector fields $X, Y \in \mathfrak{vec}(M)$ define

$$\mathsf{T}_X Y = \mathrm{H}(\nabla_{\mathrm{v}(X)}\mathrm{v}(Y)) + \mathrm{v}(\nabla_{\mathrm{v}(X)}\mathrm{H}(Y)) \tag{5.5}$$

and

$$\mathsf{S}_X Y = \mathrm{v}(\nabla_{\mathrm{H}(X)}\mathrm{H}(Y)) + \mathrm{H}(\nabla_{\mathrm{H}(X)}\mathrm{v}(Y)). \tag{5.6}$$

12) Note that O'Neill [29] considers only the Riemannian case, but his proofs rely only on the non-degeneracy of the metric. So the results generalise to the pseudo-Riemannian case. See O'Neill [30], chapter 7, and Sternberg [42], chapter 9.

Remark 5.27 $\mathsf{T}_X Y$ and $\mathsf{S}_X Y$ are tensors in X, Y.

Lemma 5.28 *If the fibres of a pseudo-Riemannian submersion $M \to B$ are totally geodesic, then*

$$\mathsf{T} = 0. \tag{5.7}$$

Proof: The first term in (5.5), $\textsc{h}(\nabla_{\mathrm{v}(X)}\mathrm{v}(Y))_p$, is the second fundamental form of the fibre F_p through $p \in M$. If F_p is totally geodesic, then the second fundamental form is 0.

Also due to geodesic completeness, for all vertical V_1, V_2 the covariant derivative $\nabla_{V_1} V_2$ is again a vertical vector field. So if H is an arbitrary horizontal vector field,

$$V_1 \underbrace{\langle V_2, H\rangle}_{=0} = \underbrace{\langle \nabla_{V_1} V_2, H\rangle}_{=0} + \langle V_2, \nabla_{V_1} H\rangle.$$

So $\nabla_{V_1} H$ is horizontal, and the second term in (5.5) is $\mathrm{v}(\nabla_{\mathrm{v}(X)}\textsc{h}(Y)) = 0$. ■

Riemannian submersions with totally geodesic fibres were studied for example by Hermann [22] and Escobales [15]. Their results generalise to the pseudo-Riemannian case. The following proposition is Corollary 1.5 in Escobales [15]:

Proposition 5.29 (Escobales) *If S is parallel, then $\mathsf{S} = 0$. If T is parallel, then $\mathsf{T} = 0$. Thus pseudo-Riemannian submersions with parellel S are characterised as those whose horizontal distributions are integrable, and pseudo-Riemannian submersions with parallel T as those whose fibres are totally geodesic.*

If G is a Wolf group acting on $\mathbb{R}^{r,s}$ and we assume the induced metric on the orbits F_p to be non-degenerate, then $\pi : \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}/G$ is a pseudo-Riemannian submersion (the metric on $\mathbb{R}^{r,s}/G$ is defined via the metric on $\mathbb{R}^{r,s}$ restricted to the horizontal distribution). In the following, we refer to this submersion.

Remark 5.30 The fibres F_p are affine subspaces. In particular, they are totally geodesic, so $\mathsf{T} = 0$ by Lemma 5.28.

Recall (Proposition 5.1) that $F_p = p + \{X_1.p, \ldots, X_k.p\}$, where $X_1, \ldots, X_k$ is a Malcev basis of $\mathfrak{g}$. So every vertical vector field V can be written as

$$V_p = \lambda_1(p)X_1.p + \ldots + \lambda_k(p)X_k.p \tag{5.8}$$

for certain smooth functions λ_i. Let X_i^+ denote the vector field $(X_i^+)_p = X_i.p$. As S is a tensor, in order to know $\mathsf{S}V$ it suffices to know the $\mathsf{S}X_i^+$.

Lemma 5.31 *Let X_i as above and H a horizontal vector field on $\mathbb{R}^{r,s}$. In affine coordinates, write $X_i = (A_i, v_i)$. Then for all $p \in \mathbb{R}^{r,s}$*

$$(\mathsf{S}_H X_i^+)_p = A_i H_p. \tag{5.9}$$

Proof: ∇ is the natural connection on $\mathbb{R}^{r+s}$. So

$$(\nabla_H X_i^+)_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} X_i.(p + tH_p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A_i p + tA_i H_p + v_i = A_i H_p.$$

The first term in (5.6) is 0 because X_i^+ is vertical. So $(\mathsf{S}_H X_i^+)_p = \mathrm{H}(A_i H_p)$. Because the centraliser of $\boldsymbol{G}$ has an open orbit at p, we may assume that H is a Killing field from the action of the centraliser. Then $[X_i^+, H]_p = 0$. With (5.9) we obtain

$$(\nabla_{X_i^+} H)_p = (\nabla_H X_i^+)_p + [X_i^+, H]_p = (\nabla_H X_i^+)_p = A_i H_p.$$

So $(\mathsf{T}_{X_i^+} H)_p = \mathrm{v}(A_i H_p)$. But $\mathsf{T} = 0$, so $(\mathsf{S}_H X_i^+)_p = \mathrm{H}(A_i H_p) = A_i H_p$. ■

Corollary 5.32 *Let $V, H \in \mathfrak{vec}(\mathbb{R}^{r,s})$, V vertical and H horizontal. Then*

$$\langle \mathsf{S}_H V, \mathsf{S}_H V \rangle = 0. \tag{5.10}$$

Proof: Let $X_i = (A_i, v_i)$ denote the Malcev basis elements as in (5.8), and for $p \in \mathbb{R}^{r,s}$ let $A_p = \sum_i \lambda_i(p) A_i$ denote the linear part of the affine coordinate expression (5.8) for V_p. Then A_p has totally isotropic image. ■

Lemma 5.33 $(\nabla_{V_1}\mathsf{S})(V_2, \cdot) = 0$ *for all vertical vector fields V_1, V_2.*

Proof: If V is vertical, then $\mathsf{S}_V = 0$ by (5.6). Also, $\nabla_{V_1} V_2$ is vertical because the fibre F_p is totally geodesic. Thus

$$(\nabla_{V_1}\mathsf{S})(V_2, Y) = \nabla_{V_1}(\mathsf{S}_{V_2} Y) - \mathsf{S}_{\nabla_{V_1} V_2} Y - \mathsf{S}_{V_2} \nabla_{V_1} Y = 0$$

for all vector fields Y. ■

Main Example 12 is an example of a group whose orbits have a non-degenerate induced metric. As $\mathsf{S} \neq 0$ in this example (section 12.3), the horizontal distribution is not integrable (Proposition 5.29).

6 The Lorentz Case and Low Dimensions

In this section, we present a structure theory for the fundamental groups of some special cases of flat pseudo-Riemannian homogeneous spaces.

For complete manifolds, the Lorentz case and the signature $(n-2, 2)$ were studied by Wolf [52], Corollary 3.7.13. We present his results with full proofs. For incomplete manifolds, these two cases were studied by Duncan and Ihrig [11, 13] (here, we only give their result in the Lorentz case).

Additionally, we determine the structure of the fundamental groups of manifolds with signature $(n-2, 2)$, and of the fundamental groups of complete spaces of dimensions 4 to 6.

The notation is as usual, $\mathbf{\Gamma}$ denotes the fundamental group, $\boldsymbol{G}$ its Zariski closure, $U_\Gamma = \sum_{X\in\mathfrak{g}} \operatorname{im} \mathrm{L}(X)$ (where $\mathfrak{g} = \mathfrak{Lie}(\boldsymbol{G})$) and $U_\Gamma^\perp$ equals the intersection of the kernels of all $\mathrm{L}(X)$.

We start by collecting some general facts about discrete Wolf groups.

Remark 6.1 Recall that $\mathbf{\Gamma}$, as a discrete subgroup of $\boldsymbol{G} \subset \mathbf{Iso}(\mathbb{R}^{r,s})$, is finitely generated and torsion free (Theorem G.22). Further, $\operatorname{rk} \mathbf{\Gamma} = \dim \boldsymbol{G}$.

This implies the following simple facts:

Lemma 6.2 *If $\mathbf{\Gamma}$ is abelian, then $\mathbf{\Gamma}$ is free abelian.*

Lemma 6.3 *Let $\gamma_1, \ldots, \gamma_k$ denote a Malcev basis of $\mathbf{\Gamma}$ (a minimal set of generators if $\mathbf{\Gamma}$ is abelian). If M is complete, then the translation parts $v_1, \ldots, v_k$ of the γ_i are linearly independent.*

Proof: The Zariski closure $\boldsymbol{G}$ of $\mathbf{\Gamma}$ acts freely (Lemma 1.18), so

$$k = \operatorname{rk} \mathbf{\Gamma} = \dim \boldsymbol{G} = \dim \boldsymbol{G}.0 = \dim \operatorname{span}\{v_1, \ldots, v_k\}.$$

So the v_i are linearly independent. ■

6.1 Riemann and Lorentz Metrics

Proposition 6.4 (Wolf) *If $M = \mathbb{R}^{n-1,1}/\mathbf{\Gamma}$ is a complete homogeneous flat Riemannian or Lorentz manifold, then $\mathbf{\Gamma}$ is an abelian group consisting of pure translations.*

Proof: It follows from Corollary 2.26 that $\boldsymbol{\Gamma}$ has abelian holonomy. Then the subspace U_{Γ} is isotropic, so $\dim U_{\Gamma} \leq 1$. If $\gamma = (I + A, v) \in \boldsymbol{\Gamma}$ has non-trivial linear part, then A contains a non-zero skew-symmetric submatrix, so it has rank ≥ 2. But this would imply $\dim U_{\Gamma} \geq 2$. It follows that $A = 0$ for all $\gamma \in \boldsymbol{\Gamma}$. ∎

Homogeneous Riemannian manifolds are always complete, so there is no need to consider the signature $(n, 0)$ in the incomplete case.

Proposition 6.5 (Duncan, Ihrig) *Let M be an incomplete flat homogeneous Lorentz manifold. Then there exists a subgroup $\boldsymbol{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{n-1,1})$ consisting of pure translations, an isotropic vector $z \in \mathbb{R}^{n-1,1}$ and an open domain $D = \{v \in \mathbb{R}^{n-1,1} \mid \langle v, z\rangle > 0\}$ such that $M = D/\boldsymbol{\Gamma}$.*

For a proof, see Duncan and Ihrig [11], Theorem 3.7.

6.2 Generalities on Abelian Wolf Groups

Most Wolf groups in low dimensions are abelian. Wolf [50] gave a classification of abelian Wolf groups with transitive centraliser (we adapt the statement of these theorems to our notation).

For $C \in \mathfrak{so}_m$ set $\mathbf{Sp}(C) = \{g \in \mathbf{GL}_m(\mathbb{R}) \mid gCg^{\top} = C\}$. If C is regular, then this is the usual symplectic group.

Theorem 6.6 (Wolf) *Let $U \subset \mathbb{R}^{r,s}$ be a totally isotropic subspace and* $\dim U = m$. *Further, let $\boldsymbol{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{r,s})$ be the abelian group generated by $\gamma_1, \ldots, \gamma_k$, where $\gamma_i = (I + A_i, v_i)$ with $v_1, \ldots, v_k \in U^{\perp}$ linearly independent, $A_i = \left(\begin{smallmatrix} 0 & 0 & C_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, where $C_i \in \mathfrak{so}_m$. The Witt decomposition of v_i with respect to U is $v_i = u_i + w_i + u_i^*$. For $v \in \mathbb{R}^{r,s}$, let $\mathfrak{S}_v = \{S \in \mathrm{Hom}(U^{\perp}, U) \mid S(v_i) = u_i + C_i v, i = 1, \ldots, k\}$. Then $M = \mathbb{R}^{r,s}/\boldsymbol{\Gamma}$ is a flat homogeneous pseudo-Riemannian manifold if and only if $\{S|_U \mid S \in \mathfrak{S}_v\} \cap \bigcap_i \mathbf{Sp}(C_i) \neq \emptyset$ for all $v \in \mathbb{R}^{r,s}$.*

Wolf requires one of the C_i to be invertible, but this assumption can be dropped without consequence for the proof of Theorem 6.6. Main Example 8 shows that there are examples where all C_i are singular. Also, Wolf remarks that a problem with the application of this theorem is the absence of a structure theory for $\mathbf{Sp}(C)$ when C is not regular. We give a more practical version of this theorem for signature $(n-2, 2)$ in Proposition 6.11 below.

Recall that $\mathbb{R}^{r,s}/\mathbf{\Gamma}$ and $\mathbb{R}^{r,s}/\mathbf{\Gamma}'$ are isometric if and only if $g\mathbf{\Gamma}g^{-1} = \mathbf{\Gamma}'$ for some $g \in \mathbf{Iso}(\mathbb{R}^{r,s})$. Wolf [50] gave necessary and sufficient conditions for such a g to exist.

Theorem 6.7 (Wolf) *Let $\mathbf{\Gamma}, \mathbf{\Gamma}'$ be groups as in Theorem 6.6 and $\operatorname{rk}\mathbf{\Gamma} = \operatorname{rk}\mathbf{\Gamma}' = k$. Then $\mathbb{R}^{r,s}/\mathbf{\Gamma}$ and $\mathbb{R}^{r,s}/\mathbf{\Gamma}'$ are isometric if and only if there exists $h = (h_{ij}) \in \mathbf{SL}_k^{\pm}(\mathbb{Z})$, an isomorphism $g_{11} : U_\Gamma \to U_{\Gamma'}$, a linear map $g_{12} : U_\Gamma^\perp/U_\Gamma \to U_{\Gamma'}$, an isometry $g_{22} : U_\Gamma^\perp/U_\Gamma \to U_{\Gamma'}^\perp/U_{\Gamma'}$ and $v = u + w + u^* \in \mathbb{R}^{r,s}$ such that for $i = 1, \ldots, k$*

(a) $g_{22}w_i = \sum_j h_{ij}w'_j$,

(b) $g_{11}C_i g_{11}^\top = \sum_j h_{ij}C'_j$,

(c) $g_{11}u_i + g_{12}w_i = \sum_j h_{ij}(u'_j + u^*)$.

For a proof, see section 5 in Wolf [50].

6.3 Signature $(n-2,2)$

As always, we assume $n-2 \geq 2$. The following proposition was proved by Wolf in the complete case, but this assumption is not needed in the proof.

Proposition 6.8 (Wolf) *Let $M = D/\mathbf{\Gamma}$ be a flat pseudo-Riemannian homogeneous manifold, where $D \subseteq \mathbb{R}^{n-2,2}$ is an open orbit of the centraliser of $\mathbf{\Gamma}$. Then $\mathbf{\Gamma}$ is a free abelian group. In particular, this holds if $\dim M \leq 5$.*

Proof: It follows from Corollary 2.26 that $\mathbf{\Gamma}$ has abelian holonomy. Consequently, if $\gamma = (I + A, v) \in \mathbf{\Gamma}$ such that $A \neq 0$, then

$$A = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in a Witt basis with respect to U_Γ. Here, $C \neq 0$ is a skew-symmetric 2×2-matrix, so we have $\operatorname{rk} A = 2$. Because $\operatorname{im} A \subset U_\Gamma$ and both these spaces are totally isotropic, we have $\dim \operatorname{im} A = \dim U_\Gamma = 2$. Because $Av = 0$ we get $v \in \ker A = (\operatorname{im} A)^\perp = U_\Gamma^\perp$. But then $Bv = 0$ for any $(I + B, w) \in \mathbf{\Gamma}$, and as also $BA = 0$, it follows that $[(I + B, w), (I + A, v)] = (I + 2BA, 2Bv) = (I, 0)$. Hence $\mathbf{\Gamma}$ is abelian. It is free abelian by Lemma 6.2. ■

In the rest of this section, the group Γ is always abelian, so the space $U_\Gamma = \sum_A \operatorname{im} A$ is totally isotropic. We fix a Witt decomposition with respect to U_Γ,

$$\mathbb{R}^{n-2,2} = U_\Gamma \oplus W \oplus U_\Gamma^*$$

and any $v \in \mathbb{R}^{n-2,2}$ decomposes into $v = u + w + u^*$ with $u \in U_\Gamma$, $w \in W$, $u^* \in U_\Gamma^*$.

Remark 6.9 As seen in the proof of Proposition 6.8, if $\dim U_\Gamma = 2$, then $U_\Gamma = \operatorname{im} A$ for any $\gamma = (I + A, v)$ with $A \neq 0$.

We can make Proposition 6.8 more precise:

Proposition 6.10 *Let $M = \mathbb{R}^{n-2,2}/\Gamma$ be a complete flat pseudo-Riemannian homogeneous manifold. Then:*

(a) Γ *is generated by elements $\gamma_i = (I + A_i, v_i)$, $i = 1, \ldots, k$, with linearly independent translation parts $v_1, \ldots, v_k$.*

(b) *If there exists $(I + A, v) \in \Gamma$ with $A \neq 0$, then in a Witt basis with respect to U_Γ,*

$$\gamma_i = (I + A_i, v_i) = \left(\begin{pmatrix} I_2 & 0 & C_i \\ 0 & I_{n-4} & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} u_i \\ w_i \\ 0 \end{pmatrix} \right) \tag{6.1}$$

with $C_i = \left(\begin{smallmatrix} 0 & c_i \\ -c_i & 0 \end{smallmatrix}\right)$, $c_i \in \mathbb{R}$, $u_i \in \mathbb{R}^2$, $w_i \in \mathbb{R}^{n-4}$.

(c) $\sum_i \lambda_i w_i = 0$ *implies* $\sum_i \lambda_i C_i = 0$ *(equivalently $\sum_i \lambda_i A_i = 0$) for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.*

Proof: We know from Proposition 6.8 that Γ is free abelian. Let $\gamma_1, \ldots, \gamma_k$ denote a minimal set of generators.

(a) Lemma 6.3.

(b) If $A \neq 0$ exists, then $U_\Gamma = \operatorname{im} A$ is a 2-dimensional totally isotropic subspace. Then the matrix representation is already known. As Γ is abelian, we have $A_i v_j = 0$ for all i, j. So $v_j \in \bigcap_i \ker A_i = U_\Gamma^\perp$ for all j.

(c) Assume $\sum_i \lambda_i w_i = 0$ and set $C = \sum_i \lambda_i C_i$. Then $\sum_i \lambda_i (A_i, v_i) = (A, u)$, where $u \in U_\Gamma$. If $A \neq 0$, then G would have a fixed point (see Remark 6.9). So $A = 0$, which implies $C = 0$. ■

Conversely, every group of the form described in the previous proposition defines a homogeneous space:

Proposition 6.11 *Let U be a 2-dimensional totally isotropic subspace of $\mathbb{R}^{n-2,2}$, and let $\mathbf{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{n-2,2})$ be a subgroup generated by matrices $\gamma_1, \ldots, \gamma_k$ of the form (6.1) with linearly independent translation parts. Further, assume that $\sum_i \lambda_i w_i = 0$ implies $\sum_i \lambda_i C_i = 0$ (equivalently $\sum_i \lambda_i A_i = 0$) for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then $\mathbb{R}^{n-2,2}/\mathbf{\Gamma}$ is a complete flat pseudo-Riemannian homogeneous manifold.*

Proof:

(i) From the matrix form (6.1) it follows that $\mathbf{\Gamma}$ is free abelian, and the linear independence of the translation parts implies that it is a discrete subgroup of $\mathbf{Iso}(\mathbb{R}^{n-2,2})$.

(ii) We check that the centraliser of $\mathbf{\Gamma}$ in $\mathbf{Iso}(\mathbb{R}^{n-2,2})$ acts transitively: Because of the signature, the subspace W in the Witt decomposition is definite. Hence, an element of $\mathfrak{iso}(\mathbb{R}^{n-2,2})$ is of the form (A.3). Consider elements of the form

$$S = \left(\begin{pmatrix} 0 & -B^\top & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \in \mathfrak{iso}(\mathbb{R}^{n-2,2}), \quad x, z \in \mathbb{R}^2, y \in \mathbb{R}^{n-2}.$$

We will show that, given any x, y, z, we can determine B so that S centralises $\log(\mathbf{\Gamma})$. Writing out the commutator equation $[S, (A_i, v_i)]$, we see that $[S, (A_i, v_i)] = 0$ is equivalent to

$$-B^\top w_i = C_i z.$$

For simplicity, assume that $w_1, \ldots, w_j$ form a maximal linearly independent subset of $w_1, \ldots, w_k$ ($j \leq k$). As $-B^\top$ is a $2 \times (n-2)$-matrix, the linear system

$$\begin{aligned} -B^\top w_1 &= C_1 z \\ &\vdots \\ -B^\top w_j &= C_j z \end{aligned}$$

consists of $2j$ linearly independent equations and $2(n-2)$ variables. As $\dim W = n - 2 \geq j$, this system is always solvable.

So S can be determined such that it commutes with $\gamma_1, \ldots, \gamma_j$. It remains to check that S also commutes with $\gamma_{j+1}, \ldots, \gamma_k$. By assumption, each w_l ($l > j$) is a linear combination $w_l = \sum_{i=1}^{j} \lambda_i w_i$. Now $w_l - \sum_{i=1}^{j} \lambda_i w_i = 0$ implies $C_l - \sum_{i=1}^{j} \lambda_i C_i = 0$. But this means

$$-B^\top w_l = \sum_{i=1}^{j} \lambda_i \underbrace{(-B^\top w_i)}_{=C_i z} = \Big(\sum_{i=1}^{j} \lambda_i C_i\Big) z = C_l z,$$

so $[(A_l, v_l), S] = 0$.

The elements $\exp(S)$ generate a unipotent subgroup of the centraliser of $\boldsymbol{\Gamma}$, so its open orbit at 0 is closed and hence all of $\mathbb{R}^{n-2,2}$. Consequently, $\boldsymbol{\Gamma}$ has transitive centraliser.

(iii) Because the centraliser is transitive, the action free everywhere. It follows from Proposition 7.8 that $\boldsymbol{\Gamma}$ acts properly discontinuously. Now $\mathbb{R}^{n-2,2}/\boldsymbol{\Gamma}$ is homogeneous by Corollary C.2, and it is complete again by the transitivity of the centraliser. ∎

In the following we speak of a *rough classification* of certain spaces if there is a structure theorem for these spaces together with conditions under which an isometry between two such spaces exist.[13)] The incomplete flat pseudo-Riemannian homogeneous manifolds M of signature $(n-2, 2)$ were studied by Duncan and Ihrig [13]. They give a rough classification of the incomplete flat pseudo-Riemannian homogeneous manifolds $M = D/\boldsymbol{\Gamma}$ of signature $(n-2, 2)$ under the condition that $D \subseteq \mathbb{R}^{n-2,2}$ is a translationally isotropic domain. Also, they mistakenly claim that if M is complete, then $\boldsymbol{\Gamma}$ is a group of pure translations, citing an unspecified article by Wolf as source. In fact, Wolf never claimed this and even constructed a class of examples of signature $(n-2, 2)$ with non-trivial holonomy (Wolf [48], section 6). The characterisation of fundamental groups for the complete case is given by Proposition 6.8. Additionally, the condition that D be translationally isotropic is void due to Theorem 3.17. So we can reformulate Duncan and Ihrig's classification as follows:

Theorem 6.12 *Let $M = D/\boldsymbol{\Gamma}$ be a flat pseudo-Riemannian homogeneous manifold of signature $(n-2, 2)$ where $D \subset \mathbb{R}^{n-2,2}$ is an open orbit of $Z_{\mathbf{Iso}(\mathbb{R}^{n-2,2})}(\boldsymbol{\Gamma})$. A rough classification of incomplete manifolds M of this type is given Duncan and Ihrig [13], Theorems 3.1 and 3.2. For complete manifolds M of this type, it is given by Proposition 6.10 and Theorem 6.7.*

13) This notion of a classification falls somewhat short of a classification by a complete set of invariants.

6.4 Dimension ≤ 5

Proposition 6.13 (Wolf) *Let $M = \mathbb{R}^{r,s}/\boldsymbol{\Gamma}$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension ≤ 4. Then $\boldsymbol{\Gamma}$ is a free abelian group consisting of pure translations.*

Proof: Assume $r \geq s$. If $s = 0$ or $s = 1$, we have the Riemann or Lorentz case. So assume $s = 2$. By Proposition 6.8, $\boldsymbol{\Gamma}$ is free abelian.

