Compact pseudo-Riemannian homogeneous spaces

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I Introduction

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- A proper subspace $U \subset T_p M$ can be totally isotropic, that is, $g_p|_U = 0$.
- The index s of (M, g) is the maximal dimension of a totally isotropic subspace $U \subset T_p M$.
  - Riemannian $s = 0$ (positive definite).
  - Lorentzian $s = 1$ (“lightlike lines”).

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- G acts transitively and by isometries (in particular volume-preserving),
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Question:

- Which Lie groups G can be isometry groups of such M?
- Which subgroups $H \subset G$ can be stabilizers of such actions?
- How is geometry of G and M related?

- Zimmer's and Gromov's work in the 1980s on rigid geometric structures.
- Adams & Stuck (1997), Zeghib (1998):
  Classification of isometry groups of compact Lorentzian manifolds.
  (Higher indices are much more difficult.)
- Zeghib (1998):
  Classification of compact homogeneous Lorentzian manifolds.
- Quiroga-Barranco (2006):
  Structure of compact M with arbitrary index and topologically transitive action of a simple G.

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The metric g on M induces a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then:

- $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}_{\mathfrak{g}}(H)$-invariant (and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$-invariant),

$$\langle \mathrm{Ad}_{\mathfrak{g}}(h)x, \mathrm{Ad}_{\mathfrak{g}}(h)y\rangle = \langle x, y\rangle \quad \text{for all } h \in H,$$
$$\langle \mathrm{ad}_{\mathfrak{g}}(h')x, y\rangle = -\langle x, \mathrm{ad}_{\mathfrak{g}}(h')y\rangle \quad \text{for all } h' \in \mathfrak{h}.$$

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- The kernel of $\langle\cdot,\cdot\rangle$ is

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid \langle x, \cdot\rangle = 0\} = \mathfrak{h}.$$

$\mathsf{G} \leq \mathrm{GL}_n(\mathbb{C})$ is a linear algebraic group (given by polynomial equations).
For a subgroup of $H \leq \mathsf{G}$, let $\overline{H}^{\mathrm{z}}$ denote the Zariski closure of H in G:

Recall: Zariski closure

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For a subgroup of $H \leq \mathsf{G}$, let $\overline{H}^{\mathrm{z}}$ denote the Zariski closure of H in G:

- $\overline{H}^{\mathrm{z}}$ is the smallest algebraic subgroup of G that contains H.
- H is Zariski-dense in G if $\overline{H}^{\mathrm{z}} = \mathsf{G}$.

Mostow Density Theorem (1971)

Let

- G *be a connected Lie group,*
- H *a closed subgroup,*
- μ *a finite* G*-invariant finite measure on* G/H,
- $\sigma : G \to \mathrm{GL}(V)$ *a representation, where* $\dim V < \infty$.

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  and $\overline{\sigma(H)}^{z} \supset \overline{\sigma(N)}^{z}$, where N is the nilradical of G.
- $\overline{\sigma(H)}^{z}$ contains every $\mathbb{R}$-diagonalizable subgroup of $\overline{\sigma(G)}^{z}$.

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Since $\langle \cdot, \cdot \rangle$ is a quadratic function on $\mathfrak{g}$, Mostow's density theorem applied to $\sigma = \mathrm{Ad}_{\mathfrak{g}}$ implies:

Corollary

1. $\langle \cdot, \cdot \rangle$ *is* $\overline{\mathrm{Ad}_{\mathfrak{g}}(H)}^{\mathrm{Z}}$*-invariant.*
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In particular, $\langle\cdot,\cdot\rangle$ is invariant under all nilpotent elements in of $\mathfrak{Lie}(\overline{\mathrm{Ad}_{\mathfrak{g}}(H)}^{\mathrm{z}})$.
We say $\langle\cdot,\cdot\rangle$ is nil-invariant.

II Compact pseudo-Riemannian solvmanifolds

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- To understand G, study solvable Lie algebras $\mathfrak{g}$ with nil-invariant $\langle \cdot, \cdot \rangle$.
- Recall: $\mathfrak{g}^\perp = \mathfrak{h}$.
- If G acts almost effectively, then $\mathfrak{g}^\perp$ contains no ideal $\neq \mathbf{0}$ in $\mathfrak{g}$.

A solvable Lie algebra with Lorentzian (nil-)invariant product is the oscillator algebra

$$\mathfrak{g} = \mathfrak{osc}(\alpha) = \mathbb{R} \ltimes \mathfrak{hei}_{2n+1}$$

where the Heisenberg algebra is

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Define Lorentzian $\langle \cdot, \cdot \rangle$ on $\mathfrak{osc}(\alpha)$ by a definite scalar product on $\mathbb{R}^{2n}$ and a dual pairing of totally isotropic subspaces $\mathbb{R}$ and $\mathbb{R}$.

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Example: Oscillator algebra

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- For index ≥ 2, Kath and Olbrich (2004) gave a classification scheme for Lie algebras with invariant scalar product, and a classification for index 2.
- For index ≥ 3 classification becomes extremely complicated.

