From symmetries of crystals to lattices in Lie groups

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Symmetry

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- Geometric: Part of an object is mapped onto another part by a transformation.
- Symmetric patterns are abundant in nature and in art.

Space-filling patterns

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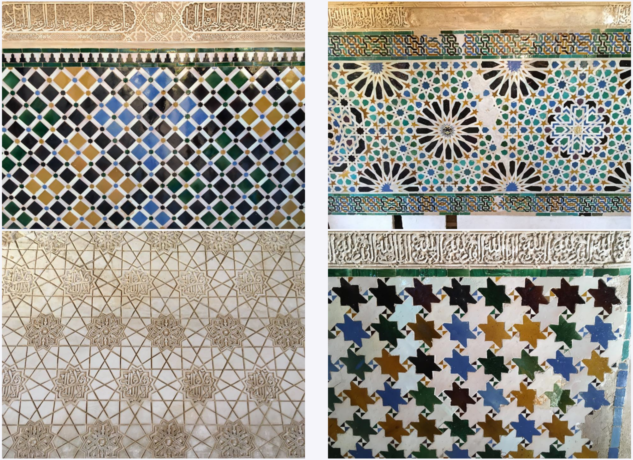
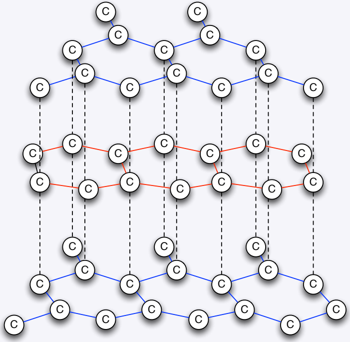
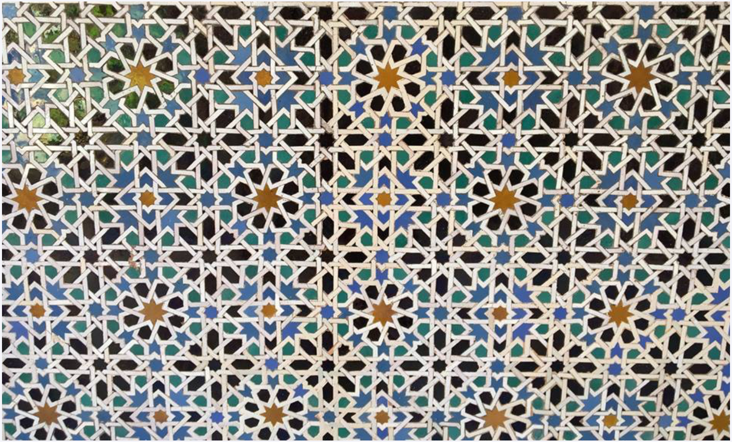
Ancient Egyptian and medieval Moorish artists created elaborate ornamentations, thereby realizing all of the 17 possible symmetry classes.



- Analogue in three dimensions: Crystals.



- Start with a single shape in $\mathbb{R}^n$



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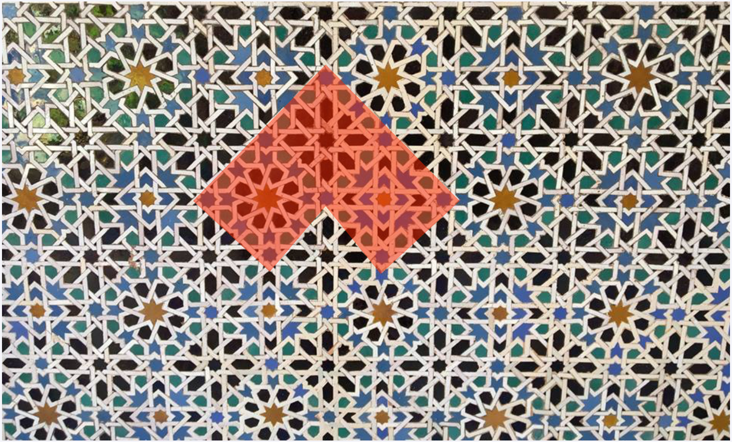
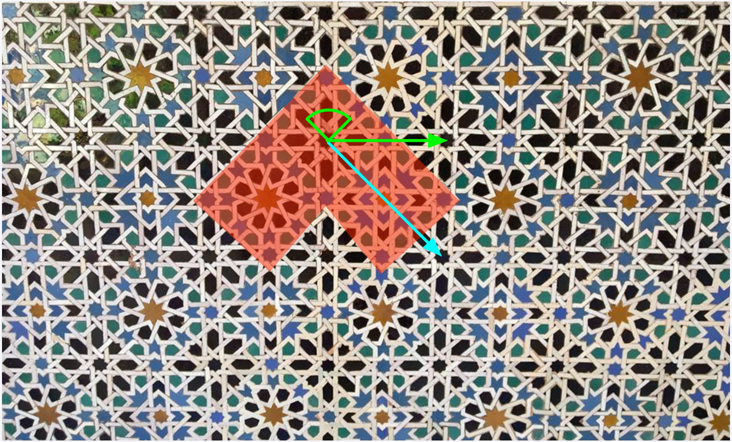