Assume there exists $\gamma = (I + A, v) \in \boldsymbol{\Gamma}$ with $A \neq 0$. Then $\dim U_{\boldsymbol{\Gamma}} = 2$, and $U_{\boldsymbol{\Gamma}}$ is a maximal totally isotropic subspace. Hence $U_{\boldsymbol{\Gamma}} = U_{\boldsymbol{\Gamma}}^{\perp}$. But then $Av = 0$ implies $v \in U_{\boldsymbol{\Gamma}}^{\perp} = U_{\boldsymbol{\Gamma}} = \operatorname{im} A$, so there exists $w \in \mathbb{R}^{2,2}$ such that $Aw = v$. Then $-w$ is a fixed point for γ. This contradicts the freeness of the $\boldsymbol{\Gamma}$-action on $\mathbb{R}^{2,2}$. Consequently, such a γ does not exist, and $\boldsymbol{\Gamma}$ consists of pure translations only. ∎

Proposition 6.14 *Let $M = \mathbb{R}^{r,s}/\boldsymbol{\Gamma}$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 5. Then $\boldsymbol{\Gamma}$ is a free abelian group. Depending on the signature of M, we have the following possibilities:*

(a) *Signature $(5,0)$ or $(4,1)$: $\boldsymbol{\Gamma}$ is a group of pure translations.*

(b) *Signature $(3,2)$: $\boldsymbol{\Gamma}$ is either a group of pure translations, or there exists $\gamma_1 = (I + A_1, v_1) \in \boldsymbol{\Gamma}$ with $A_1 \neq 0$. In the latter case, $\operatorname{rk}\boldsymbol{\Gamma} \leq 3$, and if $\gamma_1, \ldots, \gamma_k$ ($k = 1,2,3$) are generators of $\boldsymbol{\Gamma}$, then $v_1, \ldots, v_k$ are linearly independent, and $w_i = \frac{c_i}{c_1} w_1$ in the notation of (6.1) ($i = 1, \ldots, k$).*

Proof: $\boldsymbol{\Gamma}$ is free abelian by Proposition 6.8. The statement for signatures $(5,0)$ and $(4,1)$ follows from Proposition 6.4.

Let the signature be $(3,2)$ and assume $\boldsymbol{\Gamma}$ is not a group of pure translations. Then $U_{\boldsymbol{\Gamma}} = \operatorname{im} A$ is 2-dimensional (where $(I + A, v) \in \boldsymbol{\Gamma}$, $A \neq 0$). By Lemma 6.3, the translation parts of the generators of $\boldsymbol{\Gamma}$ are linearly independent elements of $U_{\boldsymbol{\Gamma}}^{\perp}$, which is 3-dimensional. So $\operatorname{rk}\boldsymbol{\Gamma} \leq 3$. Now, $U_{\boldsymbol{\Gamma}}^{\perp} = U_{\boldsymbol{\Gamma}} \oplus W$ with $\dim W = 1$. So the W-components of the translation parts are multiples of each other, and it follows from part (c) of Proposition 6.10 that $w_1 \neq 0$ and $w_i = \frac{c_i}{c_1} w_1$. ∎

Example 6.15 Let $\boldsymbol{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{3,2})$ be the discrete group generated by $\gamma_1, \gamma_2, \gamma_3$, where

$$\gamma_i = \left(\begin{pmatrix} I_2 & 0 & C_i \\ 0 & 1 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} e_{i-1} \\ c_i \\ 0 \end{pmatrix} \right),$$

where we set $e_0 = 0$, e_i the ith unit vector for $i = 1, 2$, and $c_i \in \mathbb{R}^\times$. If $C_i = c_i C_1 \neq 0$, where $c_1 = 1, c_2, c_3 \in \mathbb{R}^\times$ are linearly independent over $\mathbb{Q}$, then $\mathbf{\Gamma}$ is a discrete Wolf group on $\mathbb{R}^{3,2}$ such that every element of $\mathbf{\Gamma}\backslash\{I\}$ has non-zero linear part.

6.5 Dimension 6

Abelian and non-abelian Wolf groups appear for signature $(3, 3)$. Before we determine them, we introduce the following notation: For $x \in \mathbb{R}^3$, let

$$T(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Then for any $y \in \mathbb{R}^3$,

$$T(x)y = x \times y,$$

where $\times$ denotes the vector cross product on $\mathbb{R}^3$.

Lemma 6.16 *Let $\mathbf{\Gamma}$ be a Wolf group acting on $\mathbb{R}^{3,3}$. An element $X \in \log(\mathbf{\Gamma})$ has the form*

$$X = \left(\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u \\ u^* \end{pmatrix} \right) \tag{6.2}$$

with respect to the Witt decomposition $\mathbb{R}^{3,3} = U_{\mathbf{\Gamma}} \oplus U_{\mathbf{\Gamma}}^*$. Furthermore,

$$C = \alpha_X T(u^*)$$

for some $\alpha_X \in \mathbb{R}$. If $[X_1, X_2] \neq 0$ for $X_1, X_2 \in \log(\mathbf{\Gamma})$, then $\alpha_{X_1} = \alpha_{X_2} \neq 0$.

Proof: The holonomy is abelian by Corollary 2.26, so (6.2) follows.
For $X \in \log(\mathbf{\Gamma})$ we have $Cu^* = 0$, that is

$$Cu^* = \alpha u^* \times u^* = 0.$$

If X is non-central, then $C \neq 0$ and $u^* \neq 0$. Now let $x, y \in \mathbb{R}^3$ such that u^*, x, y form a basis of $\mathbb{R}^3$. Because C is skew,

$$u^{*\top} C x = -u^{*\top} C^\top x = -(Cu^*)^\top x = 0.$$

Also,

$$x^\top C x = -x^\top C^\top x \quad \text{and} \quad x^\top C x = (x^\top C x)^\top = x^\top C^\top x,$$

hence $x^\top Cx = 0$. So Cx is perpendicular to the span of x, u^* in the Euclidean sense[14]. This means there is a $\alpha \in \mathbb{R}$ such that

$$Cx = \alpha u^* \times x.$$

In the same way we get $Cy = \beta u^* \times y$ for some $\beta \in \mathbb{R}$. As neither x nor y is in the kernel of C (which is spanned by u^*), $\alpha, \beta \neq 0$.

As y is not in the span of u^*, x, we have

$$\begin{aligned} 0 \neq x^\top Cy &= \beta x^\top (u^* \times y) \\ &= -y^\top Cx = -\alpha y^\top (u^* \times x) = -\alpha x^\top (y \times u^*) = \alpha x^\top (u^* \times y), \end{aligned}$$

where the last line uses standard identities for the vector product. So $\alpha = \beta$, and C and $\alpha T(u^*)$ coincide on a basis of $\mathbb{R}^3$.

Now assume $[X_1, X_2] \neq 0$. Then

$$\alpha_2 u_2^* \times u_1^* = C_2 u_1^* = -C_1 u_2^* = -\alpha_1 u_1^* \times u_2^* = \alpha_1 u_2^* \times u_1^*,$$

and this is $\neq 0$ because $C_1 u_2^* = \tau(\frac{1}{2}[X_1, X_2]) \neq 0$. So $\alpha_1 = \alpha_2$. ■

Proposition 6.17 *Let $M = \mathbb{R}^{r,s}/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume Γ is abelian. Then Γ is free abelian. Depending on the signature of M, we have the following possibilities:*

(a) *Signature $(6,0)$ or $(5,1)$: Γ is a group of pure translations.*

(b) *Signature $(4,2)$: Γ is either a group of pure translations, or Γ contains elements $\gamma = (I + A, v)$ with $A \neq 0$ subject to the constraints of Proposition 6.10. Further, $\operatorname{rk} \Gamma \leq 4$.*

(c) *Signature $(3,3)$: If $\dim U_\Gamma < 3$, then Γ is one of the groups that may appear for signature $(4,2)$. There is no abelian Γ with $\dim U_\Gamma = 3$.*

Proof: Γ is free abelian by Lemma 6.2. The statement for signatures $(6,0)$ and $(5,1)$ follows from Proposition 6.4.

If the signature ist $(4,2)$ and Γ is not a group of pure translations, then the statement follows from Proposition 6.10. In this case, $U_\Gamma^\perp$ contains the linearly independent translation parts and is of dimension 4. So $\operatorname{rk} \Gamma \leq 4$.

Consider signature $(3,3)$. If $\dim U_\Gamma = 0$ or $= 2$, then Γ is a group as in the case for signature $(4,2)$. Otherwise, $\dim U_\Gamma = 3$. We show that in the latter

[14] That is, with respect to the canonical positive definite inner product on $\mathbb{R}^3$.

case the centraliser of $\mathbf{\Gamma}$ does not act with open orbit: Any $\gamma \in \mathbf{\Gamma}$ can be written as

$$\gamma = (I + A, v) = \left(\begin{pmatrix} I_3 & C \\ 0 & I_3 \end{pmatrix}, \begin{pmatrix} u \\ u^* \end{pmatrix} \right),$$

where $C \in \mathfrak{so}_3$ and $u, u^* \in \mathbb{R}^3$. In fact, we have $\mathbb{R}^{3,3} = U_{\mathbf{\Gamma}} \oplus U_{\mathbf{\Gamma}}^*$ and $U_{\mathbf{\Gamma}}^{\perp} = U_{\mathbf{\Gamma}}$. We will show that $u^* = 0$:

(i) Because $\operatorname{rk} C = 2$ for every $C \in \mathfrak{so}_3$, $C \neq 0$, but $U_{\mathbf{\Gamma}} = \sum \operatorname{im} A$ is 3-dimensional, there exist $\gamma_1, \gamma_2 \in \mathbf{\Gamma}$ such that the skew matrices C_1 and C_2 are linearly independent. So, for every $u^* \in U_{\mathbf{\Gamma}}^*$, there is an element $\gamma = (I + A, v)$ such that $Au^* \neq 0$.

(ii) $\mathbf{\Gamma}$ abelian implies $A_1 u_2^* = 0$ for every $\gamma_1, \gamma_2 \in \mathbf{\Gamma}$. With the argument above, this implies $u_2^* = 0$. So the translation part of every $\gamma = (I + A, v) \in \mathbf{\Gamma}$ is an element $v = u \in U_{\mathbf{\Gamma}}$.

Step (ii) implies $C_1 = \alpha_1 T(u_1^*) = 0$ by Lemma 6.16, but $C_1 \neq 0$ was required in step (i). Contradiction; so $\mathbf{\Gamma}$ is not a Wolf group. ■

Proposition 6.18 *Let $M = \mathbb{R}^{r,s}/\mathbf{\Gamma}$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume $\mathbf{\Gamma}$ is non-abelian. Then the signature of M is $(3,3)$, and $\mathbf{\Gamma}$ is one of the following:*

(a) *$\mathbf{\Gamma} = \mathbf{\Lambda} \times \mathbf{\Theta}$, where $\mathbf{\Lambda}$ is a discrete Heisenberg group and $\mathbf{\Theta}$ a discrete group of pure translations in U_0. Then $3 \leq \operatorname{rk} \mathbf{\Gamma} = 3 + \operatorname{rk} \mathbf{\Theta} \leq 5$.*

(b) *$\mathbf{\Gamma}$ is a lattice in a butterfly group of rank 6 (see Definition 5.19). In this case, M is compact.*

Proof: If the signature was anything but $(3,3)$ or $\dim U_0 < 3$, then $\mathbf{\Gamma}$ would have to be abelian due to the previous results in this chapter. The holonomy is abelian by Corollary 2.26.

For the following it is more convenient to work with the Zariski closure $\boldsymbol{G}$ of $\mathbf{\Gamma}$ and its Lie algebra $\mathfrak{g}$. As $\mathfrak{g}$ is 2-step nilpotent, $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{v}$ is a vector subspace of $\mathfrak{g}$ of dimension ≥ 2 spanned by non-central elements. Set $\mathfrak{v}_{\mathbf{\Gamma}} = \mathfrak{v} \cap \log(\mathbf{\Gamma})$.

(i) Assume there are $X_i = (A_i, v_i) \in \mathfrak{v}$, $\lambda_i \in \mathbb{R}$, $v_i = u_i + u_i^*$ (for $i = 1, \ldots, m$), such that $\sum_i \lambda_i u_i^* = 0$. Then $\sum_i \lambda_i X_i = (\sum_i \lambda_i A_i, \sum_i \lambda_i v_i) = (A, u) \in \mathfrak{v}$, where $u \in U_0$. For all $(A', v') \in \mathfrak{g}$, the commutator with (A, u) is $[(A', v'), (A, u)] = (0, 2A'u) = (0, 0)$. Thus $(A, u) \in \mathfrak{v} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$.

The above means that if $X_1, \dots, X_m \in \mathfrak{v}$ are linearly independent, then $u_1^*, \dots, u_m^* \in U_0^*$ are linearly independent (and by Lemma 6.16 the $C_1, \dots, C_m$ are too). But $\dim U_0^* = 3$, so $\dim \mathfrak{v} \leq 3$.

(ii) If $Z \in \mathfrak{z}(\mathfrak{g})$, then $C_Z = 0$ and $u_Z^* = 0$: As Z commutes with X_1, X_2, we have $C_Z u_1^* = 0 = C_Z u_2^*$. By step (i), u_1^*, u_2^* are linearly independent. So $\dim \ker C_Z = 2$, which implies $C_Z = 0$ because C_Z is a skew 3×3-matrix. Also, $C_1 u_Z^* = 0 = C_2 u_Z^*$, so $u_Z^* = \ker C_1 \cap \ker C_2 = \{0\}$.

(iii) Assume $\dim \mathfrak{v} = 2$. Then $\mathfrak{v}$ is spanned by X_1, X_2, and $Z_{12} = [X_1, X_2]$ is a pure translation by an element of U_0. The elements X_1, X_2, Z_{12} span a Heisenberg algebra $\mathfrak{h}_3$ contained in $\mathfrak{g}$. If $\dim \mathfrak{g} > 3$, then $\mathfrak{z}(\mathfrak{g})$ contains a subalgebra $\mathfrak{t}$ of pure translations by elements of U_0 by step (ii), and $Z_{12} \notin \mathfrak{t}$. So $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{t}$ with $0 \leq \dim \mathfrak{t} < \dim U_0 = 3$. This gives part (a) of the proposition.

(iv) Now assume $\dim \mathfrak{v} = 3$. We show that $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{v}, \mathfrak{v}]$ and $\dim \mathfrak{z}(\mathfrak{g}) = 3$: Let $X_1 = (A_1, v_1), X_2 = (A_2, v_2) \in \mathfrak{v}_\Gamma$ such that $[X_1, X_2] \neq 0$. By Lemma 6.16, $C_1 = \alpha T(u_1^*)$ and $C_2 = \alpha T(u_2^*)$ for some number $\alpha \neq 0$. There exists $X_3 \in \mathfrak{v}_\Gamma$ such that X_1, X_2, X_3 form basis of $\mathfrak{v}$. By step (i), u_1^*, u_2^*, u_3^* are linearly independent. For $i = 1, 2$, $\ker C_i = \mathbb{R} u_i^*$, and u_3^* is proportional to neither u_1^* nor u_2^*. This means $C_1 u_3^* \neq 0 \neq C_2 u_3^*$, which implies $[X_1, X_3] \neq 0 \neq [X_2, X_3]$. By Lemma 6.16, $C_3 = \alpha T(u_3^*)$.

The non-zero entries of the translation parts of the commutators $[X_1, X_2]$, $[X_1, X_3]$ and $[X_2, X_3]$ are

$$C_1 u_2^* = \alpha u_1^* \times u_2^*, \qquad C_1 u_3^* = \alpha u_1^* \times u_3^*, \qquad C_2 u_3^* = \alpha u_2^* \times u_3^*.$$

Linear independence of u_1^*, u_2^*, u_3^* implies that these are linearly independent. Hence the commutators $[X_1, X_2], [X_1, X_3], [X_2, X_3]$ are linearly independent in $\mathfrak{z}(\mathfrak{g})$. Because $\dim \mathfrak{g} = \dim \mathfrak{v} + \dim \mathfrak{z}(\mathfrak{g}) \leq 6$, it follows that $\mathfrak{z}(\mathfrak{g})$ is spanned by these three commutators, that is $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{v}, \mathfrak{v}]$. So $\mathfrak{g}$ is a 6-dimensional butterfly algebra. This gives part (b) of the proposition. ■

We have a converse statement:

Proposition 6.19 *Let* $\mathbf{\Gamma}$ *be a subgroup of* $\mathbf{Iso}(\mathbb{R}^{3,3})$. *Then* $M = \mathbb{R}^{3,3}/\mathbf{\Gamma}$ *is a complete flat pseudo-Riemannian homogeneous manifold if there exists 3-dimensional totally isotropic subspace* U *and in a Witt basis with respect to* U, $\mathbf{\Gamma}$ *is (conjugate in* $\mathbf{Iso}(\mathbb{R}^{3,3})$ *to) a group of one of the following types:*

(a) $\mathbf{\Gamma} = \mathbf{\Lambda} \times \mathbf{\Theta}$, *where* $\mathbf{\Lambda}$ *is a discrete Heisenberg group with Lie algebra generators* X_1, X_2 *as in Lemma 6.16.* $\mathbf{\Theta}$ *is a discrete group generated by translations in* U *linearly independent to* $\mathrm{T}([X_1, X_2])$.

(b) $\mathbf{\Gamma}$ *is a lattice in a butterfly group (Definition 5.19), such that there exist linearly independent non-central elements* $X_1, X_2, X_3 \in \log(\mathbf{\Gamma})$ *as in Lemma 6.16.*

Proof: Both cases can be treated simultaneously. The number $\alpha \neq 0$ from Lemma 6.16 is necessarily the same for X_1, X_2 (and X_3).

(i) The group $\mathbf{\Gamma}$ is discrete because the translation parts of the generators $\exp(X_1), \exp(X_2)$ and those of the generators of $\mathrm{Z}(\mathbf{\Gamma})$ form a linearly independent set.

(ii) We show that the centraliser of $\mathbf{\Gamma}$ is transitive. Consider elements of the form
$$S = \left(\begin{pmatrix} 0 & -\alpha T(z) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \in \mathfrak{iso}(\mathbb{R}^{3,3})$$
with $x, z \in \mathbb{R}^3$ arbitrary. Then $[X_i, S] = 0$ for $i = 1, 2, 3$, because
$$C_i z = \alpha u_i^* \times z = -\alpha z \times u_i^* = -\alpha T(z) u_i^*.$$
Clearly, S also commutes with any translation by a vector from U. So in both cases (a) and (b), $\mathbf{\Gamma}$ has a centraliser with an open orbit at 0. The exponentials of the elements of S clearly generate a unipotent subgroup of $\mathbf{Iso}(\mathbb{R}^{3,3})$, hence the open orbit is also closed and thus all of $\mathbb{R}^{3,3}$.

(iii) From the transitivity of the centraliser, it also follows that the action is free and thus proper (Proposition 7.8). ■

In the situation of Proposition 6.18 it is natural to ask whether the statement can be simplified by claiming that $\mathbf{\Gamma}$ is always a subgroup of a lattice in a butterfly group. But this is not always the case, as Example 6.22 shows. However, the following Remark 6.20 shows that $\mathbf{\Gamma}$ can be taken as a discrete subgroup of the (Zariski closed) butterfly group $\mathbf{B}_6$. Additionally, if $\mathbf{\Gamma} = \mathbf{\Lambda}$ is a discrete Heisenberg group, it can indeed be embedded as a subgroup of a lattice in $\mathbf{B}_6$.

Remark 6.20 In the situation of part (a) in Proposition 6.18, the group $\mathbf{\Gamma} = \mathbf{\Lambda} \times \mathbf{\Theta}$ can be embedded in a butterfly group $\mathbf{B}_6$: Let $\gamma_1 = I + X_1, \gamma_2 = I + X_2$

be the group generators of the discrete Heisenberg group $\mathbf{\Lambda}$ and use the notation from the proof of Proposition 6.18. Choose $u_3^* \in U_0^*$ such that u_3^* is linearly independent to u_1^*, u_2^*. Set

$$X_3 = \left(\begin{pmatrix} 0 & \alpha T(u_3^*) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_3^* \end{pmatrix} \right).$$

As in part (iv) of the proof of Proposition 6.18, the elements X_1, X_2, X_3 generate a butterfly algebra. In particular, their commutators are translations spanning U_0, so it contains $\log(\mathbf{\Theta})$ for any possible $\mathbf{\Theta}$.

Remark 6.21 If $\mathbf{\Gamma} = \mathbf{\Lambda}$, then $\mathbf{\Lambda}$ can be embedded in a discrete butterfly group (that is, a lattice in $\mathbf{B}_6$). Just choose X_3 as in the previous remark. Then the group generated by $\mathbf{\Lambda}$ and $\exp(X_3)$ is a lattice in a butterfly group.

Although $\mathbf{\Lambda}$ can be embedded in a lattice of a butterfly group, a more general group $\mathbf{\Lambda} \times \mathbf{\Theta}$ (as in Proposition 6.18) cannot in general be embedded in a lattice in a butterfly group:

Example 6.22 We choose the generators $\gamma_i = (I + A_i, v_i)$, $i = 1, 2$, of a discrete Heisenberg group $\mathbf{\Lambda}$ as follows: If we write $v_i = u_i + u_i^*$ (where $u_i \in U_0$, $u_i^* \in U_0^*$), let $u_i = 0$, $u_i^* = e_i^*$, $\alpha = 1$ (with α as in the proof of Proposition 6.18 and e_i^* refers to the ith unit vector taken as an element of U_0^*). Then $\gamma_3 = [\gamma_1, \gamma_2] = (I, v_3)$, where $u_3 = e_3$, $u_3^* = 0$. Let $\gamma_4 = (I, u_4)$ be the translation by $u_4 = \sqrt{2} e_1 + \sqrt{3} e_2 \in U_0$. Let $\mathbf{\Theta} = \langle \gamma_4 \rangle$ and $\mathbf{\Gamma} = \mathbf{\Lambda} \cdot \mathbf{\Theta}$ ($\cong \mathbf{\Lambda} \times \mathbf{\Theta}$). Assume there exists $X = (A, v)$ of the form (6.2) not commuting with X_1, X_2. We then have

$$\mathrm{T}([X_1, X]) = e_1 \times u^* = \begin{pmatrix} 0 \\ -\eta_3 \\ \eta_2 \end{pmatrix}, \quad \mathrm{T}([X_2, X]) = e_2 \times u^* = \begin{pmatrix} \eta_3 \\ 0 \\ -\eta_1 \end{pmatrix}$$

where η_i are the components of u^*, and $\eta_3 \neq 0$ due to the fact that X and the X_i do not commute. If $\mathbf{\Gamma}$ could be embedded into into a discrete butterfly group, such X would have to exist. But by construction u_4 is not contained in the $\mathbb{Z}$-span of $e_3, e_1 \times u^*, e_2 \times u^*$, but it is contained in the $\mathbb{R}$-span. So the group generated by $\mathbf{\Gamma}$ and $\exp(X)$ is not discrete in $\mathbf{Iso}(\mathbb{R}^{r,s})$.

Part II
Main Examples

In this part, we present our main examples which illustrate the results described in this thesis. Any reference to a "Main Example" in combination with a number will mean the chapter number. For lack of better naming, the chapter titles sum up the characteristic properties of these examples. Of these examples, Main Example 9 is the only compact space, and Main Example 10 is the only incomplete space.

Due to the large dimensions of these examples, it is impractical to do any computations by hand. The computations were done using the computer algebra system MUPAD[15)], and the reader who wishes to reproduce the computations can find the programme files on the author's homepage [16].

7 Miscellanea

In this chapter we collect some results which do not fit into any other chapter, but are quite helpful when constructing new examples.

7.1 On Open Orbits

The following lemma allows us to do most of our computations on the Lie algebra level.

Lemma 7.1 *If the action of some affine Lie algebra* $\mathfrak{h}$ *at a point* $p \in \mathbb{R}^n$ *is of maximal rank* n*, then the Lie group* $\boldsymbol{H}$ *generated by* $\exp(\mathfrak{h})$ *has an open orbit* $\boldsymbol{H}.p$*.*

PROOF: The tangent space of $\boldsymbol{H}.p$ at p is generated by the tangent action of $\mathfrak{h}$ on p. So the tangent space is of maximal dimension n. Then the orbit $\boldsymbol{H}.p$ is a submanifold of dimension n, hence open. ∎

So, in order to check whether the centraliser $\boldsymbol{L}$ of some potential Wolf group $\boldsymbol{G}$ has an open orbit, it is sufficient to check whether the there is a

15)Sadly, MUPAD is no longer available as independent software. It has been integrated into the numerics software MATLAB, and must be run from the MATLAB command line by typing `mupad`.

point p such that $\mathfrak{Lie}(\boldsymbol{L})$ has an orbit of dimension n at p. This is particularly convenient at $p = 0$, as one only has to check whether the Lie algebra $\mathfrak{Lie}(\boldsymbol{L})$ has $\mathbb{R}^{r+s}$ as the set of its translation parts. Note that this does not guarantee that the action of the centraliser is transitive. But the above lemma also provides a tool to find those points which are not contained in an open orbit.