Proposition

Let $\langle \cdot, \cdot \rangle$ be a symmetric nil-invariant bilinear form on a solvable Lie algebra $\mathfrak{g}$.

- *Then $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$.*
- *If $\mathfrak{g}^\perp$ contains no non-trivial ideals of $\mathfrak{g}$, then $\mathfrak{g}^\perp = \mathbf{0}$.*

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Theorem A

Assume G is solvable.
Then G acts almost freely on M.

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Theorem A

Assume G is solvable.
Then G acts almost freely on M.

Meaning:

- The stabiliser $\Gamma = G_x$ is discrete.
- The metric g on M pulls back to a left-invariant pseudo-Riemannian metric g_G on G.
- g_G is invariant under conjugation by Γ.

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Proof

- Let $\mathfrak{j}$ be a totally isotropic central ideal in $\mathfrak{g}$. Define the reduction $\overline{\mathfrak{g}} = \mathfrak{j}^{\perp}/\mathfrak{j}$.
- $\langle\cdot,\cdot\rangle$ induces a nil-invariant scalar product $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ on $\overline{\mathfrak{g}}$.

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- Assume that $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ is invariant on $\overline{\mathfrak{g}}$.
- It is now easily verified that $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is invariant if $\langle\cdot,\cdot\rangle_{\overline{\mathfrak{g}}}$ is.
- Iterated reduction to the abelian case and induction yield the result. □

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Theorem B

Assume G is solvable.
Then g pulls back to a bi-invariant pseudo-Riemannian metric g_G on G.

Bi-invariant metric for solvable G

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Corollary A+B

The universal cover $\widetilde{M}$ of M is a pseudo-Riemannian symmetric space.
In particular, M is locally symmetric.

No larger isometry groups

Theorem C

Assume G is solvable and effective.

Then $G = \mathrm{Iso}(M)^\circ$.

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Implications:

- Johnson (1972) showed that any solvmanifold has presentations by connected solvable groups of arbitrary dimension.
- Theorem A shows that most of them cannot act isometrically.
- Theorem C shows that no larger non-solvable group can act isometrically.

III Compact pseudo-Riemannian homogeneous spaces for arbitrary Lie groups

Now let G be an arbitrary connected Lie group, and K, S, R as before,

- K compact semisimple,
- S semisimple without compact factors,
- R the solvable radical of G.

We study Lie algebras

$$\mathfrak{g} = (\mathfrak{k} \times \mathfrak{s}) \ltimes \mathfrak{r}$$

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with nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$.

- $\langle\cdot,\cdot\rangle$ is a priori invariant under $\mathrm{Ad}_{\mathfrak{g}}(S)$ and $\mathrm{Ad}_{\mathfrak{g}}(R)_{\mathrm{split}}$ (maximal trigonalisable).
- From the solvable case it follows that *the restriction* $\langle\cdot,\cdot\rangle_{\mathfrak{s}\ltimes\mathfrak{r}}$ *is invariant under* $\mathrm{Ad}_{\mathfrak{g}}(SR)$.

If the index of $\langle \cdot, \cdot \rangle$ is 1, then $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$, with either $\mathfrak{s} = \mathbf{0}$ or $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$.

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Zeghib's Theorem (1998)

Let M be a homogeneous Lorentzian manifold of finite volume.
Then, up to "Riemannian type factors", M is one of the following:

1. $\mathrm{Iso}(M, \mathrm{g})$ *contains a cover of* $\mathrm{PSL}_2(\mathbb{R})$.
   $\widetilde{M} = \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times L$, *where L is a compact Riemannian homogeneous space.*
2. $\mathrm{Iso}(M, \mathrm{g})$ *contains an oscillator group* $\mathrm{Osc}(\alpha)$.
   $\widetilde{M} = \mathrm{Osc}(\alpha) \times_{\mathbf{S}^1} L$, *and* $M = \widetilde{M}/\Gamma$, *where Γ is isomorphic to a lattice in* $\mathrm{Osc}(\alpha)$.

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Theorem D

1. $\langle \cdot, \cdot \rangle$ *is* $\mathrm{Ad}_{\mathfrak{g}}(SR)$*-invariant.*
2. $\langle \cdot, \cdot \rangle_{\mathfrak{s} \ltimes \mathfrak{r}}$ *is* $\mathrm{Ad}_{\mathfrak{g}}(G)$*-invariant.*

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Theorem E

If the index of $\langle\cdot,\cdot\rangle$ is ≤ 2 and $\mathfrak{g}^{\perp}$ does not contain a non-trivial ideal of $\mathfrak{g}$, then:

1. *$\mathfrak{g}$ is a direct sum of ideals $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$.*
2. *$\mathfrak{g}^{\perp} \subset \mathfrak{z}(\mathfrak{r}) \times \mathfrak{k}$ and $\mathfrak{g}^{\perp} \cap (\mathfrak{s} \times \mathfrak{r}) = \mathbf{0}$.*

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This allows us to

- classify Lie algebras with $\langle\cdot,\cdot\rangle$ for index ≤ 2,
- recover Zeghib's Theorem (Lorentzian case) by algebraic methods,
- prove an analogue of Zeghib's Theorem for index 2.