- Start with a single shape in $\mathbb{R}^n$
  $\rightsquigarrow$ **translate**, **reflect**, **rotate**, to fill up all of $\mathbb{R}^n$.



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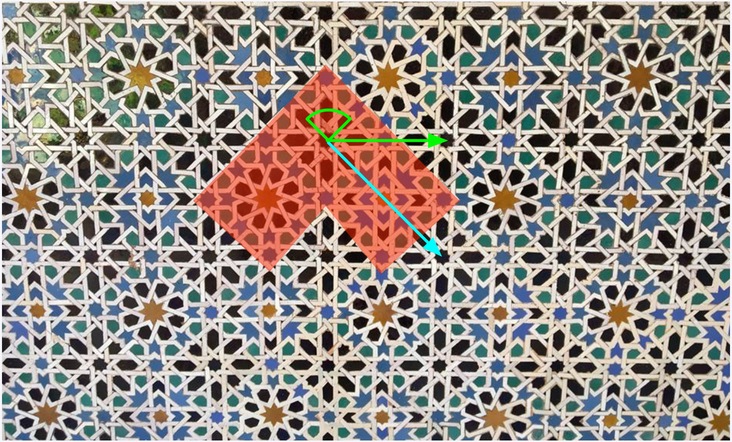

- Not any translation or rotation will do! $\rightsquigarrow$ symmetry group

A space-filling pattern is characterized by its symmetry group

$$\begin{aligned} \Gamma &= \{\gamma : \mathbb{R}^n \to \mathbb{R}^n \mid \gamma \text{ rigid transformation preserving the pattern}\} \\ &\subset \mathrm{O}_n \ltimes \mathbb{R}^n \end{aligned}$$

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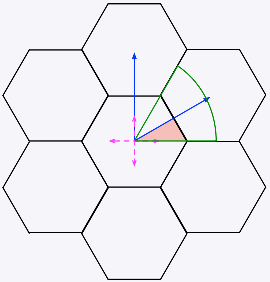
and a fundamental domain $\mathcal{F}$:

- $\mathcal{F}$ open subset of $\mathbb{R}^n$,
- $(\gamma \cdot \mathcal{F}) \cap \mathcal{F} \neq \emptyset$ if and only if $\gamma = \mathrm{id}$,
- $\Gamma \cdot \overline{\mathcal{F}} = \mathbb{R}^n$.

The symmetry group Γ of the hexagonal pattern



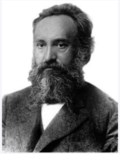
is generated by

$$(\text{reflections})\quad \sigma_1 = \left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),\quad \sigma_2 = \left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),$$

$$(\text{rotation})\quad \varrho = \left(\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),$$

$$(\text{translations})\quad \tau_1 = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right),\quad \tau_2 = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}(\sqrt{6}+\sqrt{2})/4\\ (\sqrt{6}-\sqrt{2})/4\end{pmatrix}\right).$$

Theorem (E. Fedorov, 1891)
There exist 17 *(classes of) wallpaper groups in dimension* 2*.*



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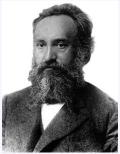

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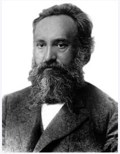

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Wait…is it 230 or 219?

Hilbert's 18^{th} Problem

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Definition

A crystallographic group Γ is a subgroup of $\mathrm{O}_n \ltimes \mathbb{R}^n$ (rigid transformations) that

1. is discrete in $\mathrm{O}_n \ltimes \mathbb{R}^n$,
2. has a fundamental domain $\mathcal{F}$ with compact closure $\overline{\mathcal{F}}$.

Theorem (L. Bieberbach, 1911)
Let $\Gamma \subset \mathrm{O}_n \ltimes \mathbb{R}^n$ *be a crystallographic group.*
Then:

1. *The linear parts of* Γ *form a finite group.*
2. *The translation subgroup* $\Gamma \cap \mathbb{R}^n$ *is a lattice in* $\mathbb{R}^n$*.*

With respect to a basis in $\Gamma \cap \mathbb{R}^n$*, the linear parts of* Γ *are represented by integer matrices.*

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With respect to a basis in $\Gamma \cap \mathbb{R}^n$, the linear parts of Γ are represented by integer matrices.