In order to obtain a homogeneous space from the centraliser's open orbit D, we need to be sure that $\boldsymbol{\Gamma}$ acts on D. This is ensured by the following lemma.

Lemma 7.2 *Assume the centraliser* $\boldsymbol{L}$ *of* $\boldsymbol{\Gamma}$ *in* $\mathbf{Iso}(\mathbb{R}^{r,s})$ *acts with open orbit* D *on* $\mathbb{R}^{r,s}$. *Then* $\boldsymbol{\Gamma}.D = D$.

Proof: As $\boldsymbol{\Gamma}$ commutes with $\boldsymbol{L}$, it permutes the open $\boldsymbol{L}$-orbits. The $\boldsymbol{L}$-action is algebraic, so there are only finitely many such orbits. So a subgroup $\boldsymbol{\Gamma}_0$ of finite index in $\boldsymbol{\Gamma}$ fixes a certain open orbit D. Then the identity component $\boldsymbol{G}^\circ$ of its Zariski closure $\boldsymbol{G}$ (which is also the Zariski closure of $\boldsymbol{\Gamma}$) also fixes D. But $\boldsymbol{G}$ is unipotent, so that $\boldsymbol{G} = \boldsymbol{G}^\circ$ holds. So $\boldsymbol{\Gamma}.D = D$. ■

7.2 Formulae

Let $\boldsymbol{G} \subset \mathbf{Aff}(\mathbb{R}^n)$ be a group of affine transformations. Assume that for all $g_i = (I + A_i, v_i) \in \boldsymbol{G}$ the following relations hold:

$$A_iA_jA_l = 0, \quad A_iA_j = -A_jA_i, \quad A_iA_jv_k = 0, \quad A_iv_j = -A_jv_i. \tag{7.1}$$

In particular, this holds if $\boldsymbol{G}$ is a Wolf group.

Remark 7.3 The relations (7.1) immediately imply $g_i = \exp(A_i, v_i)$, $A_i^2 = 0$ and $A_iv_i = 0$.

Lemma 7.4 *Let* $g_1, \ldots, g_k \in \boldsymbol{G}$, *where* $g_i = (I + A_i, v_i)$. *Then*

$$g_1 \cdots g_k = \left(I + \sum_{i=1}^{k} A_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} A_iA_j,\ \sum_{i=1}^{k} v_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} A_iv_j \right). \tag{7.2}$$

Proof: The proof is by induction on k, where one repeatedly uses the relations (7.1). ■

Corollary 7.5 *Equation (7.2) determines the Baker-Campbell-Hausdorff formula for* $\mathbf{G}$*: For* $g_1, \dots, g_k \in \mathbf{G}$*,* $g_i = \exp(A_i, v_i)$*, we have*

$$\log(g_1 \cdots g_k) = \Big(\sum_{i=1}^{k} A_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} A_i A_j,\ \sum_{i=1}^{k} v_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} A_i v_j\Big). \tag{7.3}$$

Proof: All $g = (I + A, v) \in \mathbf{G}$ satisfy $\log(g) = (A, v)$. So (7.3) follows from (7.2). ■

Lemma 7.6 *Let* $\mathbf{\Gamma} \subset \mathbf{Aff}(\mathbb{R}^n)$ *is finitely generated, and the relations (7.1) are assumed to hold for a set of generators of* $\mathbf{\Gamma}$*. Then these relations hold for all elements* $\gamma \in \mathbf{\Gamma}$*. In particular,* $\mathbf{\Gamma}$ *is unipotent.*

Proof: The equations (7.2) and (7.3) hold for any product involving the generators of $\mathbf{\Gamma}$.

Let $\gamma = g_1 \cdots g_k$, where the $g_i = (I + A_i, v_i)$ are (not necessarily disctinct) generators of $\mathbf{\Gamma}$. By assumption, the A_i, v_i satisfy (7.1). So for any $\gamma = (I + A, v)$, $\log(\gamma) = (A, v)$ is of the form (7.3).

Consider $\gamma_1 = (I + A, v) = g_1^{m_1} \cdots g_k^{m_k}$ and $\gamma_2 = (I + B, w) = g_1^{n_1} \cdots g_k^{n_k}$. Then A, B are determined by (7.3). It follows that,

$$AB = \sum_{i,j} m_i n_j A_i A_j = -\sum_{i,j} m_i n_j A_j A_i = -BA,$$

because all products involving the double sums in (7.3) are 0 due to $A_i A_j A_k = 0$. In a similar way we conclude $Aw = -Bv$.

If $\gamma_3 = (I + C, u) \in \mathbf{\Gamma}$, then $ABC = 0$ because this expression involves only triple products of the A_i. In a similar way we conclude $ABu = 0$.

In particular, $(A, v)^2 = (A^2, Av) = (0, 0)$. So all $\gamma \in \mathbf{\Gamma}$ are unipotent. ■

7.3 A Criterion for Properness

The criterion in this section is due to Oliver Baues and was first published in Baues and Globke [3]. This criterion relies on the transitivity of the centraliser of $\mathbf{G}$.

Lemma 7.7 *Let* $M = \mathbf{L}/\mathbf{L}_p$ *be a homogeneous space, where* $\mathbf{L}$ *is a Lie group and* $\mathbf{L}_p$ *is a closed subgroup, the stabiliser of some* $p \in M$*. Then there exists a surjective homomorphism* $\Phi : \mathrm{N}_{\mathbf{L}}(\mathbf{L}_p) \to \mathrm{Z}_{\mathbf{Diff}(M)}(\mathbf{L})$ *which is continuous with respect to the compact open topology.*

Proof: The right-action of the normaliser $\mathrm{N}_{\boldsymbol{L}}(\boldsymbol{L}_p)$ on $\boldsymbol{L}$ induces a map

$$\Phi : \mathrm{N}_{\boldsymbol{L}}(\boldsymbol{L}_p) \to \mathrm{Z}_{\mathbf{Diff}(M)}(\boldsymbol{L}), \quad n \mapsto \varphi_n^{-1},$$

where $\varphi_n \in \mathbf{Diff}(M)$ is defined by $\varphi_n(l\boldsymbol{L}_p) = ln\boldsymbol{L}_p = l\boldsymbol{L}_p n$ (with $l \in \boldsymbol{L}$).

(i) Φ is a homomorphism: For all $n_1, n_2 \in \mathrm{N}_{\boldsymbol{L}}(\boldsymbol{L}_p)$ and $l \in \boldsymbol{L}$ we have

$$\begin{aligned}\Phi(n_1n_2)(l\boldsymbol{L}_p) &= \varphi_{n_1n_2}^{-1}(l\boldsymbol{L}_p) = l\boldsymbol{L}_p(n_1n_2)^{-1} = l\boldsymbol{L}_p n_2^{-1}n_1^{-1} \\ &= \varphi_{n_1}^{-1}(\varphi_{n_2}^{-1}(l\boldsymbol{L}_p)) = \Phi(n_1)(\Phi(n_2)(l\boldsymbol{L}_p)).\end{aligned}$$

(ii) Φ is continuous: The right- and left-multiplication on $\boldsymbol{L}$ are continuous, so the expression $l\boldsymbol{L}_p n$ depends continuously on l and n. Inversion is continuous as well, so $\Phi(n) = \varphi_n^{-1}$ depends continuously on n.

(iii) Φ is surjective: Let $\varphi \in \mathrm{Z}_{\mathbf{Diff}(M)}(\boldsymbol{L})$ and let $a \in \boldsymbol{L}$ such that $\varphi(\boldsymbol{L}_p) = a\boldsymbol{L}_p$. As φ commutes with left-multiplication by $\boldsymbol{L}$ we get $\varphi(l\boldsymbol{L}_p) = l\varphi(\boldsymbol{L}_p) = la\boldsymbol{L}_p$ for all $l \in \boldsymbol{L}$. As all $h \in \boldsymbol{L}_p$ fix $\boldsymbol{L}_p$ under left-multiplication, there exists $b \in \boldsymbol{L}$ such that

$$a\boldsymbol{L}_p = \varphi(\boldsymbol{L}_p) = \varphi(h\boldsymbol{L}_p) = ha\boldsymbol{L}_p = bh\boldsymbol{L}_p = b\boldsymbol{L}_p,$$

so the cosets $a\boldsymbol{L}_p$ and $b\boldsymbol{L}_p$ coincide if and only if a centralises $\boldsymbol{L}_p$. So indeed $\varphi = \varphi_a$ for $a \in \mathrm{N}_{\boldsymbol{L}}(\boldsymbol{L}_p)$. ■

Proposition 7.8 (Baues) *Let $M = \boldsymbol{L}/\boldsymbol{L}_p$ be a homogeneous space, where $\boldsymbol{L}$ is a Lie group and $\boldsymbol{L}_p$ is a closed subgroup, the stabiliser of some $p \in M$. Let $\boldsymbol{G} \subset \mathbf{Diff}(M)$ be a group of diffeomorphisms of M which centralises $\boldsymbol{L}$. Then $\boldsymbol{G}$ acts properly on M if and only if $\boldsymbol{G}$ is a closed subgroup of $\mathbf{Diff}(M)$ with respect to the compact-open topology.*

Proof: By assumption $\boldsymbol{G} \subset \mathrm{Z}_{\mathbf{Diff}(M)}(\boldsymbol{L})$. Let $\Phi : \mathrm{N}_{\boldsymbol{L}}(\boldsymbol{L}_p) \to \mathrm{Z}_{\mathbf{Diff}(M)}(\boldsymbol{L})$ be the surjective homomorphism from Lemma 7.7 and let $\boldsymbol{G}_0 = \Phi^{-1}(\boldsymbol{G})$. In particular, if $\boldsymbol{G}$ is closed in $\mathbf{Diff}(X)$, then $\boldsymbol{G}_0$ is closed in $\boldsymbol{L}$. Note that $M/\boldsymbol{G} = \boldsymbol{L}/\boldsymbol{G}_0$ is a Hausdorff space if and only if the subgroup $\boldsymbol{G}_0$ is closed in $\boldsymbol{L}$. Since $\boldsymbol{G}$ acts freely on M, $M/\boldsymbol{G}$ is a Hausdorff space if and only if $\boldsymbol{G}$ acts properly on M. ■

We can apply this criterion in the affine situation, as follows:

Corollary 7.9 *Let $\boldsymbol{G} \subset \mathbf{Aff}(\mathbb{R}^n)$ be a subgroup whose centraliser in $\mathbf{Aff}(\mathbb{R}^n)$ acts transitively on $\mathbb{R}^n$. Then the action of $\boldsymbol{G}$ on $\mathbb{R}^n$ is proper if and only if $\boldsymbol{G}$ is a closed subgroup of $\mathbf{Aff}(\mathbb{R}^n)$.*

Similarly, assume that the centraliser $\boldsymbol{L}$ of $\boldsymbol{G}$ in $\mathbf{Aff}(\mathbb{R}^n)$ has an open orbit $D = \boldsymbol{L}.p$ which is preserved by $\boldsymbol{G}$. Then $\boldsymbol{G}$ acts freely on D, and the action is proper if and only if $\boldsymbol{G}$ is closed in $\mathbf{Diff}(D)$. Since $\mathbf{Diff}(D) \cap \mathbf{Aff}(\mathbb{R}^n)$ is closed in $\mathbf{Aff}(\mathbb{R}^n)$ (see Baues [2], Lemma 6.9), the above corollary generalises to:

Corollary 7.10 *Let $\boldsymbol{G} \subset \mathbf{Aff}(\mathbb{R}^n)$ be a subgroup whose centraliser in $\mathbf{Aff}(\mathbb{R}^n)$ acts transitively on an open subset D of $\mathbb{R}^n$. If $\boldsymbol{G}.D = D$, then the action of $\boldsymbol{G}$ on D is proper if and only if $\boldsymbol{G}$ is a closed subgroup of $\mathbf{Aff}(\mathbb{R}^n)$.*

Corollary 7.11 *A Wolf group acts properly discontinuously on the open orbit of its centraliser.*

Remark 7.12 The condition that the centraliser acts transitively is crucial. Püttman [33] gives an example of a free action of the abelian group $(\mathbb{C}^2, +)$ by unipotent affine transformations, such that the quotient is not a Hausdorff space. Hence the action is not proper.

8 Abelian Holonomy, Complete, Signature (3,5)

Many of Wolf's examples (see Wolf [51]) assume that for an abelian Wolf group there exists at least one element such that the skew-symmetric matrix block C in the matrix representation (1.9) is regular. Our first example serves the purpose of showing that in general this is not the case.

Consider $\mathbb{R}^{3,5}$ and choose a totally isotropic subspace U of dimension 3 to play the role of $U_{\boldsymbol{\Gamma}}$. In a Witt basis with respect to U, define the following transformations:

$$X_1 = \left(\begin{pmatrix} 0 & 0 & C_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \\ 0 \end{pmatrix} \right), \quad X_2 = \left(\begin{pmatrix} 0 & 0 & C_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2 \\ 0 \end{pmatrix} \right),$$

where e_i denotes the ith unit vector in $\mathbb{R}^2$ ($i = 1, 2$), and

$$C_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From (A.3) it follows that $\mathrm{L}(X_i) \in \mathfrak{so}_{3,5}$. Clearly $X_i^2 = 0$ and $[X_1, X_2] = 0$, and so the elements $\gamma_i = \exp(X_i) = I + X_i$ (for $i = 1, 2$) generate a discrete abelian subgroup of $\mathbf{Iso}(\mathbb{R}^{3,5})$.

As the C_i are skew 3×3-matrices, no linear combination of them can have full rank. But clearly

$$U = \operatorname{im} \mathrm{L}(X_1) + \operatorname{im} \mathrm{L}(X_2),$$

so indeed $U = U_{\boldsymbol{\Gamma}}$.

The elements X_i commute with the following elements of $\mathfrak{iso}(\mathbb{R}^{3,5})$:

$$S = \left(\begin{pmatrix} 0 & S_1 & 0 \\ 0 & 0 & -S_1^{\top} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right),$$

where $x = (x_1, x_2, x_3)^{\top}$, $y = (y_1, y_2)^{\top}$, $z = (z_1, z_2, z_3)^{\top}$ are arbitrary, and

$$S_1 = \begin{pmatrix} -z_2 & -z_3 \\ z_1 & 0 \\ 0 & z_1 \end{pmatrix}.$$

The elements $\exp(S) \in \mathbf{Iso}(\mathbb{R}^{3,5})$ clearly generate a unipotent subgroup of the centraliser of $\boldsymbol{\Gamma}$ in $\mathbf{Iso}(\mathbb{R}^{3,5})$, and as x, y, z are arbitrary, this subgroup acts transitively on $\mathbb{R}^{3,5}$. By Corollary 7.9, $\boldsymbol{\Gamma}$ acts properly discontinuously, so it is in fact a discrete complete Wolf group.

9 Non-Abelian, Abelian Holonomy, Complete, Compact, Signature (3,3)

This example was first given by Baues [2], Corollary 4.10 and Example 4.3. It was the first known example of a complete flat pseudo-Riemannian homogeneous space with non-abelian fundamental group.

The approach is somewhat different; Wolf treats homogeneous manifolds as quotients $\mathbb{R}^{r,s}/\mathbf{\Gamma}$, whereas Baues constructed his examples as quotients $N/\mathbf{\Lambda}$, where N is a nilpotent Lie group and $\mathbf{\Lambda}$ a lattice in N.

9.1 A Nilpotent Lie Group with Flat Bi-Invariant Metric

Consider the Lie group

$$N = \mathbf{H}_3 \ltimes_{\mathrm{Ad}^*} \mathfrak{h}_3^*,$$

where $\mathbf{H}_3$ is the Heisenberg group

$$\mathbf{H}_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

and $\mathfrak{h}_3^*$ the dual space of its Lie algebra (considered here as an abelian Lie group). The group multiplication is given by

$$(g, x^*)(h, y^*) = (gh, \mathrm{Ad}^*(g)y^* + x^*), \tag{9.1}$$

where $\mathrm{Ad}^*(g)$ is the coadjoint representation defined by

$$(\mathrm{Ad}^*(g)y^*)(x) = y^*(\mathrm{Ad}(g)^{-1}x)$$

for all $x \in \mathfrak{h}_3$, $y^* \in \mathfrak{h}_3^*$ and $g \in \mathbf{H}_3$. So N is not abelian, but 2-step nilpotent.

We define an inner product on its Lie algebra $\mathfrak{n} = \mathfrak{h}_3 \oplus_{\mathrm{ad}^*} \mathfrak{h}_3^*$:

$$\langle (x, x^*), (y, y^*) \rangle_{\mathfrak{n}} = x^*(y) + y^*(x) \tag{9.2}$$

for all $x, y \in \mathfrak{h}_3$, $x^*, y^* \in \mathfrak{h}_3^*$. Then $\mathfrak{h}_3$ and $\mathfrak{h}_3^*$ are totally isotropic subspaces and dual to each other. The metric has signature $(3, 3)$.

The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ induces a left-invariant metric $\langle \cdot, \cdot \rangle_N$ on N via left-multiplication. One checks that for all $X, Y, Z \in \mathfrak{n}$,

$$\langle [X, Y], Z \rangle_{\mathfrak{n}} + \langle Y, [X, Z] \rangle_{\mathfrak{n}} = 0,$$

which means that $\langle\cdot,\cdot\rangle_N$ is in fact a bi-invariant metric (O'Neill [30], chapter 11, Proposition 9). This means the induced curvature tensor is given by

$$\mathsf{R}(X,Y)Z = \frac{1}{4}[X,[Y,Z]].$$

Because $\boldsymbol{N}$ is 2-step nilpotent, $\boldsymbol{N}$ is flat.

9.2 A Lattice in G

Now,

$$\mathbf{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & m & k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, m,n,k \in \mathbb{Z} \right\}$$

is a 2-step nilpotent lattice in $\mathbf{H}_3$, and its representation

$$\mathrm{Ad}^*(\mathbf{H}_3(\mathbb{Z})) \cong \left\{ \begin{pmatrix} 1 & -m & 0 \\ 0 & 1 & 0 \\ 0 & n & 1 \end{pmatrix} \,\middle|\, m,n \in \mathbb{Z} \right\}$$

preserves a lattice $\mathbf{\Theta} \subset \mathfrak{h}_3^*$ isomorphic to $\mathbb{Z}^3$. Then

$$\mathbf{\Lambda} = \mathbf{H}_3(\mathbb{Z}) \ltimes_{\mathrm{Ad}^*} \mathbf{\Theta}$$

is a lattice in $\boldsymbol{N}$ which is 2-step nilpotent (but not abelian).

Thus, the manifold

$$M = \boldsymbol{N}/\mathbf{\Lambda} = (\mathbf{H}_3 \ltimes_{\mathrm{Ad}^*} \mathfrak{h}_3^*)/(\mathbf{H}_3(\mathbb{Z}) \ltimes_{\mathrm{Ad}^*} \mathbf{\Theta})$$

is a compact homogeneous space, and as such it is automatically complete. Also, it inherits a flat pseudo-Riemannian structure from $\boldsymbol{N}$. It is connected, as $\mathbf{H}_3$ and $\mathfrak{h}_3$ are.

9.3 The Development Representation of G

We shall now make explicit the correspondence of M with a homogeneous space $\mathbb{R}^{3,3}/\mathbf{\Gamma}$, where the fundamental group $\mathbf{\Gamma} \subset \mathbf{Iso}(\mathbb{R}^{3,3})$ is isomorphic to the lattice $\mathbf{\Lambda}$.

Identify $\mathfrak{n}$ with $\mathbb{R}^{3,3}$. Then the affine development representation δ' of $\mathfrak{n}$ on $\mathbb{R}^{3,3}$ at the point 0 takes the form

$$\delta'(X).y = \frac{1}{2}\mathrm{ad}(X)y + X,$$

for $X \in \mathfrak{n}$, $y \in \mathbb{R}^{3,3} \cong \mathfrak{n}$. Written in matrix form, this becomes

$$\delta'(X) = \left(\begin{array}{c||c} \frac{1}{2}\mathrm{ad}(X) & X \\ \hline\hline 0 & 0 \end{array}\right) \in \mathfrak{aff}(\mathfrak{n}). \tag{9.3}$$

More precisely, if a Witt basis of $\mathfrak{n} = \mathfrak{h}_3 \oplus_{\mathrm{ad}^*} \mathfrak{h}_3^*$ is given by

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}_3$$

and their dual elements

$$X_1^*, X_2^*, X_3^* \in \mathfrak{h}_3^*,$$

and some element $X = (x, x^*) \in \mathfrak{n}$ is given by

$$x = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \quad x^* = \mu_1 X_1^* + \mu_2 X_2^* + \mu_3 X_3^*$$

with $\lambda_i, \mu_i \in \mathbb{R}$, then $\delta'(X)$ is represented by

$$\delta'(X) = \frac{1}{2}\left(\begin{array}{cccccc||c} 0 & 0 & 0 & 0 & 0 & 0 & 2\lambda_1 \\ -\lambda_2 & 0 & \lambda_1 & 0 & 0 & 0 & 2\lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\lambda_3 \\ 0 & 0 & -\mu_2 & 0 & \lambda_3 & 0 & 2\mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\mu_2 \\ \mu_2 & 0 & 0 & 0 & -\lambda_1 & 0 & 2\mu_3 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{9.4}$$

with respect to this basis. The upper left 6×6-block is the linear part $A = \mathrm{L}(X)$. By exponentiation, we get the corresponding representation δ for $\boldsymbol{N}$. An element $g = \delta(\exp(X)) \in \boldsymbol{N}$ is represented by a matrix

$$g = \left(\begin{array}{cccccc||c} 1 & 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ -\frac{\lambda_2}{2} & 1 & \frac{\lambda_1}{2} & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & -\frac{\mu_2}{2} & 1 & \frac{\lambda_3}{2} & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \mu_2 \\ \frac{\mu_2}{2} & 0 & 0 & 0 & -\frac{\lambda_1}{2} & 1 & \mu_3 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{9.5}$$

With respect to the chosen Witt basis, the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ is represented by the matrix

$$Q = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}.$$

Now one checks

$$A^{\top}Q + QA = 0,$$

so $\boldsymbol{G} = \delta(\boldsymbol{N})$ is a group of isometries for $\mathfrak{n}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$.

Let θ_0 denote the orbit map at $0 \in \mathfrak{n}$ for $\boldsymbol{N}$,

$$\theta_0 : \boldsymbol{N} \to \mathfrak{n}, \quad n \mapsto \delta(n).0.$$

From the matrix form of $g = \delta(n)$ it is clear that this is a diffeomorphism. For $X, Y \in \mathfrak{n} = \mathrm{T}_{1_N}\boldsymbol{N}$ we have by definition of $\langle\cdot,\cdot\rangle_{\boldsymbol{N}}$

$$\langle X, Y\rangle_{\boldsymbol{N}}|_{1_N} = \langle X, Y\rangle_{\mathfrak{n}},$$

and because $\delta(n)$ is an isometry and $\delta'(X)$ its differential (for $n = \exp(X)$),

$$\langle X, Y\rangle_{\mathfrak{n}} = \langle \delta'(X).0, \delta'(Y).0\rangle_{\mathfrak{n}} = \langle \theta_0'(X), \theta_0'(Y)\rangle_{\mathfrak{n}},$$

where θ_0' is the differential of θ_0. So θ_0 is an isometry from $\boldsymbol{N}$ to $\mathfrak{n} \cong \mathbb{R}^{3,3}$. Under this correspondence, the lattice $\boldsymbol{\Lambda} \subset \boldsymbol{N}$ maps to the lattice

$$\boldsymbol{\Gamma} = \delta(\boldsymbol{\Lambda}) \subset \delta(\boldsymbol{N}) = \boldsymbol{G}$$

whose elements are represented by matrices (9.5) with $\lambda_i, \mu_i \in \mathbb{Z}$ (note that when multiplying two matrices of this type, no denominator other than 1 or 2 appears, so $\boldsymbol{\Gamma}$ is indeed closed under multiplication).