IV Application: Compact pseudo-Riemannian Einstein solvmanifolds

(M, g) is called Einstein manifold if

$$\mathrm{Ric} = \lambda \mathrm{g}$$

for some constant $\lambda \in \mathbb{R}$.

For a bi-invariant metric g on G,

$$\mathrm{Ric} = -\frac{1}{4}\kappa$$

where $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ is the Killing form of $\mathfrak{g}$.

Let (M, g_M) be a compact pseudo-Riemannian solvmanifold.

Recall:

- $M = G/\Gamma$ for a lattice $\Gamma \leq G$.
- g_M pulls back to bi-invariant g_G.
- $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is non-degenerate and invariant.

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If (M, g_M) is Einstein, then

$$\mathrm{Ric} = 0 = \kappa.$$

Example

If $\mathfrak{g}$ is nilpotent, then $\kappa = 0$.

Question: Are there solvable $\mathfrak{g}$, not nilpotent, with

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Example

- Take the abelian Lie algebra $\mathfrak{ab}_1^4$ with $\dim \mathfrak{ab}_1^4 = 4$ and a Lorentzian scalar product.
- Take extension
$$\mathfrak{g} = \mathfrak{j}^* \ltimes (\mathfrak{ab}_1^4 \times_\omega \mathfrak{j})$$
where $\dim \mathfrak{j} = \dim \mathfrak{j}^* = 1$ and $a \in \mathfrak{j}^*$ acts on $\mathfrak{ab}_1^4$ by
$$\delta_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
Define $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{j}^* \times \mathfrak{j}$ by dual pairing. A cocycle ω corrects the Lie bracket on $\mathfrak{ab}_1^4 \times \mathfrak{j}$ such that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ becomes invariant with signature $(4, 2)$.

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$\mathfrak{g}$ solvable with nilradical $\mathfrak{n}$, then (as vector space)

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}.$$

- $\mathrm{ad}(x)$ for $x \in \mathfrak{n}$ is nilpotent, so $\kappa(x,x) = 0$.
- For $a \in \mathfrak{a}$, the Einstein condition (Ricci-flat, $\kappa = 0$) becomes

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where $\lambda_i \in \mathbb{R}$ and $\zeta_j = \alpha_j + \mathrm{i}\beta_j \in \mathbb{C}\backslash\mathbb{R}$ are the eigenvalues of $\mathrm{ad}(a)$.

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Theorem F
Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a solvable Lie algebra with invariant Einstein scalar product. If $\mathfrak{g}$ is not nilpotent, then $\dim \mathfrak{g} \geq 6$ and the index of $\langle\cdot,\cdot\rangle$ is ≥ 2.

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The characteristic polynomial of $\exp(\mathrm{ad}(a))$ has coefficients in $\mathbb{Z}$ for certain $a \in \mathfrak{a}$…
which means that all eigenvalues e^{λ_i}, e^{ζ_j} of $\exp(\mathrm{ad}(a))$ are algebraic numbers.

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Gelfond-Schneider Theorem (1935)

Let $\alpha \in \mathbb{C} \setminus \{0, 1\}$ *and let* $\beta \in \mathbb{C}$ *be irrational.*
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Generalizations:

- Baker's Theorem (1966/67) on linear forms in logarithms of algebraic numbers (Fields Medal 1970).
- Schanuel's Conjecture, stating:
  Let $\alpha_1, \ldots, \alpha_d$ be complex numbers that are linearly independent over $\mathbb{Q}$.
  Then the transcendence degree over $\mathbb{Q}$ of the extension field $\mathbb{Q}(\alpha_1, \ldots, \alpha_d, e^{\alpha_1}, \ldots, e^{\alpha_d})$ is at least d.

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Conjecture

If Schanuel's Conjecture is true, then the Algebraic Lemma holds without "$n \leq 5$".

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Conjecture

Every compact pseudo-Riemannian Einstein solvmanifold is a nilmanifold.

References

- S. Adams, G. Stuck,
  The isometry group of a compact Lorentz manifold I,
  Inventiones Mathematicae 129, 1997
- O. Baues, W. Globke,
  Rigidity of compact pseudo-Riemannian homogeneous spaces for solvable Lie groups,
  to appear in International Mathematics Research Notices, arXiv:1507.02575
- O. Baues, W. Globke, A. Zeghib,
  The structure of compact homogeneous spaces with indefinite of low metric index,
  in preparation
- W. Globke, Y. Nikolayevsky,
  Compact pseudo-Riemannian homogeneous Einstein manifolds of low dimension,
  to appear in Differential Geometry and Applications, arXiv:1611.08662
- A. Zeghib,
  Sur les espaces-temps homogènes,
  Geometry and Topology Monographs 1: The Epstein Birthday Schrift, 1998