Remark

- Every lattice in $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$.
- So every Γ is essentially given by integer translations and a finite group of integer rotations.
- This phenomenon is called arithmeticity of Γ.

Theorem (L. Bieberbach, 1912)

Let $\Gamma_1, \Gamma_2 \subset \mathrm{O}_n \ltimes \mathbb{R}^n$ *be crystallographic groups.*

$\Gamma_1 \cong \Gamma_2$ *if and only if* $\Gamma_1 = A\Gamma_2 A^{-1}$ *for some affine transformation* A *of* $\mathbb{R}^n$*.*

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- If $\Gamma_1 \cong \mathbb{Z}^n$ (no linear symmetry), then any isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ extends to a continuous isomorphism $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^n$.
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  We say lattices in $\mathbb{R}^n$ are rigid.
- If Γ_1 has non-trivial rotation parts, then conjugation by A does not always preserve the ambient group $\mathrm{O}_n \ltimes \mathbb{R}^n$,

$$A(\mathrm{O}_n \ltimes \mathbb{R}^n)A^{-1} \neq \mathrm{O}_n \ltimes \mathbb{R}^n.$$

Crystallographic groups are not rigid in general.

The answer to Hilbert's question:

Theorem (L. Bieberbach, 1912)
For a given dimension n, there exist only finitely many (affine equivalence classes of) crystallographic groups.

Geometric meaning of Bieberbach's theorems

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If M is a compact complete connected flat Riemannian manifold, then $M = \mathbb{R}^n/\Gamma$, where Γ is a crystallographic group without fixed points.

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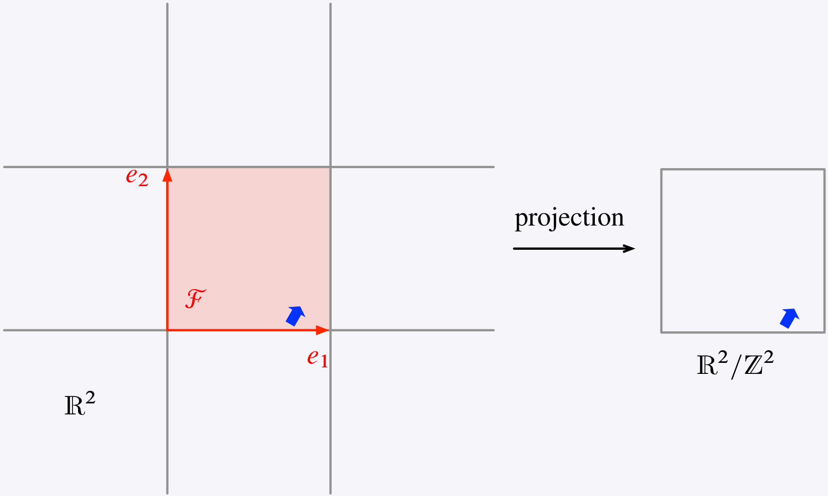
Here, the quotient space $\mathbb{R}^n/\Gamma$ is the space obtained by identifying all points in $\mathbb{R}^n$ that differ only by an element of Γ:

$$x_1 \sim x_2 \quad \Leftrightarrow \quad x_1 = \gamma \cdot x_2 \text{ for some } \gamma \in \Gamma.$$

Quotient space: Torus, $\Gamma = \mathbb{Z}^2$



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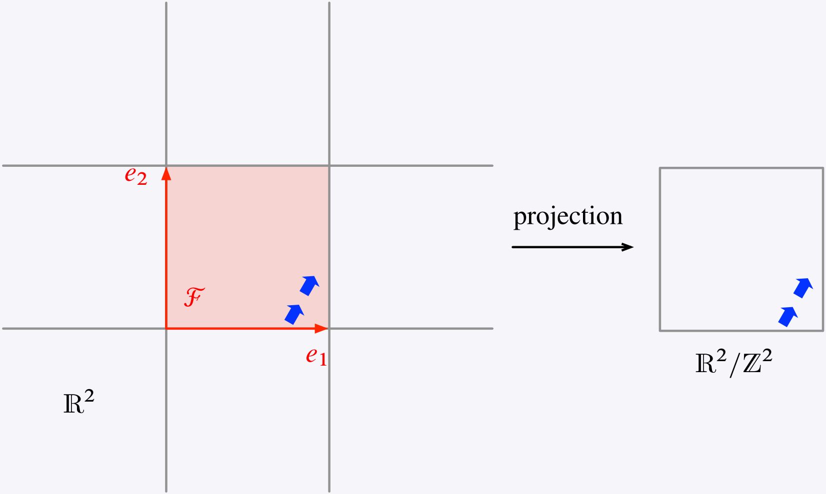