Since $\boldsymbol{N}$ is connected, $\boldsymbol{G}$ is connected as well. Then $\boldsymbol{G}$ centralises the action of $\boldsymbol{\Gamma}$. This means

$$\theta_0(n\lambda) = \delta(n\lambda).0 = \delta(\lambda)\delta(n).0 = \delta(\lambda)\theta_0(n)$$

for all $\lambda \in \boldsymbol{\Lambda}$, $n \in \boldsymbol{N}$. So the right-action of $\boldsymbol{\Lambda}$ on $\boldsymbol{N}$ corresponds to the action of $\boldsymbol{\Gamma}$ on $\mathbb{R}^{r,s}$ under the isometry θ_0. Now θ_0 induces an isometry

$$\boldsymbol{N}/\boldsymbol{\Lambda} = M \to \mathbb{R}^{3,3}/\boldsymbol{\Gamma}, \quad n\boldsymbol{\Lambda} \mapsto \boldsymbol{\Gamma}.\theta_0(n).$$

9.4 The Linear Holonomy Group

The linear part $I + A$ of an element of $\boldsymbol{G}$ is given by the upper 6×6-block in (9.5), so

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_2 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_2 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & -\lambda_1 & 0 \end{pmatrix}.$$

From (9.3) if follows that the space U_{Γ} defined in (1.7) is

$$U_{\Gamma} = \sum_{\gamma \in \Gamma} \operatorname{im} A = \sum_{X \in \log(\Gamma)} \operatorname{im} \operatorname{ad}(X) = [\mathfrak{n}, \mathfrak{n}] = \mathbb{R}X_2 \oplus \mathbb{R}X_1^* \oplus \mathbb{R}X_3^*,$$

with basis elements X_i, X_i^* defined as above. Now U_{Γ} is totally isotropic with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ (this is immediate from (9.2)). Further, $U_{\Gamma}^{\perp} = U_{\Gamma}$, as U_{Γ} is of maximal dimension 3.

Changing to a representation for a Witt basis for U_{Γ}, for example

$$\{\underbrace{X_2, X_1^*, X_3^*}_{\in U_{\Gamma}}, \underbrace{X_2^*, X_1, X_3}_{\in U_{\Gamma}^*}\},$$

the matrix A transforms to

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & -\lambda_3 & \lambda_1 \\ 0 & 0 & 0 & \lambda_3 & 0 & -\mu_2 \\ 0 & 0 & 0 & -\lambda_1 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this representation it is immediate that the linear holonomy of $\mathbf{\Gamma}$ is abelian. Of course, this was to be expected from Theorem 4.4.

Note that, as in Main Example 8, the skew-symmetric upper right block in these matrices is always singular.

10 Non-Abelian Holonomy, Incomplete, Signature (4,4)

The group $\mathbf{\Gamma}$ in this example is a faithful unipotent representation of the discrete Heisenberg group in $\mathbf{Iso}(\mathbb{R}^{4,4})$. It has a centraliser $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{4,4})}(\mathbf{\Gamma})$ with open orbit D_0 through 0. The $\mathbf{\Gamma}$-action has a fixed point set of codimension 2, so the homogeneous space $D_0/\mathbf{\Gamma}$ is not complete.

10.1 The Group Generators

Choose a totally isotropic subspace U of dimension 2 in $\mathbb{R}^{4,4}$. We give the generators $\gamma_1 = (I + A_1, v_1)$, $\gamma_2 = (I + A_2, v_2)$ of $\mathbf{\Gamma}$ in a Witt basis with respect to U:

$$\gamma_1 = \left(\begin{pmatrix} I_2 & -B_1^\top \tilde{I} & 0 \\ 0 & I_4 & B_1 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 \\ w_1 \\ 0 \end{pmatrix}\right), \quad \gamma_2 = \left(\begin{pmatrix} I_2 & -B_2^\top \tilde{I} & 0 \\ 0 & I_4 & B_2 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix}\right).$$

Here,

$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

and $\tilde{I} = \left(\begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix}\right)$ is the signature matrix of W_0 (as in (2.3)). Their commutator is

$$\gamma_3 = [\gamma_1, \gamma_2] = \left(\begin{pmatrix} I_2 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix}\right),$$

with

$$C_3 = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ -4 \end{pmatrix},$$

and further for $i = 1, 2$

$$[\gamma_i, \gamma_3] = I.$$

So $\mathbf{\Gamma}$ is isomorphic to a discrete Heisenberg group, and U in fact coincides with the subspace $U_0(\mathbf{\Gamma})$ from (2.1).

Write $\gamma_i = (I + A_i, v_i)$. Since $A_1 A_2 = -A_2 A_1$, it follows from Lemma 7.6 that $\mathbf{\Gamma}$ is unipotent and 2-step nilpotent, and

$$\exp(A, v) = (I + A, v).$$

The elements $X = (A, v)$ generate the Lie algebra $\mathfrak{g}$ of the Zariski closure $\boldsymbol{G}$ of $\boldsymbol{\Gamma}$. In the chosen basis, the pseudo-scalar product is represented by the matrix

$$Q = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & \tilde{I} & 0 \\ I_2 & 0 & 0 \end{pmatrix}.$$

The elements A_i are by definition of the form (A.2), so $A_i \in \mathfrak{so}_{4,4}$ and so $\boldsymbol{\Gamma} \subset \boldsymbol{G} \subset \mathbf{Iso}(\mathbb{R}^{4,4})$.

Note though that $\boldsymbol{\Gamma}$ has e_7 (the 7th unit vector in $\mathbb{R}^8$) as a fixed point. Therefore, $\boldsymbol{\Gamma}$ does not act properly on $\mathbb{R}^8$. But we will find an open subset $D_0 \subset \mathbb{R}^8$ on which $\boldsymbol{\Gamma}$ acts properly, so that $M = D_0/\boldsymbol{\Gamma}$ is a manifold.

10.2 The Centraliser

Let $\boldsymbol{L} = \mathrm{Z}_{\mathbf{Iso}(\mathbb{R}^{4,4})}(\boldsymbol{\Gamma})$. Its Lie algebra $\mathfrak{l}$ consists of the elements

$$S = \left(\begin{pmatrix} S_1 & -S_2^\top \tilde{I} & S_3 \\ 0 & S_4 & S_2 \\ 0 & 0 & -S_1^\top \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right), \tag{10.1}$$

where $x = (x_1, x_2)^\top, y = (y_1, y_2, y_3, y_4)^\top, z = (z_1, z_2)^\top$ are arbitrary and

$$S_1 = \begin{pmatrix} z_1 & z_2 \\ z_2 - 2a & -z_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -y_1 & y_3 - y_2 + b \\ -y_2 & y_1 + y_4 - c \\ -y_3 & c \\ -y_4 & b \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & a & a - z_2 & -z_1 \\ -a & 0 & z_1 & a - z_2 \\ a - z_2 & z_1 & 0 & -a \\ -z_1 & a - z_2 & a & 0 \end{pmatrix},$$

with free parameters $a, b, c, d \in \mathbb{R}$.

As x, y, z are arbitrary, $\boldsymbol{L}$ has an open orbit $\boldsymbol{L}.0 = D_0$. But as noted above, $\boldsymbol{\Gamma}$ does not act freely on $\mathbb{R}^{4,4}$. Consequently, $\boldsymbol{L}$ does not act transitively on all of $\mathbb{R}^{4,4}$, that is $D_0 \neq \mathbb{R}^{4,4}$.

By Proposition 7.2, the $\boldsymbol{\Gamma}$-action preserves D_0, so $\boldsymbol{\Gamma}$ acts freely on D_0 and by Corollary 7.10, $\boldsymbol{\Gamma}$ acts properly discontinuously. So

$$M = D_0/\boldsymbol{\Gamma}$$

is an incomplete flat pseudo-Riemannian homogeneous space with signature $(4,4)$. We will study D_0 in more detail in the next section.

We compute the Chevalley decomposition (G.1) of $\boldsymbol{L}$. If ξ runs through the parameters $x_i, y_j, z_k, a, b, c, d$, let S_ξ denote the element S in (10.1), with $\xi = 1$ and all other parameters $= 0$. Further, set $S_0 = -\frac{1}{4}S_a - \frac{1}{2}S_{z_2}$. Then

$$\{S_{x_1}, S_{x_2}, S_{y_1}, S_{y_2}, S_{y_3}, S_{y_4}, S_b, S_c, S_d, S_{z_1}, S_0, S_a\}$$

is a basis of $\mathfrak{l}$.

Lemma 10.1 *Let $\mathfrak{s}$ denote the subalgebra generated by S_0, S_{z_1}, S_a Then*

$$\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R}).$$

Proof: One computes

$$[S_{z_1}, S_0] = 2S_0, \quad [S_{z_1}, S_a] = -2S_a, \quad [S_0, S_a] = S_{z_1}.$$

These are the relations defining $\mathfrak{sl}_2(\mathbb{R})$. ■

Lemma 10.2 *Let $\mathfrak{u}$ denote the subalgebra generated by $S_{x_1}, S_{x_2}, S_{y_1}, S_{y_2}, S_{y_3}, S_{y_4}, S_b, S_c, S_d$. Then $\mathfrak{u}$ is a nilpotent ideal consisting of upper triangular matrices.*

Proof: The upper triangular form is immediate from (10.1). Hence $\mathfrak{u}$ is nilpotent. By tedious computations[16)] one checks that $\mathfrak{u}$ is a subalgebra and that $[S_0, \mathfrak{u}], [S_a, \mathfrak{u}], [S_{z_1}, \mathfrak{u}] \subset \mathfrak{u}$, so $\mathfrak{u}$ is an ideal. ■

Proposition 10.3 *Let $\boldsymbol{U} = \exp(\mathfrak{u})$ and $\boldsymbol{S} \subset \boldsymbol{L}$ a Lie subgroup isomorphic to $\mathbf{SL}_2(\mathbb{R})$ with Lie algebra $\mathfrak{s}$. The Chevalley decomposition of $\boldsymbol{L}^\circ$ is*

$$\boldsymbol{L}^\circ = \boldsymbol{S} \cdot \boldsymbol{U}.$$

Proof: We have $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{u}$ (semidirect sum). Further, $\mathfrak{u}$ is a maximal nilpotent ideal in $\mathfrak{l}$ (otherwise, $\mathfrak{l}/\mathfrak{u}$ would have a nilpotent ideal, but $\mathfrak{l}/\mathfrak{u} \cong \mathfrak{sl}_2(\mathbb{R})$ by Lemma 10.1). As $\mathfrak{u}$ consists of upper triangular matrices, $\boldsymbol{U} = \exp(\mathfrak{u})$ is a maximal unipotent normal subgroup.

There exists a Lie subgroup $\boldsymbol{S}$ of $\boldsymbol{L}$ generated by $\exp(\mathfrak{s})$. It is isomorphic to $\mathbf{SL}_2(\mathbb{R})$ by Lemma 10.1. Together, $\boldsymbol{S}$ and $\boldsymbol{U}$ generate $\boldsymbol{L}^\circ$. ■

Remark 10.4 The point e_7 is a common fixed point for all elements of $\boldsymbol{S}$, and it is a fixed point for all elements of $\boldsymbol{U}$ whose translation part is in W_0. This is immediate from (10.1).

16) Or with the help of mupad.

10.3 The Open Orbit of the Centraliser

Use coordinates x, y, z for $\mathbb{R}^8$ as in the translation part of (10.1). Let D_L denote the union of open orbits of $\boldsymbol{L}$, and let P denote the closed subset in $\mathbb{R}^8$ given by

$$P = \{(x, y, z)^\top \in \mathbb{R}^8 \mid z_1 = 1, z_2 = 0\}.$$

We shall prove that the open orbit D_0 of $\boldsymbol{L}$ through 0 is the only open orbit ($D_0 = D_{\boldsymbol{L}}$), and that

$$D_0 = \mathbb{R}^8 \backslash P. \tag{10.2}$$

Lemma 10.5 *P does not intersect any open orbit of $\boldsymbol{L}$, that is $P \subset \mathbb{R}^8 \backslash D_{\boldsymbol{L}}$.*

Proof: A computation in affine coordinates (the representation of $\boldsymbol{\Gamma}$ from section 10.1) shows that every $p \in P$ is a fixed point for the non-trivial element $\gamma_3 \in \boldsymbol{\Gamma}$. Hence p is not contained in an open orbit of $\boldsymbol{L}$ (see Remark 1.15). ■

Lemma 10.6 *Let $p = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)^\top \in \mathbb{R}^8$. Then the tangent action $S \mapsto S.p = \mathrm{L}(S)p + \mathrm{T}(S)$ of $\mathfrak{l} = \mathfrak{Lie}(\boldsymbol{L})$ has maximal rank at p if $(p_7, p_8) \neq (1, 0)$.*

Proof: The action of $\mathfrak{l}$ at p has full rank if for any $q \in \mathbb{R}^8$ one can find an element $S \in \mathfrak{l}$ as in (10.1) such that $S.p = q$.

(i) First, assume $p_7 = 1, p_8 \neq 0$, and set the free parameters a, d in S equal to 0. We write out the equation $S.p = q$:

$$\begin{pmatrix} x_1 + p_3y_1 + p_4y_2 - p_5y_3 - p_6y_4 + p_1z_1 + p_2z_2 \\ x_2 + bp_6 + cp_5 + p_1z_2 - p_2z_1 - p_3(b - y_2 + y_3) - p_4(y_1 + y_4 - c) \\ p_8(b - y_2 + y_3) - p_6z_1 - p_5z_2 \\ p_5z_1 - p_6z_2 + p_8(y_1 + y_4 - c) \\ p_8c - p_3z_2 + p_4z_1 \\ p_8b - p_3z_1 - p_4z_2 \\ -p_8z_2 \\ p_8z_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix}.$$

As $p_8 \neq 0$, the last two rows can be solved directly for z_1 and z_2. Plugging these into rows 5 and 6, one then solves for c and b. Then rows 3 and 4 can be solved for y_1, y_2, with y_3, y_4 arbitrary (or the other way round). Finally, x_1 and x_2 can be determined from the first two rows.

(ii) Now, let $p_7 \neq 1$. We assume the free parameters a, b, c, d in S set to 0 and write out the equation $S.p = q$:

$$\begin{pmatrix} x_1 + p_3 y_1 + p_4 y_2 - p_5 y_3 - p_6 y_4 + p_1 z_1 + p_2 z_2 \\ x_2 - p_4(y_1 + y_4) + p_1 z_2 - p_2 z_1 + p_3(y_2 - y_3) \\ (1 - p_7) y_1 - p_8 y_2 - p_5 z_2 - p_6 z_1 + p_8 y_3 \\ (1 - p_7) y_2 + p_8 y_1 + p_8 y_4 + p_5 z_1 - p_6 z_2 \\ (1 - p_7) y_3 - p_3 z_2 + p_4 z_1 \\ (1 - p_7) y_4 - p_3 z_1 - p_4 z_2 \\ (1 - p_7) z_1 - p_8 z_2 \\ p_8 z_1 + (1 - p_7) z_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix}.$$

The last two rows of the above equation yield a system of two linear equations for the unknowns z_1, z_2. This system has a solution if and only if $(1 - p_7)^2 + p_8^2 \neq 0$, which is the case if $p_7 \neq 1$. Once z_1, z_2 are determined, the rows 5 and 6 can be solved directly for y_3 and y_4 (because $p_7 \neq 1$). Then rows 3 and 4 again yield a system of linear equations for y_1 and y_2, which again is solvable precisely if $p_7 \neq 1$. Plugging in all known variables in the first two rows, we can solve directly for x_1 and x_2.

So for any p with $(p_7, p_8) \neq (1, 0)$, the tangent action is of maximal rank. ■

Theorem 10.7 *Let $D_0 = \mathbf{L}.0$. Then*

$$D_{\mathbf{L}} = D_0 = \mathbb{R}^8 \backslash P.$$

In particular, D_0 is the only open orbit of $\mathbf{L}$.

Proof: Combining Lemma 10.5 and Lemma 10.6, the points $p \in \mathbb{R}^8 \backslash P$ are precisely the points in $D_{\mathbf{L}}$. As P is an affine subspace of codimension 2, the set $\mathbb{R}^8 \backslash P$ is connected. Hence it consists of a single orbit, D_0. ■

Remark 10.8 Clearly the set D_0 is invariant under translation by a vector v if and only if $v \in U_0^{\perp}$. So D_0 is translationally isotropic.

Remark 10.9 Note that the set D_0 is not simply connected. In fact, it is diffeomorphic to $\mathbb{R}^6 \times (\mathbb{R}^2 \backslash \{0\})$. So the affine holonomy group $\mathbf{\Gamma}$ is not the fundamental group of M.

10.4 The Complement of the Open Orbit

In this section, we study how $\mathbf{\Gamma}$ and its centraliser $\boldsymbol{L}$ act on the complement P of the open orbit of $\boldsymbol{L}$.

We write $V = U_0^{\perp}$. Recall that the space $\mathbb{R}^8$ decomposes as

$$\mathbb{R}^8 = V \oplus U_0^*,$$

where V is spanned by the first six unit vectors (that is, it is the subspace with coordinates x, y), and U_0^* is spanned by the 7th and 8th unit vector (that is, it has coordinates z). The complement of the centraliser's open orbit is the 6-dimensional affine subspace $P = e_7 + V$. The induced metric on P is degenerate of signature $(2, 2, 2)$.

Proposition 10.10 *The action of* $\mathrm{Z}(\mathbf{\Gamma}) = \langle \gamma_3 \rangle$ *on* P *is trivial. The induced action of* $\mathbf{\Lambda} = \mathbf{\Gamma}/\mathrm{Z}(\mathbf{\Gamma})$ *on* P *is linear.*

Proof: It is immediate from the matrix representation of $\gamma_1, \gamma_2, \gamma_3$ that e_7 is fixed by all of $\mathbf{\Gamma}$ and all of P is fixed by γ_3. So the action of $\mathbf{\Gamma}$ on P is linear with origin e_7 and γ_3 acts trivially. ■

If we choose e_7 as the origin, we may identify P with the vector space V. The induced linear action of $\mathbf{\Lambda}$ is represented by matrices

$$\lambda_{s,t} = \begin{pmatrix} I_2 & -(sB_1^{\top} + tB_2^{\top})\tilde{I} \\ 0 & I_4 \end{pmatrix} \in \mathbf{GL}(V) \tag{10.3}$$

with B_1, B_2 as defined in section 10.1. By $h_{s,t}$ we denote the respective elements of the Zariski closure $\boldsymbol{H}$ of $\mathbf{\Lambda}$.

Remark 10.11 The action of $\mathbf{\Lambda}$ on V stabilises the degenerate subspace U_0.

Proposition 10.12 *For each non-trivial* $\lambda = \lambda_{s,t} \in \mathbf{\Lambda}$*, the fixed point set* $\operatorname{fix}\lambda$ *of* λ *in* V *is a linear subspace of* V *of dimension* 4*.*

Proof: A point $v \in V$ is fixed by $\lambda_{s,t}$ if and only if it satisfies the linear system

$$\begin{aligned} sv_3 - tv_4 - tv_5 - sv_6 &= 0, \\ tv_3 + sv_4 - sv_5 + tv_6 &= 0, \end{aligned} \tag{10.4}$$

which is solvable for all $s, t \in \mathbb{R}$. It is of rank 2 unless $s = t = 0$. ■

Note that the system (10.4) is independent of the first two coordinates v_1, v_2.

Corollary 10.13 *More precisely,*

$$\operatorname{fix}\lambda = U_0 \oplus N_{s,t},$$

where $N_{s,t} \subset W_0$ is the kernel of the linear system (10.4).

For $r \in \mathbb{R}$, set

$$Q_r = \{v \in V \mid v_3^2 + v_4^2 - v_5^2 - v_6^2 = r\} \subset V.$$

Clearly, $U_0 \cong \mathbb{R}^2 \subset Q_r$ for all r. More precisely, Q_r is a hyperquadric in V, and we have

$$Q_r = \begin{cases} \mathbb{R}^2 \times \mathbf{S}_2^3(r) & \text{if } r > 0, \\ \mathbb{R}^2 \times \mathbf{C} & \text{if } r = 0, \\ \mathbb{R}^2 \times \mathbf{H}_1^3(r) & \text{if } r < 0, \end{cases} \tag{10.5}$$

where $\mathbf{S}_2^3(r)$, $\mathbf{H}_2^3(r)$ are pseudo-spherical and pseudo-hyperbolic spaces in $W_0 \cong \mathbb{R}^{2,2}$, and $\mathbf{C}$ is the light cone in W_0 (see O'Neill [30], Definition 23 in chapter 4).

Proposition 10.14 *Let $\boldsymbol{H}$ denote the Zariski closure of $\boldsymbol{\Lambda}$. Then*

$$Q_0 = \bigcup_{h \in \boldsymbol{H}\setminus\{I\}} \operatorname{fix} h = \bigcup_{s,t \in \mathbb{R}} U_0 \oplus N_{s,t}$$

and $\boldsymbol{\Lambda}$ acts freely on $E = V \setminus Q_0$, which is an open subset of V. Furthermore, E is a disjoint union

$$E = E_+ \cup E_-$$

of the set $E_+ = \bigcup_{r>0} Q_r$ of spacelike vectors and the set $E_- = \bigcup_{r<0} Q_r$ of timelike vectors.

Proof: If one fixes $v \in V$, then (10.4) can be seen as a linear system for the variables $s, t \in \mathbb{R}$,

$$\begin{aligned} s(v_3 - v_6) - t(v_4 + v_5) &= 0, \\ s(v_4 - v_5) + t(v_3 + v_6) &= 0, \end{aligned} \tag{10.6}$$

and this system has non-trivial real solutions precisely if $v_3^2 + v_4^2 - v_5^2 - v_6^2 = 0$. These v are precisely the isotropic vectors in V with respect to the induced metric.

It follows that every vector in E is not isotropic, so it is either spacelike or timelike. ■

Corollary 10.15 *If $v \notin Q_0$, then $\boldsymbol{H}.v = v + U_0 \subsetneq Q_r$. In particular, the quotient for the $\boldsymbol{H}$-action on Q_r is $Q_r/\boldsymbol{H} = \mathbf{S}_2^3(r)$ for $r > 0$ and $Q_r/\boldsymbol{H} = \mathbf{H}_1^2(r)$ for $r < 0$.*

Proof: For $v \notin Q_0$, the system (10.6) is regular. So any inhomogeneous system associated to it has a unique solution. In other words, for every $w \in v + U_0$ we can find s, t such that $h_{s,t}.v = w$.

The assertion on the quotients follows from (10.5). ∎

Corollary 10.16 *If $v \in Q_0 \backslash \{0\}$, then $\boldsymbol{H}.v$ is a 1-dimensional affine subspace of $v + U_0$.*

Proof: From the system (10.6) it follows that $\boldsymbol{H}$ does not act trivially on $v \neq 0$, but also that some $h \in \boldsymbol{H}$ fix v. Write v_x, v_y for the components of v corresponding to the x- and y-coordinates. If $h_{s,t}$ is an element not fixing v, then $\boldsymbol{H}.v = \{h_{\alpha s, \alpha t}.v \mid \alpha \in \mathbb{R}\}$, and

$$h_{\alpha s,\alpha t}.v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} - \begin{pmatrix} \alpha(sB_1^\top + tB_2^\top)\tilde{I}v_y \\ 0 \end{pmatrix}.$$

By varying α through all of $\mathbb{R}$, we see $\boldsymbol{H}.v$ is an affine line through v. ∎

Let $\boldsymbol{G}$ denote the Zariski closure of $\boldsymbol{\Gamma}$. Then the restriction of the $\boldsymbol{G}$-action of P is represented by $\boldsymbol{H}$. The following theorem sums up the results on the orbits of $\boldsymbol{G}$ and $\boldsymbol{H}$.

Theorem 10.17 $\mathbb{R}^{4,4}$ *has a partition into $\boldsymbol{G}$-orbits of the following type:*

(a) *3-dimensional affine subspaces $\boldsymbol{G}.p \subset D_0$.*

(b) *2-dimensional affine subspaces $v+U_0$ parameterised by the $v \in \mathbf{S}_2^3(r) \cup \mathbf{H}_1^2(r) \subset W_0$.*

(c) *1-dimensional affine lines in Q_0.*

(d) *A fixed point e_7 for $\boldsymbol{G}$.*

We go on to study the action of the centraliser $\boldsymbol{L}$ on the space V. By comparing with the representation (10.1) of elements in $\mathfrak{l} = \mathfrak{Lie}(\boldsymbol{L})$ applied to P, we find that an element S of $\mathfrak{l}$ acts by

$$S.v = \begin{pmatrix} S_1 & -S_2^\top \tilde{I} \\ 0 & S_4 \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} x \\ 0 \end{pmatrix}, \tag{10.7}$$

where $x, v_x \in U_0$, $v_y \in W_0$ and S_1, S_2, S_4 as in (10.1). So $\boldsymbol{L}$ clearly does not act transitively on V. In fact, $S_4 \in \mathfrak{so}_{2,2}$, so for all r

$$\exp(S).Q_r = Q_r.$$

So we have to study the action of $\boldsymbol{L}$ on the Q_r. Note that the affine connection on Q_r induced by the natural connection on V is not flat. So the situation is not in analogy to the action of a Wolf group on some open subset of $\mathbb{R}^n$.

Remark 10.18 The centraliser $\boldsymbol{L}'$ of $\boldsymbol{\Lambda}$ in $\mathbf{Aff}(V)$ is larger than the restriction of $\boldsymbol{L}$ to V. But the only difference is that in the matrix representation (10.7), the submatrix $-S_2^\top \tilde{I}$ can be replaced by an arbitrary matrix. This affects only the x-coordinates of the action. But as both $\boldsymbol{L}'$ and $\boldsymbol{L}$ contain the translations by elements of U_0, their orbits are identical.

As $\boldsymbol{L}$ contains translations by elements of U_0, in order to understand the action of $\exp(S)$, we need to study the action of the submatrix S_4 on W_0. This amounts to studying the action of the subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$ from Lemma 10.1. With the S_4, a, z_1, z_2 from (10.1),

$$S_4 \cdot \begin{pmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} v_4 + v_5 & -v_6 & -v_5 \\ v_6 - v_3 & v_5 & -v_6 \\ v_3 - v_6 & v_4 & -v_3 \\ v_4 + v_5 & -v_3 & -v_4 \end{pmatrix} \cdot \begin{pmatrix} a \\ z_1 \\ z_2 \end{pmatrix}. \tag{10.8}$$

Given $v \in Q_r$, one checks that the 3×3-minors of this matrix are rv_3, $-rv_4$, $-rv_5$, rv_6.

Proposition 10.19 *If $r \neq 0$, then the centraliser $\boldsymbol{L}$ acts transitively on Q_r.*

Proof: If $r \neq 0$, at least one 3×3-minor of the matrix (10.8) is $\neq 0$, so the matrix is of rank 3. Together with the action of the translations by U_0, it follows that the centraliser $\boldsymbol{L}$ has an open orbit at every point in Q_r.

By (10.5) and Lemma 25 in chapter 4 of O'Neill [30], Q_r is diffeomorphic to the connected space $\mathbb{R}^4 \times \mathbf{S}^1$. So there is only one open orbit. ■

Assume $r = 0$. We know from Proposition 10.14 that $\boldsymbol{H}$ has fixed points on Q_0, so the centraliser cannot act transitively on Q_0.

Proposition 10.20 *Let $v \in Q_0$. Then the orbit $\boldsymbol{L}.v$ is one of the following:*

(a) *$\boldsymbol{L}.v = U_0$ if $v \in U_0$.*

(b) *$\boldsymbol{L}.v$ is a 4-dimensional submanifold of V, and $U_0 \subset \boldsymbol{L}.v$.*

Proof: $\boldsymbol{L}$ contains the translations by U_0. From the matrix representation of $S \in \mathfrak{Lie}(\boldsymbol{L})$ it is clear that $\boldsymbol{L}.U_0 = U_0$. If $v \in Q_0 \backslash U_0$, one checks that the matrix (10.8) is of rank 2. So $\boldsymbol{L}.v$ is an orbit of dimension 4. ■

The following theorem sums up the above discussion.

Theorem 10.21 $\mathbb{R}^{4,4}$ *has a partition into* $\mathbf{L}$*-orbits of the following type:*

(a) *The unique open orbit* $D_0 = \mathbb{R}^8 \backslash P$ *of dimension* 8*.*

(b) *For every* $r \in \mathbb{R}\backslash\{0\}$*, a hyperquadric* $Q_r \subset P$ *of dimension* 5*.*

(c) *A submanifold* $\mathbf{L}.v$ *for* $v \in Q_0 \backslash U_0$ *of dimension* 4*.*

(d) *The subspace* U_0 *of dimension* 2*.*

11 Non-Abelian Holonomy, Complete, Signature (7,7)

The group $\mathbf{\Gamma}$ in this example is a faithful unipotent representation of the discrete Heisenberg group in $\mathbf{Iso}(\mathbb{R}^{7,7})$.

11.1 The Group Generators

Choose a totally isotropic subspace U of dimension 5 in $\mathbb{R}^{7,7}$. We give the generators $\gamma_1 = (I + A_1, v_1)$, $\gamma_2 = (I + A_2, v_2)$ of $\mathbf{\Gamma}$ in a Witt basis with respect to U:

$$\gamma_1 = \left(\begin{pmatrix} I_5 & -B_1^\top \tilde{I} & C_1 \\ 0 & I_4 & B_1 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_1^* \end{pmatrix} \right), \quad \gamma_2 = \left(\begin{pmatrix} I_5 & -B_2^\top \tilde{I} & C_2 \\ 0 & I_4 & B_2 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_2^* \end{pmatrix} \right).$$

Here,

$$B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $\tilde{I} = \left(\begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix}\right)$ is the signature matrix of W_0 (as in (2.3)). Their commutator is

$$\gamma_3 = [\gamma_1, \gamma_2] = \left(\begin{pmatrix} I_5 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix} \right),$$

with

$$C_3 = \begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix},$$

and further for $i = 1, 2$

$$[\gamma_i, \gamma_3] = I.$$

So $\mathbf{\Gamma}$ is isomorphic to a discrete Heisenberg group, and U in fact coincides with the subspace $U_0(\mathbf{\Gamma})$ from (2.1).

Write $\gamma_i = (I + A_i, v_i)$. Since $A_1 A_2 = -A_2 A_1$, it follows from Lemma 7.6 that $\mathbf{\Gamma}$ is unipotent and 2-step nilpotent, and

$$\exp(A, v) = (I + A, v).$$

The elements $X = (A, v)$ generate the Lie algebra $\mathfrak{g}$ of the Zariski closure $\boldsymbol{G}$ of $\mathbf{\Gamma}$. In the chosen basis, the pseudo-scalar product is represented by the matrix

$$Q = \begin{pmatrix} 0 & 0 & I_5 \\ 0 & \tilde{I} & 0 \\ I_5 & 0 & 0 \end{pmatrix}.$$

From (A.2) it follows that $A_1, A_2 \in \mathfrak{so}_{7,7}$, that is $\mathbf{\Gamma} \subset \boldsymbol{G} \subset \mathbf{Iso}(\mathbb{R}^{7,7})$.

11.2 The Centraliser

The following elements $S \in \mathfrak{iso}(\mathbb{R}^{7,7})$ commute with (A_1, v_1) and (A_2, v_2):

$$S = \left(\begin{pmatrix} S_1 & -S_2^\top \tilde{I} & S_3 \\ 0 & 0 & S_2 \\ 0 & 0 & -S_1^\top \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right), \tag{11.1}$$

where $x = (x_1, \ldots, x_5)^\top, y = (y_1, \ldots, y_4)^\top, z = (z_1, \ldots, z_5)^\top$ are arbitrary and

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -2z_2 \\ 0 & 0 & 0 & 0 & 2z_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & -z_2 & z_1 & 0 \\ 0 & 0 & z_1 & z_2 & 0 \\ 0 & 0 & -z_1 & z_2 & 0 \\ 0 & 0 & z_2 & z_1 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & -y_2 - y_3 & y_4 - y_1 & 0 \\ 0 & 0 & y_1 + y_4 & y_3 - y_2 & 0 \\ y_2 + y_3 & -y_1 - y_4 & 0 & z_5 & -z_4 \\ y_1 - y_4 & y_2 - y_3 & -z_5 & 0 & z_3 \\ 0 & 0 & z_4 & -z_3 & 0 \end{pmatrix}.$$

The linear part of such a matrix S is conjugate to a strictly upper triangular matrix via conjugation with the matrix

$$T = (e_1,\ e_2,\ e_3,\ e_4,\ e_7 + e_8,\ e_5,\ e_6,\ e_9,\ e_{10},\ e_{11},\ e_{12},\ e_{13},\ e_{14},\ e_7 - e_8),$$

where e_i denotes the ith unit vector in $\mathbb{R}^{14}$. Hence, the elements $\exp(S)$ generate a unipotent group of isometries whose translation parts contain all of $\mathbb{R}^{14}$. Therefore, the centraliser of $\mathbf{\Gamma}$ in $\mathbf{Iso}(\mathbb{R}^{7,7})$ acts transitively by Proposition 3.1.

By Remark 1.15, $\mathbf{\Gamma}$ acts freely on $\mathbb{R}^{7,7}$. By Corollary 7.9, $\mathbf{\Gamma}$ acts properly discontinuously on $\mathbb{R}^{7,7}$. Hence

$$M = \mathbb{R}^{7,7}/\mathbf{\Gamma}$$

is a complete flat homogeneous pseudo-Riemannian manifold.

Remark 11.1 It can be verified that the set of all matrices S forms a 3-step nilpotent Lie subalgebra of the centraliser Lie algebra. Since they are conjugate to upper triangular matrices, the set of all $\exp(S)$ forms a unipotent group of isometries acting simply transitively on $\mathbb{R}^{7,7}$.

11.3 The Orbits of $\mathbf{\Gamma}$

We write $X_i = (A_i, v_i)$. Then the elements $\gamma_i(t_i) = \exp(t_i X_i)$ (for $i = 1, 2, 3$) form a Malcev basis for the Zariski closure $\boldsymbol{G}$ of $\mathbf{\Gamma}$.

With Malcev coordinates t_1, t_2, t_3, any element of $\boldsymbol{G}$ is written as

$$g(t_1,t_2,t_3) = \left(\begin{array}{ccccc|cccc|ccccc||c}
1&0&0&0&0&t_1&-t_2&-t_2&-t_1&0&-4t_3-2t_1t_2&0&0&0&0\\
0&1&0&0&0&t_2&t_1&-t_1&t_2&4t_3+2t_1t_2&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&-t_1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&-t_2&0\\
0&0&0&0&1&0&0&0&0&0&0&t_1&t_2&0&2t_3+t_1t_2\\
\hline
0&0&0&0&0&1&0&0&0&-t_1&-t_2&0&0&0&0\\
0&0&0&0&0&0&1&0&0&t_2&-t_1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&-t_2&-t_1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-t_1&t_2&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&t_2\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&-t_1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
\hline\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

Fix a point

$$p = \begin{pmatrix} x\\ y\\ z\end{pmatrix} \in \mathbb{R}^{14} \quad (x, z \in \mathbb{R}^5, y \in \mathbb{R}^4).$$

Applying the transformation $g(t_1, t_2, t_3)$ with the t_i varying through all values in $\mathbb{R}$ yields the orbit $\boldsymbol{G}.p$. Explicitely, every element in $\boldsymbol{G}.p$ looks like

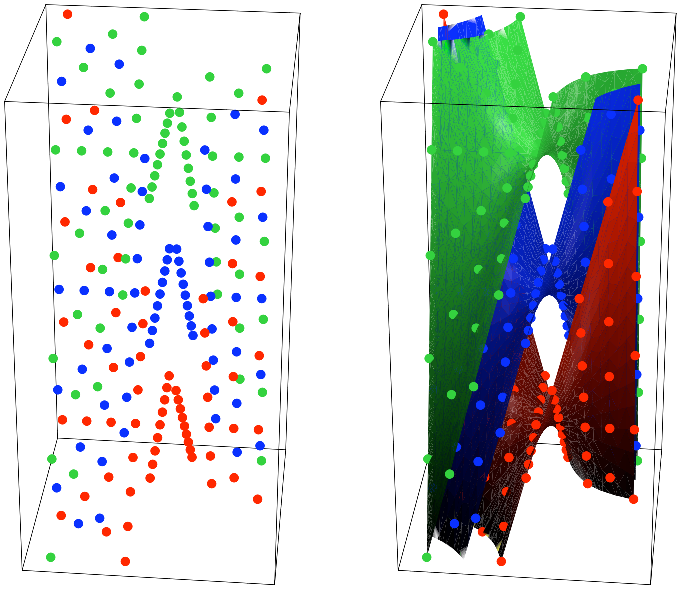
$$g(t_1, t_2, t_3).p = \begin{pmatrix} x_1 + t_1 y_1 - t_1 y_4 - t_2 y_2 - t_2 y_3 - (4t_3 + 2t_1 t_2) z_2 \\ x_2 + t_1 y_2 - t_1 y_3 + t_2 y_1 + t_2 y_4 + (4t_3 + 2t_1 t_2) z_1 \\ x_3 - t_1 z_5 \\ x_4 - t_2 z_5 \\ x_5 + (2t_3 + t_1 t_2) + t_1 z_3 + t_2 z_4 \\ y_1 - t_1 z_1 - t_2 z_2 \\ y_2 - t_1 z_2 + t_2 z_1 \\ y_3 - t_1 z_2 - t_2 z_1 \\ y_4 - t_1 z_1 + t_2 z_2 \\ z_1 \\ z_2 \\ z_3 + t_2 \\ z_4 - t_1 \\ z_5 \end{pmatrix}. \tag{11.2}$$

All terms in this parametrisation are linear, except for the occasional quadratic term $t_1 t_2$, which is always accompanied by an independent term $2t_3$. So varying t_1 and t_2 (with fixed t_3) yields a 2-dimensional surface parameterised by quadratic polynomials. Then varying t_3 translates this surface along an axis in such a way that it sweeps out a 3-dimensional affine subspace of $\mathbb{R}^{14}$.

The images below visualise this phenomenon for the respective orbits of $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$ through the point $p = 0$. Here we identify the orbit with $\mathbb{R}^3$ via the parametrisation

$$g(t_1, t_2, t_3).0 = \begin{pmatrix} 2t_3 + t_1 t_2 \\ t_2 \\ -t_1 \end{pmatrix}$$

Note that this amounts to simply deleting those rows from the column vector (11.2) where the group $\boldsymbol{G}$ acts trivially.



The left image shows the orbit $\boldsymbol{\Gamma}.0$, where dots of the same colour correspond to the orbit through 0 for a fixed t_3 and t_1, t_2 varying in $\mathbb{Z}$. The right image additionally contains 2-dimensional surfaces traced out by varying t_1, t_2 in $\mathbb{R}$ for a fixed t_3. One sees that by translating one of these surfaces along the vertical axis (the t_3-axis), one runs through the whole space $\mathbb{R}^3$.

11.4 A Global Slice

Let F_p denote the orbit $\boldsymbol{G}.p$ through p and let E denote the subspace orthogonal to F_0 with respect to the canonical Euclidean inner product on $\mathbb{R}^{14}$ (note that this is not related to our pseudo-Euclidean inner product which makes $\mathbb{R}^{14}$ into $\mathbb{R}^{7,7}$). Recall that F_p is an affine subspace of $\mathbb{R}^{14}$ (for $p = 0$ even a linear one), and $\dim F_p = \dim \boldsymbol{G}$ because $\boldsymbol{G}$ acts freely. So

$$\dim E = \dim \mathbb{R}^{14} - \dim F_0 = 14 - 3 = 11.$$

Consider (11.2) for $p = 0$ (that is, set all $x_i = y_j = z_k = 0$), the points contained in F_0. We find that the points $q \in E$ are precisely the points

$$q = (u_1, u_2, u_3, u_4, 0, v_1, v_2, v_3, v_4, v_5, w_1, w_2, 0, 0, w_5)^\top \in \mathbb{R}^{14}. \qquad (11.3)$$

We show that for arbitrary $p \in \mathbb{R}^{14}$, the intersection $F_p \cap E$ contains exactly one point q. To this end, equate (11.3) with (11.2). Considering the rows 12, 13 and 5 yields the identities

$$t_1 = z_4, \quad t_2 = -z_3, \quad t_3 = \frac{z_3 z_4 - x_5}{2}, \tag{11.4}$$

and plugging this into (11.2) yields the values for the non-zero entries of q, and so q is uniquely determined by the entries of p.

So E is a submanifold intersecting each orbit F_p in exactly one point, which means E is a global slice for the action of $\boldsymbol{G}$ on $\mathbb{R}^{7,7}$ and thus the orbit space $\mathbb{R}^{14}/\boldsymbol{G}$ can be identified as a set with the affine space E. But E is even a quotient in the category of affine varieties, since the orbit projection map

$$\pi : \mathbb{R}^{14} \to E, \quad p \mapsto g(t_1, t_2, t_3).p$$

with t_1, t_2, t_3 depending on p as in (11.4) is a polynomial map.

The space $\mathbb{R}^{14}$ is isomorphic (as an algebraic principal bundle with structure group $\boldsymbol{G}$) to the trivial bundle $E \times \boldsymbol{G} \cong \mathbb{R}^{11} \times \boldsymbol{G}$.

Considering the action of $\boldsymbol{\Gamma}$ rather than that of $\boldsymbol{G}$, we have a fibre bundle

$$\boldsymbol{G}/\boldsymbol{\Gamma} \to \mathbb{R}^{14} \to \mathbb{R}^{11}.$$

12 Non-Degenerate Orbits, Complete, Signature (7,7)

The group $\mathbf{\Gamma}$ in this example is a lattice in a faithful representation of the butterfly group in $\mathbf{Iso}(\mathbb{R}^{7,7})$. In this example the induced metric on the orbits of the Zariski closure $\boldsymbol{G}$ of $\mathbf{\Gamma}$ is non-degenerate, but the quotient map $\pi : \mathbb{R}^{7,7} \to \mathbb{R}^{7,7}/\boldsymbol{G}$ is not a trivial pseudo-Riemannian submersion.

12.1 The Group Generators

Choose a totally isotropic subspace U of dimension 5 in $\mathbb{R}^{7,7}$. We give the generators $\gamma_1 = (I + A_1, v_1)$, $\gamma_2 = (I + A_2, v_2)$, $\gamma_3 = (I + A_3, v_3)$ of $\mathbf{\Gamma}$ in a Witt basis with respect to U:

$$\gamma_i = \left(\begin{pmatrix} I_5 & 0 & C_i \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u_i^* \end{pmatrix} \right), \qquad i = 1, 2, 3.$$

Here,

$$C_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, u_1^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, u_2^* = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, u_3^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The commutators $\gamma_4, \gamma_5, \gamma_6$ are

$$\gamma_{1+i+j} = [\gamma_i, \gamma_j] = \left(\begin{pmatrix} I_5 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \begin{pmatrix} u_{1+i+j} \\ 0 \\ 0 \end{pmatrix} \right), \qquad i, j = 1, 2, 3, \ i \neq j$$

with

$$u_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

and further for $i = 1, 2$

$$[\gamma_i, \gamma_3] = I.$$

So $\mathbf{\Gamma}$ is isomorphic to a lattice in a butterfly group, and U in fact coincides with the subspace $U_{\mathbf{\Gamma}} = \sum \operatorname{im} A_i$.

As in the previous examples one can check that $\mathbf{\Gamma}$ is indeed a group of isometries.

Remark 12.1 The induced metric on the orbits $F_p = \boldsymbol{G}.p$ of $\boldsymbol{G}$ is non-degenerate of signature $(3,3)$. This can be seen immediately for the orbit F_0 which is spanned by the translation parts of the γ_i (the pseudo-scalar product on $\mathbb{R}^{r,s}$ is represented by the matrix $Q = \left(\begin{smallmatrix} 0 & 0 & I_5 \\ 0 & I_4 & 0 \\ I_5 & 0 & 0 \end{smallmatrix}\right)$).

12.2 The Centraliser

The following elements $S \in \mathfrak{iso}(\mathbb{R}^{7,7})$ commute with the (A_i, v_i):

$$S = \left(\begin{pmatrix} 0 & 0 & S_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right), \tag{12.1}$$

where $x = (x_1, \ldots, x_5)^\top, y = (y_1, \ldots, y_4)^\top, z = (z_1, \ldots, z_5)^\top$ are arbitrary and

$$S_3 = \begin{pmatrix} 0 & \frac{z_3}{2} & -\frac{z_2}{2} & 0 & 0 \\ -\frac{z_3}{2} & 0 & \frac{z_1}{2} & 0 & 0 \\ \frac{z_2}{2} & -\frac{z_1}{2} & 0 & -z_5 & z_4 \\ 0 & 0 & z_5 & 0 & 0 \\ 0 & 0 & -z_4 & 0 & 0 \end{pmatrix}.$$

These S are strictly upper triangular matrices. So their exponentials generate a unipotent group of isometries which acts transitively on $\mathbb{R}^{7,7}$.

By Remark 1.15, $\mathbf{\Gamma}$ acts freely on $\mathbb{R}^{7,7}$. By Corollary 7.9, $\mathbf{\Gamma}$ acts properly discontinuously on $\mathbb{R}^{7,7}$. Hence

$$M = \mathbb{R}^{7,7}/\mathbf{\Gamma}$$

is a complete flat homogeneous pseudo-Riemannian manifold.

12.3 The Tensor S

We compute the tensor S as defined in Definition 5.26 (recall from Lemma 5.28 that the tensor $\mathsf{T} = 0$). Let $p = (p_1, \ldots, p_{14})^\top \in \mathbb{R}^{7,7}$. The coordinate vector fields with respect to the chosen Witt basis are denoted by $\partial_1, \ldots, \partial_{14}$. For better readability we omit the index p when ∂_i is evaluated at p, as the vectors $(\partial_i)_p$ always correspond to the unit vectors in affine coordinates. The tangent space of F_p at p is spanned by the elements $X_i.p$, $i = 1, \ldots, 6$. These are

$$\begin{aligned}
X_1.p &= -\frac{p_{11}}{2}\partial_1 + \frac{p_{10}}{2}\partial_2 + p_{14}\partial_4 - p_{13}\partial_5 + \partial_{12},\\
X_2.p &= \frac{p_{12}}{2}\partial_1 - \frac{p_{10}}{2}\partial_3 + \partial_{11},\\
X_3.p &= -\frac{p_{12}}{2}\partial_2 + \frac{p_{11}}{2}\partial_3 + \partial_{10},\\
X_4.p &= -\partial_1,\\
X_5.p &= \partial_2,\\
X_6.p &= -\partial_3.
\end{aligned}$$

So every vertical vector field V is of the form

$$V = \alpha_1\partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3 + p_{14}\alpha_{12}\partial_4 - p_{13}\alpha_{12}\partial_5 + \alpha_{10}\partial_{10} + \alpha_{11}\partial_{11} + \alpha_{12}\partial_{12}$$

where the α_i are smooth functions. The horizontal distribution is generated by the vectors fields

$$H = (-p_{14}\beta_{13} + p_{13}\beta_{14})\partial_3 + \beta_4\partial_4 + \beta_5\partial_5 + \beta_6\partial_6 + \beta_7\partial_7 + \beta_8\partial_8 + \beta_9\partial_9 + \beta_{13}\partial_{13} + \beta_{14}\partial_{14}$$

where the β_i are smooth functions. One checks that at every $p \in \mathbb{R}^{7,7}$,

$$\langle H_p, V_p \rangle = 0.$$

To compute the tensor $\mathsf{S}_H V$ it is more convenient to write the vertical field V in the form (5.8),

$$V_p = \lambda_1(p)X_1.p + \ldots + \lambda_k(p)X_k.p$$

with smooth λ_i. We compute (5.9) in affine coordinates (represent ∂_i by e_i):

$$(\mathsf{S}_H V)_p = \Big(\sum_{i=1}^{6} \lambda_i(p)A_i\Big)H_p = \lambda_1(p)\beta_{14}(p)\partial_4 - \lambda_1(p)\beta_{13}(p)\partial_5.$$

To completely determine S, we need to compute the vertical component of $\nabla_{H_1}H_2$ for horizontal fields H_1, H_2. Denote the coefficient functions of H_1 by $\beta_i^{H_1}$ and those of H_2 by $\beta_i^{H_2}$. A tedious computation shows that

$$\mathrm{v}(\nabla_{H_1}H_2) = \left(\beta_{13}^{H_1}\beta_{14}^{H_2} - \beta_{14}^{H_1}\beta_{13}^{H_2}\right)\partial_3.$$

Note that this is an isotropic vector field.

For arbitrary vector fields $X, Y \in \mathfrak{vec}(\mathbb{R}^{7,7})$ we now have

$$\begin{aligned} \mathsf{S}_X Y &= \mathrm{v}(\nabla_{\mathrm{H}(X)}\mathrm{H}(Y)) + \mathrm{H}(\nabla_{\mathrm{H}(X)}\mathrm{v}(Y)) \\ &= \left(\beta_{13}^{\mathrm{H}(X)}\beta_{14}^{\mathrm{H}(Y)} - \beta_{14}^{\mathrm{H}(X)}\beta_{13}^{\mathrm{H}(Y)}\right)\partial_3 + \lambda_1^{\mathrm{v}(Y)}\beta_{14}^{\mathrm{H}(X)}\partial_4 - \lambda_1^{\mathrm{v}(Y)}\beta_{13}^{\mathrm{H}(X)}\partial_5. \end{aligned}$$

A formula for the induced sectional curvature on the quotient $\mathbb{R}^{7,7}/\boldsymbol{G}$ (see O'Neill [30], chapter 7, Theorem 47, or O'Neill [29], Corollary 1) gives us the following:

Proposition 12.2 *Let H_1, H_2 be two horizontal vector fields on $\mathbb{R}^{7,7}$ which span a non-degenerate plane. Then the induced sectional curvature of the plane spanned by $\pi_* H_1$ and $\pi_* H_2$ is*

$$\mathsf{K}(\pi_* H_1, \pi_* H_2) = \frac{3\langle \mathrm{v}([H_1, H_2]), \mathrm{v}([H_1, H_2])\rangle}{4\mathrm{vol}(H_1, H_2)} = 0.$$

For this proposition have also used that $\mathbb{R}^{7,7}$ is flat and that $\mathsf{S}_{H_1} H_2 = \frac{1}{2}\mathrm{v}([H_1, H_2])$ (O'Neill [29], Lemma 2) is an isotropic vector field.

Part III

Appendix

A Pseudo-Euclidean Spaces and their Isometries

We review some facts about pseudo-Euclidean spaces.

Let $\mathbb{R}^{r,s}$ denote the space $\mathbb{R}^n$ (where $n = r + s$) endowed with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of signature (r, s). By a theorem of Sylvester, this means $\langle\cdot,\cdot\rangle$ can be represented by a matrix

$$I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}.$$

Without loss of generality, we may assume $r \geq s$.

When we are not interested in the metric properties, we shall write $\mathbb{R}^{r+s}$ or $\mathbb{R}^n$ rather than $\mathbb{R}^{r,s}$.

A.1 Isotropic Subspaces

As usual, we write $U^\perp$ for the subspace containing all vectors orthogonal to a given subspace U.

Theorem A.1 *Let U be a subspace of $\mathbb{R}^{r,s}$.*

(a) $U^{\perp\perp} = U$.

(b) $\dim U + \dim U^\perp = n$.

(c) *The form $\langle\cdot,\cdot\rangle$ is non-degenerate on U if and only if it is non-degenerate on $U^\perp$. In this case, $U \cap U^\perp = \{0\}$ and $U \oplus U^\perp = \mathbb{R}^{r,s}$.*

(d) *If $U \oplus W = \mathbb{R}^{r,s}$ for some subspace W and $W \perp U$, then $W = U^\perp$ and $\langle\cdot,\cdot\rangle$ is non-degenerate on U.*

For a proof, see Proposition 12.21 in Brieskorn [6], volume II.

Definition A.2 A vector $x \in \mathbb{R}^{r,s}$ is called **isotropic** if $\langle x, x\rangle = 0$. Let $U \subset \mathbb{R}^{r,s}$ be a vector subspace. If all elements $x, y \in U$ satisfy $\langle x, y\rangle = 0$, then U is called **totally isotropic**. A subspace W is called **anisotropic** if $\langle x, x\rangle \neq 0$ for all non-zero $x \in W$ (or equivalently, if $\langle \cdot, \cdot \rangle$ is positive or negative definite on W).

Remark A.3 Note that if $\langle \cdot, \cdot \rangle$ is not definite, then $U \cap U^{\perp} \neq \{0\}$ is possible. In particular, of U is totally isotropic, we have $U \subseteq U^{\perp}$.

Theorem A.4 *Let U be a totally isotropic subspace of $\mathbb{R}^{r,s}$. There exists another totally isotropic subspace U^* and a subspace $W \subset U^{\perp}$ such that*

(a) $\langle \cdot, \cdot \rangle$ *is non-degenerate on W,*

(b) $\dim U = \dim U^*$ *and* $\dim W = n - 2\dim U$*,*

(c) $U^{\perp} = U \oplus W$ *and* $\mathbb{R}^{r,s} = U \oplus W \oplus U^*$*,*

(d) $U \perp W$ *and* $U^* \perp W$*,*

(e) *for every $x \in U$ there exists $x^* \in U^*$ such that $\langle x, x^* \rangle = 1$, and vice versa.*

The space U^* in the theorem is called a **dual space** to U, and the decomposition $\mathbb{R}^{r,s} = U \oplus W \oplus U^*$ is called a **Witt decomposition**. The proof of this theorem is essentially that of Satz 12.38 in Brieskorn [6], volume II.

Remark A.5 Because U and U^* are totally isotropic, $U \cap U^* = \{0\}$ and U is a dual to U^*. Further, $U^{*\perp} = U^* \oplus W$. But dual spaces are not unique (though isomorphic to one another). For example, consider $\mathbb{R}^{2,2}$ and let U be the totally isotropic space generated by $e_1 + e_3$ and $e_2 + e_4$. Then $U = U^{\perp}$ and a dual U^* is generated by $e_1 - e_3$ and $e_2 - e_4$. Another dual $\tilde{U}^*$ is generated by $e_1 - e_3 + e_2 + e_4$ and $e_2 - e_4 - e_1 - e_3$.

Theorem A.6 *Let $U \subset \mathbb{R}^{r,s}$ be a totally isotropic subspace of dimension k, and U^* a dual space to U. Then*

$$k \leq \min\{r, s\}.$$

There exist isotropic subspaces of maximal dimension $\min\{r, s\}$*.*

Throughout this text, we shall assume $r \geq s$, so that s is the maximal dimension of a totally isotropic subspace. This number is also known as the **Witt index** $\mathrm{wi}(\mathbb{R}^{r,s})$ of $\mathbb{R}^{r,s}$. Clearly, $n = r + s \geq 2\mathrm{wi}(\mathbb{R}^{r,s})$.

Definition A.7 Slightly generalising the common use of the term, we shall call a basis $\{b_1, \ldots, b_r, b_{r+1}, \ldots, b_{r+s}\}$ an **orthonormal basis** of $\mathbb{R}^{r,s}$ if

$$\begin{aligned}
\langle b_i, b_i \rangle &= 1, && \text{for } i = 1, \ldots, r, \\
\langle b_j, b_j \rangle &= -1, && \text{for } j = r+1, \ldots, r+s, \\
\langle b_i, b_j \rangle &= 0, && \text{if } i \neq j \text{ for } i, j = 1, \ldots, r+s.
\end{aligned}$$

Definition A.8 Let $U \subset \mathbb{R}^{r,s}$ be a totally isotropic subspace of dimension k. A **Witt basis** (or **skew basis**) of $\mathbb{R}^{r,s}$ with respect to U is given by

$$\{u_1, \ldots, u_k, \quad w_1, \ldots, w_{n-2k}, \quad u_1^*, \ldots, u_k^*\},$$

where $\{u_1, \ldots, u_k\}$ is a basis of U, $\{w_1, \ldots, w_{n-2k}\}$ is a basis of a vector space complement W of U in $U^\perp$ (that is $U^\perp = U \oplus W$), and $\{u_1^*, \ldots, u_k^*\}$ is a basis of the dual U^* such that $\langle u_i, u_j^* \rangle = \delta_{ij}$.

Remark A.9 In a skew basis with respect to a totally isotropic subspace U, the restriction of $\langle \cdot, \cdot \rangle$ to the subspace W spanned by $w_1, \ldots, w_{n-2k}$ is non-degenerate, that is, $W \cong \mathbb{R}^{p,q}$ for some p, q with $p + q = n - 2k$. We have the following relations among the elements of a skew basis:

(a) For all $i, j = 1, \ldots, k$ and $m = 1, \ldots, n - 2k$:

$$\langle u_i, u_j^* \rangle = \delta_{ij}, \quad u_i \perp u_j, \quad u_i^* \perp u_j^*, \quad w_m \perp u_i, \quad w_m \perp u_i^*.$$

(b) Further, we may assume the $w_m \in W$ to be chosen in such a way that

$$\begin{aligned}
\langle w_i, w_i \rangle &= 1, && \text{for } i = 1, \ldots, p, \\
\langle w_j, w_j \rangle &= -1, && \text{for } j = p+1, \ldots, p+q, \\
\langle w_i, w_j \rangle &= 0, && \text{if } i \neq j \text{ for } i, j = 1, \ldots, p+q.
\end{aligned}$$

A.2 Pseudo-Euclidean Isometries

Definition A.10 The **pseudo-orthogonal group** is

$$\mathbf{O}_{r,s} = \{g \in \mathbf{GL}_n(\mathbb{R}) \mid \langle gx, gy \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^{r,s}\}.$$

The **special pseudo-orthogonal group** is

$$\mathbf{SO}_{r,s} = \{g \in \mathbf{O}_{r,s} \mid \det(g) = 1\}.$$

Lemma A.11 *If $I_{r,s}$ is the signature matrix of $\mathbb{R}^{r,s}$, then*

$$\mathbf{O}_{r,s} = \{g \in \mathbf{GL}_n(\mathbb{R}) \mid g^\top I_{r,s} g = I_{r,s}\}.$$

It Lie algebra is

$$\mathfrak{so}_{r,s} = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^\top I_{r,s} + I_{r,s} X = 0\}.$$

Theorem A.12 *The group of isometries $\mathbf{Iso}(\mathbb{R}^{r,s})$ of $\mathbb{R}^{r,s}$ is the semi-direct product*

$$\mathbf{Iso}(\mathbb{R}^{r,s}) = \mathbf{O}_{r,s} \ltimes \mathbb{R}^n, \tag{A.1}$$

where the action of $\mathbf{O}_{r,s}$ on the normal subgroup $\mathbb{R}^n$ is given by matrix-vector multiplication.

Remark A.13 Let $U \subset \mathbb{R}^{r,s}$ be a totally isotropic subspace of dimension k. In skew basis representation with respect to U, the pseudo-Euclidean form $\langle \cdot, \cdot \rangle$ is represented by the matrix

$$Q = \begin{pmatrix} 0 & 0 & I_k \\ 0 & \tilde{I} & 0 \\ I_k & 0 & 0 \end{pmatrix},$$

where $\tilde{I} = I_{p,q}$ is the signature matrix of the restriction to the space W from Definition A.8. In this representation, the elements $X \in \mathfrak{so}_{r,s}$ satisfy

$$X^\top Q + QX = 0.$$

Writing

$$X = \begin{pmatrix} A & B' & C \\ D & E & B \\ F & D' & A' \end{pmatrix},$$

this means

$$\begin{pmatrix} F^\top & D^\top \tilde{I} & A^\top \\ D'^\top & E^\top \tilde{I} & B'^\top \\ A'^\top & B^\top \tilde{I} & C^\top \end{pmatrix} + \begin{pmatrix} F & D' & A' \\ \tilde{I}D & \tilde{I}E & \tilde{I}B \\ A & B' & C \end{pmatrix} = 0$$

Comparing coefficients leads to

$$X = \begin{pmatrix} A & -B^\top \tilde{I} & C \\ D & E & B \\ F & -D^\top \tilde{I} & -A^\top \end{pmatrix}, \tag{A.2}$$

where $C, F \in \mathfrak{so}_k$ and $E \in \mathfrak{so}_{p,q}$, and $A \in \mathbb{R}^{k\times k}$, $B, D \in \mathbb{R}^{(n-2k)\times k}$ are arbitrary matrices. In particular, if U is a maximal isotropic subspace, then W is positive definite (as we assume $r \geq s$), and (A.2) takes the form

$$X = \begin{pmatrix} A & -B^\top & C \\ D & E & B \\ F & -D^\top & -A^\top \end{pmatrix} \tag{A.3}$$

with $E \in \mathfrak{so}_{n-2k}$.

B Affine Manifolds

B.1 Affine Transformations

The real affine space is denoted by $\mathbb{R}^n$.

Theorem B.1 *The group of affine transformations of* $\mathbb{R}^n$ *is*

$$\mathbf{Aff}(\mathbb{R}^n) = \mathbf{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n. \tag{B.1}$$

If one embeds $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the set $\{(p, 1) \in \mathbb{R}^{n+1} \mid p \in \mathbb{R}^n\}$, then an affine transformation $g \in \mathbf{Aff}(\mathbb{R}^n)$ of $\mathbb{R}^n$ can be represented by a matrix

$$g = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in \mathbf{GL}_{n+1}(\mathbb{R}).$$

We call this a representation in **affine coordinates**. The **linear part** of g is $\mathrm{L}(g) = A \in \mathbf{GL}_n(\mathbb{R})$, the **translation part** of g is $\mathrm{T}(g) = v \in \mathbb{R}^n$. Other notations we will use, depending on what seems most convenient, are

$$g = \left(\begin{array}{c||c} A & v \\ \hline\hline 0 & 1 \end{array}\right) \quad \text{and} \quad g = (A, v).$$

Theorem B.2 *The Lie algebra of* $\mathbf{Aff}(\mathbb{R}^n)$ *is*

$$\mathfrak{aff}(\mathbb{R}^n) = \left\{ X = \begin{pmatrix} B & w \\ 0 & 0 \end{pmatrix} \,\middle|\, B \in \mathfrak{gl}_n(\mathbb{R}), w \in \mathbb{R}^n \right\},$$

and we write $\mathrm{L}(X) = B$, $\mathrm{T}(X) = w$.

B.2 Affine Vector Fields

Let M be an affine manifold with affine connection ∇. Let $\mathfrak{vec}(M)$ denote the Lie algebra of vector fields on M.

Definition B.3 A vector field $X \in \mathfrak{vec}(M)$ is called **affine** if its local flow at each point in M preserves ∇.

Example B.4 For $X = (B, w) \in \mathfrak{aff}(\mathbb{R}^n)$, an affine vector field X_p is given by

$$p \mapsto \begin{pmatrix} B & w \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

At a fixed point p_0, the derivative of this map with respect to $v \in \mathbb{R}^n$ is given by Bv. This shows $\textsc{l}(X) = \nabla_v X_{p_0}$, where ∇ denotes the natural affine connection on $\mathbb{R}^n$.

Let $\mathscr{L}_X$ denote the Lie derivative with respect to X, and A_X the tensor field

$$\mathsf{A}_X = \mathscr{L}_X - \nabla_X. \tag{B.2}$$

Proposition B.5 *Let R_∇ denote the curvature tensor for ∇, and assume ∇ is torsion free. Then:*

(a) $\mathsf{A}_X Y = -\nabla_Y X$ *for* $X, Y \in \mathfrak{vec}(M)$.

(b) $\mathsf{R}_\nabla(X, A) = \nabla_A \mathsf{A}_X$ *for* $X, Y \in \mathfrak{aut}(M)$, $A \in \mathfrak{vec}(M)$.

(c) $\mathsf{A}_{[X,Y]} = [\mathsf{A}_X, \mathsf{A}_Y] + \mathsf{R}_\nabla(X, Y)$ *for* $X, Y \in \mathfrak{aut}(M)$.

Theorem B.6 *Let* $\mathfrak{aut}(M)$ *denote the set of affine vector fields on* M*, and let* $\mathfrak{aff}(M) = \mathfrak{Lie}(\mathbf{Aff}(M))$.

(a) $\mathfrak{aut}(M)$ *is a Lie subalgebra of the Lie algebra of vector fields on* M.

(b) *The set* $\mathfrak{aut}_c(M)$ *of complete affine vector fields is a Lie subalgebra of* $\mathfrak{aut}(M)$.

(c) *If* M *is complete with respect to* ∇*, then* $\mathfrak{aut}(M) = \mathfrak{aut}_c(M)$.

(d) $\mathfrak{aut}_c(M)$ *is anti-isomorphic to* $\mathfrak{aff}(M)$ *as a Lie algebra.*

All proofs for this section can be found in Kobayashi and Nomizu [25], volume I, chapter VI, section 2.

C Pseudo-Riemannian Manifolds

Up to isometry, every connected pseudo-Riemannian manifold M is a quotient $\tilde{M}/\mathbf{\Gamma}$ of a simply connected pseudo-Riemannian manifold $\tilde{M}$ by a discrete isometry group $\mathbf{\Gamma}$ acting properly discontinuously on $\tilde{M}$ (in the flat complete case, we may take $\tilde{M} = \mathbb{R}^{r,s}$). Two such quotients are isomorphic if and only if the respective discrete groups are conjugate.

C.1 Pseudo-Riemannian Isometries

Given the isometry group $\mathbf{Iso}(\tilde{M})$, one also knows $\mathbf{Iso}(M)$:

Proposition C.1 *Let $\tilde{M}$ be a simply connected pseudo-Riemannian manifold, and let $\mathbf{\Gamma} \subset \mathbf{Iso}(\tilde{M})$ act properly discontinuously on $\tilde{M}$. The isometry group of $M = \tilde{M}/\mathbf{\Gamma}$ is*

$$\mathbf{Iso}(M) = \mathrm{N}_{\mathbf{Iso}(\tilde{M})}(\mathbf{\Gamma})/\mathbf{\Gamma}, \qquad \text{(C.1)}$$

where $\mathrm{N}_{\mathbf{Iso}(\tilde{M})}(\mathbf{\Gamma})$ is the normaliser of $\mathbf{\Gamma}$ in $\mathbf{Iso}(\tilde{M})$.

For a proof, see O'Neill [30], Proposition 20 in chapter 9.

The centraliser of $\mathbf{\Gamma}$ contains the identity component of the normaliser of $\mathbf{\Gamma}$ in $\mathbf{Iso}(\tilde{M})$. So in the homogeneous case, we get the first part of the following result:

Corollary C.2 *M is homogeneous if and only if $\mathrm{Z}_{\mathbf{Iso}(\tilde{M})}(\mathbf{\Gamma})$ acts transitively on $\tilde{M}$.*

For a pseudo-Riemannian covering $N \to M$ where N is not simply connected, the "if"-part also requires that the covering is normal. See Wolf [52], Theorem 2.4.17, for details.

C.2 Killing Fields

Definition C.3 Let M be a pseudo-Riemannian manifold. A vector field X on M is called a **Killing field** if its local flow at each point in M is a local isometry on M.

Theorem C.4 *Isometries are affine maps, and Killing vector fields are affine vector fields.*

For a proof, see Kobayashi and Nomizu [25], volume I, Proposition 2.5 in chapter IV.

Let $\mathscr{L}_X$ and A_X as in the previous chapter.

Proposition C.5 *Let M be a pseudo-Riemannian manifold, $X \in \mathfrak{vec}(M)$. The following are equivalent:*

(a) *X is a Killing field.*

(b) $\mathscr{L}_X\langle\cdot,\cdot\rangle = 0$.

(c) $\mathscr{L}_X(\langle A, B\rangle) = \langle [X, A], B\rangle + \langle A, [X, B]\rangle$ *for* $A, B \in \mathfrak{vec}(M)$.

(d) *The tensor field A_X is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$, that is* $\langle \mathsf{A}_X A, B\rangle = -\langle A, \mathsf{A}_X B\rangle$ *for* $A, B \in \mathfrak{vec}(M)$.

See Kobayashi and Nomizu [25], volume I, chapter VI, Proposition 3.2 for a proof.

Theorem C.6 *Let $\mathfrak{kill}(M)$ denote the set of Killing fields on M, and let* $\mathfrak{iso}(M) = \mathfrak{Lie}(\mathbf{Iso}(M))$.

(a) $\mathfrak{kill}(M)$ *is a Lie subalgebra of* $\mathfrak{vec}(M)$.

(b) *The set $\mathfrak{kill}_c(M)$ of complete Killing fields is a Lie subalgebra of* $\mathfrak{kill}(M)$.

(c) *If M is complete, then* $\mathfrak{kill}(M) = \mathfrak{kill}_c(M)$.

(d) $\mathfrak{iso}(M)$ *is anti-isomorphic to $\mathfrak{kill}_c(M)$ as a Lie algebra.*

See Kobayashi and Nomizu [25], volume I, chapter VI, Theorem 3.4, for proofs of parts (a) to (c), and O'Neill [30], chapter 9, Proposition 33 for a proof of part (d).

C.3 Bi-Invariant Metrics

Definition C.7 Let G be a Lie group endowed with a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$. This metric is called **bi-invariant** if the left- and right-multiplications by elements of G are isometries.

Proposition C.8 *Let G be a connected Lie group with a left-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$. Then the following are equivalent:*

(a) $\langle\cdot,\cdot\rangle$ *is right-invariant (hence bi-invariant).*

(b) $\langle\cdot,\cdot\rangle$ *is* $\mathrm{Ad}(\boldsymbol{G})$*-invariant.*

(c) $g \mapsto g^{-1}$ *is an isometry.*

(d) $\langle [X,Y],Z\rangle = -\langle Y,[X,Z]\rangle$ *for all* $X,Y,Z \in \mathfrak{Lie}(\boldsymbol{G})$.

(e) $\nabla_X Y = \frac{1}{2}[X,Y]$ *for all* $X, Y \in \mathfrak{g}$.

(f) *The geodesics starting at* 1_G *are the one-parameter subgroups of* $\boldsymbol{G}$.

In particular, the curvature tensor on $\boldsymbol{G}$ *is given by*

$$\mathsf{R}(X,Y)Z = \frac{1}{4}[X,[Y,Z]].$$

For a proof, see O'Neill [30], chapter 11, Proposition 9.

D Discrete Groups and Proper Actions

D.1 Proper Definition of Proper Action

In order for a quotient $\mathbb{R}^n/\mathbf{\Gamma}$ to be a manifold, it is necessary for the quotient to be a Hausdorff space. This can be characterised by the properties of the $\mathbf{\Gamma}$-action, namely the *proper discontinuity*. Unfortunately, in the literature there is much ambiguity about the definition of proper discontinuity. We follow Thurston [43] to get a reasonable definition:

Definition D.1 Let $\boldsymbol{G}$ be a group acting on a topological space X by homeomorphisms. The action is called **proper** if the map

$$\boldsymbol{G} \times X \to X \times X, \quad (g, x) \mapsto (g.x, x),$$

is proper, meaning that the preimage of every compact set is compact. Equivalently, for every compact set $K \subset X$ the set $\{g \in \boldsymbol{G} \mid g.K \cap K \neq \emptyset\}$ is compact.

Definition D.2 Let $\mathbf{\Gamma}$ be a group acting on a locally compact topological space X by homeomorphisms.

(a) The action is **effective** if id_X is the only element acting trivially on X.

(b) The action is **free** if the stabiliser subgroup $\mathbf{\Gamma}_x \subset \mathbf{\Gamma}$ of every point $x \in X$ is trivial, $\mathbf{\Gamma}_x = \{\mathrm{id}_X\}$.

(c) The action is **discrete action** if it is effective and $\mathbf{\Gamma}$ is a discrete subset of the group of homeomorphisms of X with respect to the compact-open topology (in particular, if this is a matrix group, then $\mathbf{\Gamma}$ is discrete with respect to the usual topology on $\mathbb{R}^{n \times n}$).

(d) The action has **discrete orbits** if it is effective and every $x \in X$ has a neighbourhood U such that the set $\{\gamma \in \mathbf{\Gamma} \mid \gamma.x \in U\}$ is finite.

(e) The action is **wandering** if it is effective and every $x \in X$ has a neighbourhood U such that the set $\{\gamma \in \mathbf{\Gamma} \mid \gamma.U \cap U \neq \emptyset\}$ is finite.

(f) The action is **properly discontinuous** if it is discrete and proper.

Many authors define "properly discontinuous" by what we call a wandering action. This definition is inappropriate for our purposes, as the quotient $X/\boldsymbol{\Gamma}$ of a wandering action need not be a Hausdorff space. For example, let $X = \mathbb{R}^2\backslash\{0\}$ and $\boldsymbol{\Gamma} \cong \mathbb{Z}$ be the group generated by $\begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}$. Then every neighbourhood of the orbit of $(1,0)$ intersects every neighbourhood of the orbit of $(0,1)$. Thus $X/\boldsymbol{\Gamma}$ is not a Hausdorff space.

Theorem D.3 *Let* $\boldsymbol{\Gamma}$ *be a group acting properly discontinuously by diffeomorphisms on a differentiable manifold* M*. Then the quotient* $M/\boldsymbol{\Gamma}$ *has the structure of a differentiable manifold such that the projection* $\pi : M \to M/\boldsymbol{\Gamma}$ *is differentiable.*

For a proof, see Kobayashi and Nomizu [25], Proposition 4.3 in chapter I of volume I.

In many situations, the action of a discrete group is automatically properly discontinuous.

Proposition D.4 *Let* $\boldsymbol{G}$ *be a Lie group acting transitively on a manifold* M *such that the stabiliser* $\boldsymbol{G}_x$ *for any* $x \in M$ *is compact. Then any discrete subgroup* $\boldsymbol{\Gamma} \subset \boldsymbol{G}$ *acts properly discontinuously on* M*.*

For a proof, see Thurston [43], Corollary 3.5.11.

Proposition D.5 *Let* $\boldsymbol{\Gamma}$ *be a discrete group of isometries on a metric space* M*. If* $\boldsymbol{\Gamma}$ *is wandering, then* $\boldsymbol{\Gamma}$ *acts properly discontinuously on* M*.*

Unfortunately, pseudo-Riemannian homogeneous spaces do not admit a simplified definition of properness.

E Algebraic Groups

E.1 Algebraic Group Actions

We collect some facts on algebraic groups which are used throughout this thesis. These results are quoted from Borel [5], where they appear in more general form than needed in this thesis. In particular,

- the condition that a morphism is *separable* is void, as in characteristic 0 all dominant morphisms are, and
- the condition that some variety is *normal* is void, as all spaces in question are manifolds, hence smooth, and a smooth variety V is normal. This follows from the fact that its local rings are regular and thus integrally closed (even factorial), which is required for V to be normal, see Theorem 19.19 in Eisenbud [14].

Let $\mathbb{k}$ be a field, $\overline{\mathbb{k}}$ its algebraic closure and $\boldsymbol{G}$ an algebraic $\mathbb{k}$-group acting $\mathbb{k}$-morphically on a non-empty $\mathbb{k}$-variety V (here, "variety" means an affine or quasi-affine variety).

Proposition E.1 *Each orbit of the* $\boldsymbol{G}$*-action is a smooth variety which is open in its closure in* V*. Its boundary is a union of orbits of strictly lower dimension. In particular, the orbits of minimal dimension are closed.*

For a proof, see Borel [5], 1.8.

Definition E.2 The graph F of an action is the image of the morphism $\theta : \boldsymbol{G} \times V \to F \subseteq V \times V$, $(g, v) \mapsto (g.v, v)$. If the action is free, θ is bijective. If θ is an isomorphism of varieties, then the free action is called **principal**.

An action being principal means that for $(v, w) \in F$, the unique element $g \in \boldsymbol{G}$ with $g.w = v$ depends morphically on (v, w).

Definition E.3 A **(geometric) quotient** of V by $\boldsymbol{G}$ is a $\mathbb{k}$-variety W together with a **quotient map** $\pi : V \to W$ which is a $\mathbb{k}$-morphism such that:

(a) π is surjective and open.

(b) If $U \subset V$ is open, then the comorphism π^* induces an isomorphism from $\overline{\mathbb{k}}[\pi(U)]$ onto $\overline{\mathbb{k}}[U]^{\boldsymbol{G}}$.

(c) The fibres of π are the orbits of $\boldsymbol{G}$.

Theorem E.4 (Universal mapping property) *Let (W, π) be a quotient for the $\boldsymbol{G}$-action on V. If $\varphi : V \to Z$ is any morphism constant on the orbits of $\boldsymbol{G}$ there is a unique morphism $\psi : W \to Z$ such that $\varphi = \psi \circ \pi$. If φ is a $\Bbbk$-morphism of $\Bbbk$-varieties, so is ψ.*

For a proof, see Borel [5] 6.3 and 6.1.

Corollary E.5 *A bijective quotient map is an isomorphism.*

Remark E.6 In general, the orbit space $V/\boldsymbol{G}$ does not exist as a variety.

Proposition E.7 *Suppose $\pi : V \to W$ is a separable, surjective morphism such that the fibres of π are the $\boldsymbol{G}$-orbits, and assume that W is normal and that the irreducible components of V are open. Then (W, π) is the quotient of V by $\boldsymbol{G}$.*

Proposition E.8 *Suppose v is a $\Bbbk$-point of V, and let θ_v be the surjective $\Bbbk$-morphism $\theta_v : \boldsymbol{G} \to \boldsymbol{G}.v$, $g \mapsto g.v$. Then $\boldsymbol{G}.v$ is a smooth variety defined over $\Bbbk$ and locally closed in V. Moreover, the fibres of θ_v are the orbits of the stabiliser $\boldsymbol{G}_v$ for the action of $\boldsymbol{G}_v$ by right-multiplication on $\boldsymbol{G}$. The following conditions are equivalent:*

(a) *θ_v is a quotient of $\boldsymbol{G}$ by $\boldsymbol{G}_v$.*

(b) *θ_v is separable, that is $\mathrm{d}\theta_v|_{1_G} : \mathfrak{Lie}(\boldsymbol{G}) \to \mathrm{T}_v\boldsymbol{G}.v$ is surjective.*

(c) *The kernel of $\mathrm{d}\theta_v|_{1_G}$ is contained in $\mathfrak{Lie}(\boldsymbol{G}_v)$.*

When these conditions hold, $\boldsymbol{G}_v$ is defined over $\Bbbk$, and hence θ_v is a quotient of $\boldsymbol{G}$ by $\boldsymbol{G}_v$ over $\Bbbk$.

For proofs of the lemma and the two propositions, see Borel [5], 6.2, 6.6 and 6.7.

Assume now that $\boldsymbol{G}$ acts freely on V.

Definition E.9 Assume the $\Bbbk$-morphism $\pi : V \to W$ is surjective and its fibres are the orbits of $\boldsymbol{G}$. A **local cross section** over $\Bbbk$ for π is a $\Bbbk$-morphism $\sigma : U \to V$, where U is $\Bbbk$-open in W, such that

$$\pi \circ \sigma = \mathrm{id}_U.$$

A local cross section defined on all of W is called a **(global) cross section**.

Definition E.10 Assume W can be covered by $\mathbb{k}$-open sets U each admitting a $\mathbb{k}$-defined local cross section such that the $\boldsymbol{G}$-action on $\pi^{-1}(U)$ is principal. Then the fibration π is called **locally trivial**. If there is a global cross section, the fibration π is called **trivial**.

Proposition E.11 *Assume the $\mathbb{k}$-morphism $\pi : V \to W$ is surjective and its fibres are the orbits of $\boldsymbol{G}$. Further assume a $\mathbb{k}$-section $\sigma : U \to V$ exists, where U is $\mathbb{k}$-open in W.*

(a) *The map*

$$\varphi : \boldsymbol{G} \times U \to V, \quad (g, u) \mapsto g.\sigma(u)$$

is a bijective $\mathbb{k}$-morphism onto $\pi^{-1}(U)$.

(b) *The $\boldsymbol{G}$-action on $\pi^{-1}(U)$ is principal if and only if φ is an isomorphism.*

(c) *If U and $\pi^{-1}(U)$ are normal, then φ is an isomorphism.*

(d) *If $\pi : V \to W$ is locally trivial, then π maps the $\mathbb{k}$-points in V onto the $\mathbb{k}$-points in W.*

For proofs, see the discussion in Borel [5], 6.14.

Remark E.12 From the remarks at the beginning of the chapter it is clear in the cases of interest for this thesis, the action of $\boldsymbol{G}$ is always principal on $\pi^{-1}(U)$ is a section exists.

Remark E.13 Local sections do not always exist. This problem has been studied extensively by Rosenlicht [36, 37, 39]. In the cases of interest to us, the existence of local sections is known (Corollary G.26 below).

Definition E.14 Let $\boldsymbol{H} \subset \boldsymbol{G}$ be a closed subgroup and $\boldsymbol{N} \subset \boldsymbol{G}$ a closed subgroup normalised by $\boldsymbol{H}$. Then $\boldsymbol{G}$ is called the **semidirect product of subgroups $\boldsymbol{H}$ and $\boldsymbol{N}$** if the map

$$\boldsymbol{H} \times \boldsymbol{N} \to \boldsymbol{G}, \quad (h, n) \mapsto hn$$

is an isomorphism of affine varieties. In particular, $\boldsymbol{G} = \boldsymbol{H} \cdot \boldsymbol{N}$.

Remark E.15 The semidirect product of subgroups is a special case of the semidirect product $\boldsymbol{H} \ltimes \boldsymbol{N}$ of arbitrary (algebraic) groups $\boldsymbol{H}$, $\boldsymbol{N}$, where $\boldsymbol{H}$ acts on $\boldsymbol{N}$ by (algebraic) automorphisms. In Definition E.14, $\boldsymbol{H}$ acts on $\boldsymbol{N}$ by conjugation.

E.2 Algebraic Homogeneous Spaces

The existence of quotients of algebraic group actions as varieties is a difficult problem. However, if we are considering homogeneous spaces, things get somewhat easier.

Let $\boldsymbol{G}$ be an affine $\mathbb{k}$-group acting $\mathbb{k}$-morphically on a $\mathbb{k}$-variety V.

Theorem E.16 *Let $\boldsymbol{H}$ be a closed subgroup of $\boldsymbol{G}$ defined over $\mathbb{k}$. Then the quotient $\pi : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$ exists over $\mathbb{k}$, and $\boldsymbol{G}/\boldsymbol{H}$ is a smooth quasi-projective variety. If $\boldsymbol{H}$ is a normal subgroup of $\boldsymbol{G}$, then $\boldsymbol{G}/\boldsymbol{H}$ is an affine $\mathbb{k}$-group and π is a $\mathbb{k}$-morphism of $\mathbb{k}$-groups.*

See Borel [5], 6.8, for a proof.

Proposition E.17 *Let $\boldsymbol{H}$ be a closed normal subgroup of $\boldsymbol{G}$ defined over $\mathbb{k}$.*

(a) *If $V/\boldsymbol{H}$ exists over $\mathbb{k}$ and is a normal variety, then $\boldsymbol{G}/\boldsymbol{H}$ acts $\mathbb{k}$-morphically on $V/\boldsymbol{H}$. In particular, if $\boldsymbol{H}$ acts trivially on V, then $\boldsymbol{G}/\boldsymbol{H}$ acts $\mathbb{k}$-morphically on V.*

(b) *Moreover, if the quotient $V/\boldsymbol{G}$ exists and is a normal variety, then the quotient of $V/\boldsymbol{H}$ by $\boldsymbol{G}/\boldsymbol{H}$ exists and is canonically isomorphic to $V/\boldsymbol{G}$.*

Proposition E.18 *Let $\boldsymbol{N} \subset \boldsymbol{H}$ be closed subgroups of $\boldsymbol{G}$ defined over $\mathbb{k}$ such that $\boldsymbol{N}$ is a normal subgroup of $\boldsymbol{H}$. Then:*

(a) *$\boldsymbol{H}/\boldsymbol{N}$ acts $\mathbb{k}$-morphically on $\boldsymbol{G}/\boldsymbol{N}$, the quotient exists and is isomorphic to $\boldsymbol{G}/\boldsymbol{H}$.*

(b) *For each point $p \in \boldsymbol{G}/\boldsymbol{N}$, the (right-)action $\theta_p : \boldsymbol{H}/\boldsymbol{N} \to p.(\boldsymbol{H}/\boldsymbol{N})$ is an isomorphism.*

(c) *If $\boldsymbol{H}$ and $\boldsymbol{N}$ are normal subgroups of $\boldsymbol{G}$, then θ_p is an isomorphism of $\mathbb{k}$-groups.*

F Čech Cohomology and Fibre Bundles

F.1 Čech Cohomology

Cohomology is a useful tool for studying if local properties of topological spaces (manifolds, varieties) can be extended to global properties.

We give an ad hoc definition of the first Čech cohomology group of a topological space V, rather than introducing cohomology groups of arbitrary degree.

Definition F.1 Let V be a topological space, $\mathcal{U} = (U_i)_{i=1}^m$ a finite covering of V by open subsets, and let $\mathcal{S}$ be a sheaf of abelian groups on V.

(a) A **1-cocycle** β is a collection of elements $\beta_{ij} \in \mathcal{S}(U_i \cap U_j)$ for all $i, j = 1, \ldots, m$, such that

$$\beta_{ij} + \beta_{jk} = \beta_{ik} \quad \text{when restricted to } U_i \cap U_j \cap U_k.$$

(b) A **1-coboundary** is a 1-cocycle β such that there exists a collection α of elements $\alpha_i \in \mathcal{S}(U_i)$ satisfying

$$\beta_{ij} = \alpha_i - \alpha_j \quad \text{when restricted to } U_i \cap U_j.$$

(c) Let $\mathrm{Z}^1(\mathcal{U}, \mathcal{S})$ denote the set of 1-cocycles, and let $\mathrm{B}^1(\mathcal{U}, \mathcal{S})$ denote the set of 1-coboundaries. They are abelian groups, and the quotient group $\mathrm{H}^1(\mathcal{U}, \mathcal{S}) = \mathrm{Z}^1(\mathcal{U}, \mathcal{S})/\mathrm{B}^1(\mathcal{U}, \mathcal{S})$ is called the **first Čech cohomology group** of $\mathcal{S}$ relative to $\mathcal{U}$.

For the rest of this section, let V be an affine algebraic variety, and let $\mathcal{O}_V$ denote the sheaf of regular functions on V. If V is embedded in some $\mathbb{k}^n$, then, for any open subset $U \subset V$, $\mathcal{O}_V(U)$ is the algebra of rational functions defined on U. In particular, $\mathcal{O}_V(V)$ is the subring of $\mathbb{k}[x_1, \ldots, x_n]$ not vanishing on all of V.

Definition F.2 Let $A = \mathcal{O}_V(V)$, M an A-module, $f \in A$, and U_f the open subset where $f(x) \neq 0$ for all x. On U_f, define a $A[\frac{1}{f}]$-module by setting $\tilde{M}(U_f) = M \otimes_A A[\frac{1}{f}]$.

Remark F.3 The open subsets U_f define a basis of the Zariski topology of V. Because of this, and since because $\mathcal{O}_V(U_f) = A[\frac{1}{f}]$, the collection $\tilde{M}$ defines a sheaf of $\mathcal{O}_V$-modules. See Definition 7.3 and the remarks after Definition 2.3 in Perrin [32].

Definition F.4 A sheaf $\mathcal{S}$ on V is **quasi-coherent** if it is isomorphic to a sheaf of type $\tilde{M}$ from Definition F.2.

Example F.5 The sheaf $\mathcal{O}_V$ of regular functions itself is quasi-coherent.

Theorem F.6 *Let V be an affine variety and let $\mathcal{S}$ be a quasi-coherent sheaf, and $\mathcal{U}$ a finite covering of V by open subsets. Then*

$$H^1(\mathcal{U}, \mathcal{S}) = \{0\}. \tag{F.1}$$

That is, every 1-cocycle is a 1-coboundary.

A very accessible proof of this theorem is given in chapter VII, Theorem 2.5 of Perrin [32]. The statement holds more generally for all cohomology groups $H^k(\mathcal{U}, \mathcal{S})$ with $k > 0$.

F.2 Fibre Bundles and Principal Bundles

The concepts presented in this section can be defined in the category of topological spaces, differentiable manifolds and algebraic varieties (and many more). So we shall speak of "objects", "morphisms" and "isomorphisms" rather than of "topological spaces", "continuous maps" and "homeomorphism" etc.

Definition F.7 A pair of objects V, W with a morphism $\pi : V \to W$ is called a **fibre bundle** with **total space** V, **base** W, **structure group** $\boldsymbol{G}$ and **generic fibre** F, if

(i) $\boldsymbol{G}$ is a group object acting effectively on F by automorphisms,

(ii) there exists an open covering $\mathcal{U} = (U_i)_{i\in I}$ of the base W and a family of isomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ such that the fibre $\pi^{-1}(u)$ is mapped to $\{u\} \times F$,

(iii) for each index pair i, j, there exists a morphism $g_{ij} : U_i \cap U_j \to \boldsymbol{G}$, such that

$$\varphi_i \circ \varphi_j^{-1}(u, f) = (u, g_{ij}(u).f). \tag{F.2}$$

For short, we also call $\pi : V \to W$ a fibre bundle.

Definition F.8 Let $V, W, \pi, \boldsymbol{G}, F$ as in Definition F.7. If $F = \boldsymbol{G}$ and $\boldsymbol{G}$ acts by left-multiplication, then the fibre bundle $\pi : V \to W$ is called a **principal (fibre) bundle**.

Remark F.9 In the category of affine varieties, if $\pi : V \to W$ is a principal bundle, then the action of $\boldsymbol{G}$ on V is principal in the sense of definition E.2.

Definition F.10 Let $\pi : V \to W$ be a fibre bundle. Let U be an open subset of W. A isomorphism $\varphi_U : \pi^{-1}(U) \to U \times F$ is an **admissible chart** if it can be added to the collection $\mathcal{U}$ in Definition F.7, part (ii), such that part (iii) still holds.

Definition F.11 Let $\pi : V \to W$ and $\pi' : V' \to W$ be fibre bundles over the same base W. An **isomorphism of fibre bundles** is a isomorphism $\Psi : V \to V'$ such that for each $w \in W$

(i) Ψ preserves the fibre of w: $\Psi(\pi^{-1}(w)) = \pi'^{-1}(w)$,

(ii) there is an open neighbourhood U of w, a morphism $g_U : U \to \boldsymbol{G}$ and admissible charts $\varphi_U : \pi^{-1}(U) \to U \times F$, $\varphi'_U : \pi'^{-1}(U) \to U \times F$ such that
$$\varphi'_U \circ \Psi \circ \varphi_U(u, f) = (u, g_U(u).f)$$
for all $u \in U, f \in F$.

Definition F.12 A fibre bundle $\pi : V \to W$ is called **trivial** if it is isomorphic as a fibre bundle to $\pi_1 : W \times \boldsymbol{G} \to W$.

Remark F.13 If $\boldsymbol{G}$ is an abelian group, then the equivalence classes of isomorphic bundles correspond to elements of $\mathrm{H}^1(\mathcal{U}, \mathcal{G})$, where $\mathcal{G}$ is the sheaf of abelian groups defined by the morphisms $U_i \to \boldsymbol{G}$. Then a bundle is trivial if and only if $\mathrm{H}^1(\mathcal{U}, \mathcal{G}) = \{0\}$.

Theorem F.14 *A principal bundle is trivial if and only if there exists a morphism $\sigma : W \to V$ such that $\pi \circ \sigma = \mathrm{id}_W$.*

For a proof, see Steenrod [41], Theorem 8.3. The morphism σ in the proof is a **global cross section**, as defined in the algebraic case in Definition E.9.

G Unipotent Groups

G.1 Unipotent Groups as Lie Groups

Definition G.1 A subgroup $G \subset \mathbf{GL}_n(\mathbb{R})$ is called **unipotent** if all elements of G are unipotent matrices, that is all $g \in G$ satisfy $(g-I)^m = 0$ for some m. A connected linear Lie subgroup of $\mathbf{GL}_n(\mathbb{R})$ which is unipotent is called a **unipotent Lie group**.

Example G.2 The **Heisenberg group**

$$\mathbf{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\},$$

is a unipotent subgroup of $\mathbf{GL}_3(\mathbb{R})$.

Proposition G.3 *Let G be a unipotent Lie group with Lie algebra $\mathfrak{g}$. Then the matrix exponential* $\exp : \mathfrak{g} \to G$ *is polynomial diffeomorphism. In particular, a unipotent Lie group is diffeomorphic to $\mathbb{R}^d$ for $d = \dim G$.*

Corollary G.4 *A unipotent Lie group G is closed, connected and simply connected.*

Theorem G.5 *Let $G \subset \mathbf{GL}_n(\mathbb{R})$ be a unipotent Lie group. Then its Lie algebra $\mathfrak{g}$ is nilpotent, and G is conjugate to a subgroup of the group of upper triangular matrices with diagonal entries all* 1.

G.2 Unipotent Groups as Algebraic Groups

The exponential map allows a unipotent Lie group to be identified with a vector space via a polynomial map. This leads to the following theorem:

Theorem G.6 *Let $G \subset \mathbf{GL}_n(\mathbb{R})$ be a unipotent Lie group. Then G is a connected affine (linear) algebraic group.*

In the following, all topological terms refer to the Zariski topology, and all algebraic groups are assumed to be defined over $\mathbb{R}$ or $\mathbb{C}$.

Proposition G.7 *Let G be a unipotent affine algebraic group acting morphically on a quasi-affine variety V. Then all orbits are closed.*

For a proof, see Borel [5], 4.10.

In analogy to the Levi decomposition of an arbitrary Lie algebra into a semisimple and a solvable part, there exists, for linear algebraic groups, a decomposition into a reductive and a unipotent part:

Definition G.8 Let $\boldsymbol{G}$ be an arbitrary connected algebraic group. The **unipotent radical $\boldsymbol{U}$** of $\boldsymbol{G}$ is the largest unipotent normal subgroup of $\boldsymbol{G}$. Equivalently, $\boldsymbol{U}$ is the set of all unipotent elements contained in the solvable radical of $\boldsymbol{G}$.

Theorem G.9 *Let $\boldsymbol{G}$ be an arbitrary connected algebraic group and $\boldsymbol{U}$ its unipotent radical. Then $\boldsymbol{G}$ is a semidirect product*

$$\boldsymbol{G} = \boldsymbol{H} \cdot \boldsymbol{U}, \tag{G.1}$$

where $\boldsymbol{H} \cong \boldsymbol{G}/\boldsymbol{U}$ is a certain reductive subgroup of $\boldsymbol{G}$.

This decomposition is sometimes called the **Chevalley decomposition** of $\boldsymbol{G}$. For a further discussion of unipotent radicals and the Chevalley decomposition, see Onishchik and Vinberg [31], volume III, section 6.5 in chapter 1, section 5.3 in chapter 2, and Borel [5], 11.21 to 11.23.

A unipotent group has only trivial characters (algebraic homomorphisms onto $\mathbf{G}^\times$). This gives a characterisation of invariant rational functions for unipotent groups (also Rosenlicht [38], lemma on p. 220):

Lemma G.10 *Let $\boldsymbol{G}$ be a unipotent group acting morphically on a quasi-affine variety V. Then any $\boldsymbol{G}$-invariant rational function f on V is the quotient $f = \frac{p}{q}$ of two $\boldsymbol{G}$-invariant rational functions p, q defined on all of V. For affine V this means p, q are polynomials.*

G.3 Malcev Coordinates

By Proposition G.3, a unipotent Lie group $\boldsymbol{G}$ is diffeomorphic to its Lie algebra $\mathfrak{g}$. By virtue of this correspondence, one obtains a very handy set of coordinates for $\boldsymbol{G}$ coming from a vector space basis of $\mathfrak{g}$. The exposition follows Onishchik and Vinberg [31], volume III, section 4.2 in chapter 2.

Theorem G.11 *Let $\boldsymbol{G}$ be a unipotent Lie group. There exists a connected normal Lie subgroup $\boldsymbol{H}$ of codimension 1 and subgroup $\boldsymbol{A} \cong \mathbf{G}_+$ such that $\boldsymbol{G}$ is the semidirect product*

$$\boldsymbol{G} = \boldsymbol{A} \cdot \boldsymbol{H}. \tag{G.2}$$

By induction on dim G one concludes:

Corollary G.12 *Let G be a unipotent Lie group of dimension k. There exist one-parameter subgroups $A_1, \ldots, A_k$, each isomorphic to $\mathbf{G}_+$, such that the following holds:*

(a) *Every $g \in G$ has a unique representation $g = a_1 \cdots a_k$, where $a_i \in A_i$. In particular, $G = A_1 \cdot A_2 \cdots A_k$.*

(b) *Let $G_i = A_{k-i+1} \cdots A_k$. Then each G_i is a connected normal Lie subgroup of G.*

For proofs, see Onishchik and Vinberg [31], volume III, chapter 2, section 3.1 and the remarks at the beginning of section 4.2.

Definition G.13 Let $X_1, \ldots, X_k \in \mathfrak{g}$ such that the one-parameter subgroup A_i from Corollary G.12 is given by $\exp(tX_i)$. Then the set $\{X_1, \ldots, X_k\}$ is called a **Malcev basis**[17)] for $\mathfrak{g}$ (or G).

Definition G.14 By Corollary G.12 (a), for $g \in G$ and a Malcev basis $X_1, \ldots, X_k$, we have

$$g = \exp(t_1 X_1) \cdots \exp(t_k X_k)$$

for uniquely determined $t_1, \ldots, t_k \in \mathbb{R}$. These parameters are called **Malcev coordinates** of G.

Recall that the exponential map is a diffeomorphism for unipotent groups. This provides us with another set of coordinates:

Definition G.15 Let $X_1, \ldots, X_k$ be any basis of $\mathfrak{g}$. Then every $g \in G$ can be written as

$$g = \exp(s_1 X_1 + \ldots + s_k X_k)$$

for uniquely determined $s_1, \ldots, s_k \in \mathbb{R}$. These parameters are called **exponential coordinates** of G.

Proposition G.16 *Let $X_1, \ldots, X_k$ be a Malcev basis for $\mathfrak{g}$. For $g \in G$ there exist Malcev coordinates and exponential coordinates such that*

$$\exp(t_1 X_1) \cdots \exp(t_k X_k) = g = \exp(s_1 X_1 + \ldots + s_k X_k).$$

[17)] In the terminology of Corwin and Greenleaf [10], this is a *strong Malcev basis*.

Then, for $i = 1, \dots, k$,

$$s_i = t_i + f_i(t_1, \dots, t_{i-1}),$$
$$t_i = s_i + h_i(s_1, \dots, s_{i-1}),$$

where f_i, h_i *are polynomials.*

Corollary G.17 *The multiplication in* $\boldsymbol{G}$ *is polynomial with respect to the Malcev coordinates.*

Remark G.18 There is an analogue for Malcev coordinates in the context of algebraic groups: If $\boldsymbol{G}$ is a linear algebraic group defined over an arbitrary field $\Bbbk$, and $\boldsymbol{G}$ is unipotent and $\Bbbk$-solvable[18)], then a series of normal subgroups $\boldsymbol{G} \supset \boldsymbol{G}_1 \supset \dots \supset \boldsymbol{G}_k = \{1_{\boldsymbol{G}}\}$ exists such that $\boldsymbol{G}_i/\boldsymbol{G}_{i+1} = \boldsymbol{A}_{k-i}$ is $\Bbbk$-isomorphic to $\mathbf{G}_+(\Bbbk)$ and all $\boldsymbol{G}_i$, $\boldsymbol{A}_i$ are defined over $\Bbbk$. For a proof, see Lemma 1, p. 116, in Rosenlicht [39].

G.4 Lattices in Unipotent Groups

Definition G.19 Let $\boldsymbol{G}$ be a unipotent Lie group. By a **lattice** $\boldsymbol{\Gamma}$ in $\boldsymbol{G}$ we mean a discrete subgroup such that $\boldsymbol{G}/\boldsymbol{\Gamma}$ is compact.

Usually, one only requires for a lattice that $\boldsymbol{G}/\boldsymbol{\Gamma}$ is of finite volume, but for nilpotent groups, this already implies compactness.

Theorem G.20 *A unipotent Lie group* $\boldsymbol{G}$ *admits a lattice if and only if its Lie algebra* $\mathfrak{g}$ *admits a basis with rational structure constants.*

Theorem G.21 $\boldsymbol{\Gamma}$ *is a lattice in the unipotent Lie group* $\boldsymbol{G}$ *if and only if* $\boldsymbol{G}$ *is the Zariski closure of* $\boldsymbol{\Gamma}$.

Theorem G.22 *A group* $\boldsymbol{\Gamma}$ *is isomorphic to a lattice in a unipotent Lie group* $\boldsymbol{G}$ *if and only if* $\boldsymbol{\Gamma}$ *is finitely generated, torsion free and nilpotent. In this case,* $\dim \boldsymbol{G} = \operatorname{rk} \boldsymbol{\Gamma}$.

For proofs, see Raghunathan [35], Theorems 2.12, 2.3 and 2.18.

A lattice $\boldsymbol{\Gamma}$ is a rigid structure in $\boldsymbol{G}$ in the sense that many properties of $\boldsymbol{G}$ are determined once they are known for $\boldsymbol{\Gamma}$. For example, we get the following important result:

18) For non-perfect fields, *unipotent* does not imply $\Bbbk$-*solvable*, see the introduction in Rosenlicht [39] for a discussion.

Theorem G.23 $\mathbf{\Gamma}$ *is a lattice in a unipotent Lie group* $\mathbf{G}$ *if and only if there exists a Malcev basis* $X_1, \ldots, X_k$ *for* $\mathfrak{Lie}(\mathbf{G})$ *such that*

$$\mathbf{\Gamma} = \{\exp(m_1 X_1) \cdots \exp(m_k X_k) \mid m_i \in \mathbb{Z}\}.$$

See Corwin and Greenleaf [10], Theorem 5.1.6, for a proof.[19)] We will call the $X_1, \ldots, X_k$ a *Malcev basis for* $\mathbf{\Gamma}$.

G.5 Homogeneous Spaces of Unipotent Groups

In Theorem E.16 we noted that a homogeneous space of an algebraic group is a quasi-affine variety. If the group is unipotent, stronger statements can be made.

All topological terms refer to the Zariski topology.

Proposition G.24 *Let* $\mathbf{G}$ *be a unipotent algebraic group. Then every homogeneous space of* $\mathbf{G}$ *is an affine variety.*

For a proof, see Borel [5], 6.9.

The following theorem is due to Rosenlicht [36] (Theorem 10). It is formulated for an arbitrary field of definition $\Bbbk$ and for solvable groups, which of course includes the unipotent case. We give a somewhat simplified formulation more suited to our needs.

Theorem G.25 (Rosenlicht) *If a connected* $\Bbbk$*-solvable algebraic group* $\mathbf{G}$ *operates* $\Bbbk$*-morphically on an affine* $\Bbbk$*-variety* V *such that the quotient* $\pi : \mathbf{G} \to V/\mathbf{G}$ *exists as a quasi-affine variety, then there exists a local cross section* $\sigma : U \to V$ *defined on a dense open subset* $U \subseteq V/\mathbf{G}$*. If* π *and* $V/\mathbf{G}$ *are defined over* $\Bbbk$ *as well, then* σ *can be assumed to be defined over* $\Bbbk$*.*

Corollary G.26 *If the quotient in Theorem G.25 is an algebraic homogeneous space* $\mathbf{G}/\mathbf{H}$*, then* $\mathbf{G}/\mathbf{H}$ *is covered by open subsets* U *admitting a local cross section (by virtue of left-multiplication with* $\mathbf{G}$*). If all* U *and* $\pi^{-1}(U)$ *are normal, then* $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ *is a locally trivial fibration (Proposition E.11).*

Note that in our applications, the U and $\pi^{-1}(U)$ in Corollary G.26 are smooth, hence normal, and so the last statement in the corollary always holds.

[19)] Corwin and Greenleaf [10] use the term *uniform discrete subgroup* for what we call a *lattice*.

Proposition G.24 can be further refined:

Theorem G.27 (Rosenlicht) *A homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ for a unipotent algebraic $\mathbb{k}$-group $\boldsymbol{G}$ by a closed $\mathbb{k}$-subgroup $\boldsymbol{H}$ is isomorphic to $\mathbb{k}^m$ as an affine $\mathbb{k}$-variety, where $m = \dim \boldsymbol{G} - \dim \boldsymbol{H}$.*

This was proved by Rosenlicht [39] (Theorem 5) in a more general form including $\mathbb{k}$-solvable algebraic groups. The proof is considerably simpler if one assumes $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$.

Sketch of proof for $\mathbb{k} = \mathbb{R}$: The proof works by induction on $\dim \boldsymbol{G}$:

A unipotent group of $\dim \boldsymbol{G} = 1$ is itself isomorphic to $\mathbb{R}^1$. If $\dim \boldsymbol{G} > 1$, then one obtains a normal one-parameter subgroup $\boldsymbol{A}$ from Theorem G.11. If $\boldsymbol{A}$ acts trivially on $W = \boldsymbol{G}/\boldsymbol{H}$, then the $\boldsymbol{G}$-action is isomorphic to the $\boldsymbol{G}/\boldsymbol{A}$-action (Proposition E.17), and we are done by the induction hypothesis.

Otherwise, consider the $\boldsymbol{A}$-action on W. The quotient $W/\boldsymbol{A}$ is homogeneous for $\boldsymbol{G}/\boldsymbol{A}$ (Proposition E.17), hence affine by Proposition G.24. It can be covered by open sets admitting local cross sections (Corollary G.26). The smooth action of $\boldsymbol{A}$ on W is free, hence principal by PropositionE.11, so $\pi : W \to W/\boldsymbol{A}$ is a locally trivial fibration.

Since the first Čech cohomology group $\mathrm{H}^1(W/\boldsymbol{A}, \mathcal{O})$ of the sheaf of regular functions is trivial (Theorem F.6), there exists a global cross section (see the proof of Lemma G.29 for details). Hence the locally trivial fibration is algebraically isomorphic to the trivial algebraic principal bundle $(W/\boldsymbol{A})\times\boldsymbol{A}$, which itself is algebraically isomorphic to $(W/\boldsymbol{A}) \times \mathbb{R}$.

$\boldsymbol{G}/\boldsymbol{A}$ acts algebraically on $W/\boldsymbol{A}$, and applying the induction hypothesis, $W/\boldsymbol{A}$ is algebraically isomorphic to $\mathbb{R}^{m-1}$. It follows that W is algebraically isomorphic to $\mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$. ■

G.6 Algebraic Principal Bundles for Unipotent Groups

Theorem G.31 below does not seem to be as widely known as the rest of this chapter. One reference is Kraft and Schwarz [28], chapter IV, Proposition 3.4 (for $\mathbb{k} = \mathbb{C}$). To ensure that we can employ the theorem for our purposes in chapter 5, where everything is defined over $\mathbb{R}$, we give a detailed proof, first for dimension 1 (Lemma G.29) and then for the general case.

In the following, an **algebraic principal bundle** will mean a principal bundle $\pi : V \to W$ in the category of algebraic varieties, that is all objects in Definition F.8 are $\mathbb{k}$-varieties and all morphisms are algebraic $\mathbb{k}$-morphisms.

Remark G.28 Let $\pi : V \to W$ be an algebraic principal bundle with structure group $\boldsymbol{G}$. If V and W are irreducible and W is smooth (both conditions are always satisfied in the applications of this thesis), then it follows from Proposition E.7 that W is a quotient $V/\boldsymbol{G}$.

Lemma G.29 *Let $\pi : V \to W$ be an algebraic principal bundle for a unipotent algebraic action of $\mathbf{G}_+$. If V, W are smooth and W is affine, then there exists an algebraic cross section $\sigma : W \to V$.*

Proof: Let $\pi : V \to W$ denote the canonical projection. We will show the existence of a global cross section $\sigma : W \to V$ (which is equivalent to V being a trivial bundle):

(i) That V is a principal bundle for $\mathbf{G}_+$ means that $\pi : V \to W$ is locally trivial (Definition E.10). Because W is affine, we can cover W by a finite system $\mathcal{U} = (U_i)_{i=1}^m$ of dense open subsets admitting local cross sections $\sigma_i : U_i \to V$.

(ii) The action of $\mathbf{G}_+$ is principal, which means the map β which, for any $p \in V$ and $g \in \mathbf{G}_+$, is defined by $\beta(g.p, p) = g$, is a morphism. But $\mathbf{G}_+ = \mathbb{k}$, so β is in fact a regular function on its domain of definition. Hence we can define regular functions β_{ij} on each $U_{ij} = U_i \cap U_j$ by

$$\beta_{ij} : U_i \cap U_j \to \mathbb{R}, \quad p \mapsto \beta(\sigma_i(p), \sigma_j(p))$$

satisfying

$$\sigma_i|_{U_{ij}}(p) = \beta_{ij}(p).\sigma_j|_{U_{ij}}(p).$$

By definition of β, we have on $U_i \cap U_j \cap U_k$:

$$\beta_{ij}(p) + \beta_{jk}(p) = \beta(\sigma_i(p), \sigma_j(p)) + \beta(\sigma_j(p), \sigma_k(p)) = \beta(\sigma_i(p), \sigma_k(p)) = \beta_{ik}(p).$$

So the β_{ij} form a 1-cocycle in the Čech cohomology (Definition F.1) of the sheaf $\mathcal{O}$ of $\overline{\mathbb{k}}$-valued regular functions on W.

(iii) As W is an affine variety, its first Čech cohomology group $\mathrm{H}^1(W, \mathcal{O})$ is trivial (Theorem F.6). This means there exist $\overline{\mathbb{k}}$-valued regular functions α_i defined on U_i such that

$$\beta_{ij} = \alpha_i|_{U_{ij}} - \alpha_j|_{U_{ij}}.$$

(iv) The maps

$$p \mapsto -\alpha_i(p).\sigma_i(p), \quad p \mapsto -\alpha_j(p).\sigma_j(p)$$

are local cross sections defined on U_i, U_j, respectively, which coincide on the open set U_{ij}. By continuity, they define an $\Bbbk$-morphism

$$\sigma^{ij} : U_i \cup U_j \to V.$$

Further, $\pi \circ \sigma^{ij}$ coincides with the identity on the open subset $U_i \subset U_i \cup U_j$, hence it is the identity on all of $U_i \cup U_j$. So σ^{ij} is a local cross section defined on $U_i \cup U_j$.

(v) As the cover $\mathcal{U}$ is finite, one can repeat step (iv) at most m times to obtain a global cross section $\sigma : W \to V$. ■

Lemma G.30 *Let $\pi : V \to W$ be an algbraic principal $\boldsymbol{G}$-bundle for a unipotent algebraic group $\boldsymbol{G}$. Let $\boldsymbol{H}, \boldsymbol{A} \subset \boldsymbol{G}$ be as in Theorem G.11. If V, W are affine and smooth, then V is also an algebraic principal $\boldsymbol{H}$-bundle with base $W \times \boldsymbol{A}$.*

Proof: Recall that $\boldsymbol{A} \cong \mathbf{G}_+$ and $\boldsymbol{A} \cong \boldsymbol{G}/\boldsymbol{H}$ as algebraic groups. Thus there is a cross section $\boldsymbol{G}/\boldsymbol{H} \to \boldsymbol{A} \hookrightarrow \boldsymbol{G}$. It then follows that $V/\boldsymbol{H}$ exists as an affine variety, $V \to V/\boldsymbol{H}$ is an algebraic principal $\boldsymbol{H}$-bundle, and that $V/\boldsymbol{H} \to W$ is a bundle with structure group $\boldsymbol{A}$. By Lemma G.29, $V/\boldsymbol{H} \cong W \times \boldsymbol{A}$ as an algebraic principal $\boldsymbol{A}$-bundle. ■

Theorem G.31 *Let $\pi : V \to W$ be an algebraic principal bundle for a unipotent algebraic group $\boldsymbol{G}$. If V, W are affine and smooth, then $W = V/\boldsymbol{G}$ and there exists an algebraic cross section $\sigma : V/\boldsymbol{G} \to V$.*

Proof: Let $k = \dim \boldsymbol{G}$. The case $k = 1$ is Lemma G.29. The theorem follows by induction on k: Let $\boldsymbol{H}, \boldsymbol{A}$ denote the subgroups from Theorem G.11.

(i) By Lemma G.30 we may apply the induction hypothesis to $\boldsymbol{H}$, and together with Lemma G.29, we have global cross sections:

$$\sigma_{\boldsymbol{H}} : V/\boldsymbol{H} \to V,$$
$$\sigma_{\boldsymbol{A}} : (V/\boldsymbol{H})/\boldsymbol{A} \to V/\boldsymbol{H}.$$

Note here that $(V/\boldsymbol{H})/\boldsymbol{A} = (V/\boldsymbol{H})/(\boldsymbol{G}/\boldsymbol{H}) = V/\boldsymbol{G}$ by Proposition E.17.

(ii) Let $\pi : V \to V/\boldsymbol{G}$ denote the canonical projection and define a morphism $\sigma = \sigma_{\boldsymbol{H}} \circ \sigma_{\boldsymbol{A}} : V/\boldsymbol{G} \to V$: This is defined on all of $V/\boldsymbol{G}$, because $\sigma_{\boldsymbol{A}}$ and $\sigma_{\boldsymbol{H}}$ are global sections. Further, for any orbit $\boldsymbol{G}.p$, we have for certain $a \in \boldsymbol{A}, h \in \boldsymbol{H}$:

$$\begin{aligned}\pi \circ \sigma(\boldsymbol{G}.p) &= \pi \circ \sigma_{\boldsymbol{H}} \circ \sigma_{\boldsymbol{A}}(\boldsymbol{G}.p)\\ &= \pi \circ \sigma_{\boldsymbol{H}} \circ \sigma_{\boldsymbol{A}}(\boldsymbol{A}.(\boldsymbol{H}.p))\\ &= \pi \circ \sigma_{\boldsymbol{H}}(\boldsymbol{H}.(a.p))\\ &= \pi(ha.p)\\ &= \boldsymbol{G}.(ha.p) = \boldsymbol{G}.p.\end{aligned}$$

So $\pi \circ \sigma = \mathrm{id}_{V/G}$, that is σ is a global cross section for the action of $\boldsymbol{G}$. Hence the principal bundle is trivial. ■

Remark G.32 Assume $\Bbbk = \mathbb{R}$ in Theorem G.31, and that V, W, π and the action of $\boldsymbol{G}$ are all defined over $\mathbb{R}$. Then the cross section σ may be taken to be defined over $\mathbb{R}$: In the proof of Theorem G.31, we may assume $\sigma_{\boldsymbol{H}}$ to be $\mathbb{R}$-defined by the induction hypothesis. Further, $\sigma_{\boldsymbol{A}}$ may be assumed to be $\mathbb{R}$-defined, because in the proof of Lemma G.29, the local cross sections σ_i can be assumed to be $\mathbb{R}$-defined by Rosenlicht's Theorem (Theorem G.25), and the 1-cocycles α_i may be replaced by their real parts and still yield $\alpha_i - \alpha_j = \beta_{ij}$, because the latter is an $\mathbb{R}$-valued regular function which is defined over $\mathbb{R}$ if the action of $\boldsymbol{A}$ is.

References

[1] O. Baues
Personal Communication
2010 & 2011

[2] O. Baues
Prehomogeneous Affine Representations and Flat Pseudo-Riemannian Manifolds (in *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*)
EMS, 2010
http://arxiv.org/abs/0809.0824

[3] O. Baues, W. Globke
Flat Pseudo-Riemannian Homogeneous Spaces with Non-Abelian Holonomy Group
to appear in Proc. Amer. Math. Soc.
http://arxiv.org/abs/1009.3383

[4] A. Besse
Einstein Manifolds
Springer, 1987

[5] A. Borel
Linear Algebraic Groups, 2nd ed.
Springer, 1991

[6] E. Brieskorn
Lineare Algebra I & II
Vieweg, 1983 & 1985

[7] Y. Carrière
Un survol de la théorie des variétés affines
Seminaire de Théorie spectrale et géometrie 6, 9-22, 1987-1988

[8] Y. Carrière
Autour de la conjecture de L. Markus sur les variétés affines
Invent. math. 95, 615-628, 1989

[9] L. Charlap
Bieberbach Groups and Flat Manifolds
Springer, 1986

[10] L. Corwin, F. Greenleaf
Representations of Nilpotent Lie Groups and their Applications, Part 1: Basic Theory and Examples
Cambridge University Press, 1990

[11] D. Duncan, E. Ihrig
Homogeneous Spacetimes of Zero Curvature
Proc. Amer. Math. Soc. 107, no. 3, 785-795, 1989

[12] D. Duncan, E. Ihrig
Flat pseudo-Riemannian Manifolds with a Nilpotent Transitive Group of Isometries
Ann. Global Anal. Geom. 10, no. 1, 87-101, 1992

[13] D. Duncan, E. Ihrig
Translationally Isotropic Flat Homogeneous Manifolds with Metric Signature $(n, 2)$
Ann. Global Anal. Geom. 11, no. 1, 3-24, 1993

[14] D. Eisenbud
Commutative Algebra with a View Toward Algebraic Geometry
Springer, 1995

[15] R.H. Escobales jr
Riemannian Submersions with Totally Geodesic Fibres
J. Differential Geom. 10, no. 2, 253-276, 1975

[16] W. Globke
Homepage at the Karlsruhe Institute of Technology
http://www.math.kit.edu/iag2/~globke/en

[17] W. Goldman, M. Hirsch
Affine Manifolds and Orbits of Algebraic Groups
Trans. Amer. Math. Soc., 295, no. 1, 175-198, 1986

[18] W. Goldman, Y. Kamishima
The Fundamental Group of a Compact Flat Lorenzt Space Form is Virtually Polycyclic
J. Differential Geom. 19, 233-240, 1984

[19] F. Grunewald, G. Margulis
Transitive and Quasitransitive Actions of Affine Groups Preserving a Generalized Lorentz-structure
J. Geom. Phys. 5, no. 4, 493-531, 1989

[20] G. Greuel, G. Pfister
Geometric Quotients of Unipotent Group Actions
Proc. Lond. Math. Soc. 67, no. 3, 75-105, 1993

[21] S. Helgason
Differential Geometry, Lie Groups, and Symmetric Spaces
AMS, 2001

[22] R. Hermann
A Sufficient Condition that a Mapping of Riemannian Manifolds be a Fibre Bundle
Proc. Amer. Math. Soc. 11, no. 2, 236-242, 1960

[23] F. Hirzebruch
Topological Methods in Algebraic Geometry
Springer, 1978

[24] K. Jo, I. Kim
Convex affine domains and Markus conjecture
Math. Z. 248, 173-182, 2004

[25] S. Kobayashi, K. Nomizu
Foundations of Differential Geometry I & II
Wiley & Sons, 1961

[26] J.L. Koszul
Variétés localement plates et convexité
Osaka J. Math. 2, 285-290, 1965

[27] T. Kobayashi
On Discontinuous Group Actions on Non-Riemannian Homogeneous Spaces
Sugaku Expositions Amer. Math. Soc. 22, 1-19, 2009

[28] H. Kraft, G.W. Schwarz
Reductive Group Actions with One-Dimensional Quotient
Publ. Math. Inst. Hautes Études Sci. 76, 1-97, 1992
http://www.numdam.org/item?id=PMIHES_1992__76__1_0

[29] B. O'Neill
The Fundamental Equations of a Submersion
Michigan Math. J. 13, 459-469, 1966

[30] B. O'Neill
Semi-Riemannian Geometry: With Applications to Relativity
Academic Press, 1983

[31] A. Onishchik, E. Vinberg
Lie Groups and Lie Algebras I & II & III
Springer 1993 & 2000 & 1994

[32] D. Perrin
Algebraic Geometry
Springer, 2008

[33] A. Püttmann
Free Affine Actions of Unipotent Groups on $\mathbb{C}^n$
Transform. Groups 12, no. 1, 2007

[34] A. Püttmann
Biquotient Actions on Unipotent Lie Groups
Internat. J. Math. 19, no. 8, 2008
http://front.math.ucdavis.edu/math.CV/0701109

[35] M.S. Raghunathan
Discrete Subgroups of Lie Groups
Springer, 1972

[36] M. Rosenlicht
Some Basic Theorems on Algebraic Groups
Amer. J. Math. 78, no. 2, 401-443, 1956

[37] M. Rosenlicht
Extensions of Vector Groups by Abelian Varieties
Amer. J. Math. 80, no. 3, 685-714, 1958

[38] M. Rosenlicht
On Quotient Varieties and the Affine Embedding of Certain Homogeneous Spaces
Trans. Amer. Math. Soc. 101, no. 2, 211-223, 1961

[39] M. Rosenlicht
Questions of Rationality for Solvable Algebraic Groups over Nonperfect Fields
Ann. Mat. Pura Appl. 62, no. 1, 97-120, 1963

[40] R.K. Sachs, H. Wu
General Relativity for Mathematicians
Springer, 1977

[41] N. Steenrod
The Topology of Fibre Bundles
Princeton University Press, 1951

[42] S. Sternberg
Semi-Riemannian Geometry and General Relativity
Online Book, 2003
http://www.math.harvard.edu/~shlomo/

[43] W. Thurston
Three-Dimensional Geometry and Topology
Princeton University Press, 1997

[44] J. Vey
Sur la division des domaines de Siegel
Ann. scient. Éc. Norm. Sup. 4. série 3, 479-506, 1970

[45] E. Vinberg
The Theory of Convex Homogeneous Cones
Transl. Moscow Math. Soc. 12, 340-303, 1963

[46] F. Warner
Foundations of Differentiable Manifolds and Lie Groups
Springer, 1971

[47] J.A. Wolf
Homogeneous Manifolds of Constant Curvature
Comment. Math. Helv. 36, 112-147, 1961

[48] J.A. Wolf
Homogeneous Manifolds of Zero Curvature
Trans. Amer. Math. Soc. 104, 462-469, 1962

[49] J.A. Wolf
Isotropic Manifolds of Indefinite Metric
Comment. Math. Helv. 39, 21-64, 1964

[50] J.A. Wolf
Flat Homogeneous Pseudo-Riemannian Manifolds
Geometriae Dedicata 57, 111-120, 1994

[51] J.A. Wolf
Isoclinic Spheres and Flat Homogeneous Pseudo-Riemannian Manifolds (in *Crystallographic Groups and Their Generalizations*)
AMS, 2000

[52] J.A. Wolf
Spaces of Constant Curvature, 6th ed.
AMS, 2011

[53] K. Yagi
On Compact Homogeneous Affine Manifolds
Osaka J. Math 7, 457-475, 1970

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