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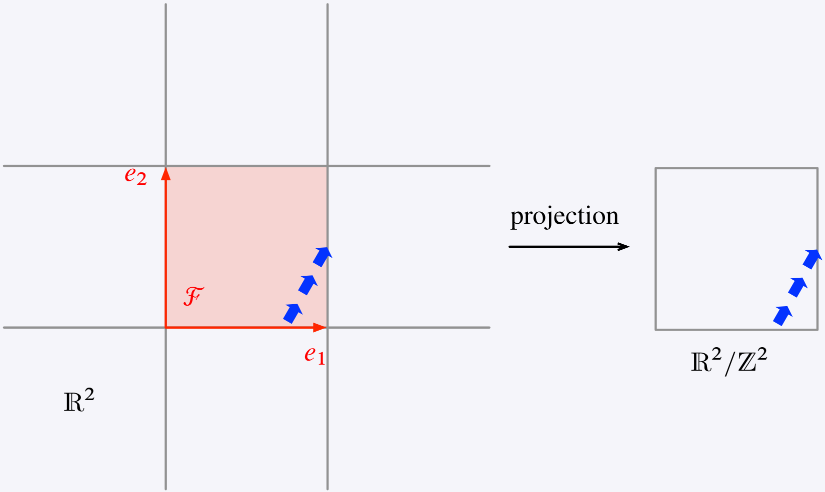

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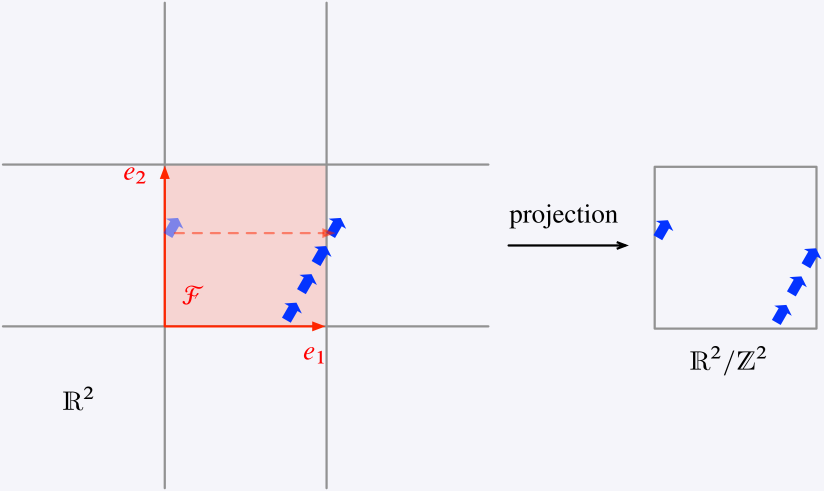

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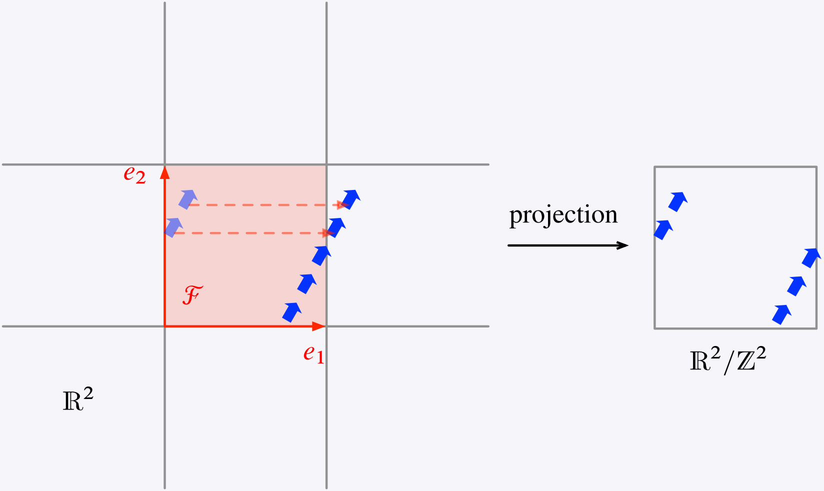

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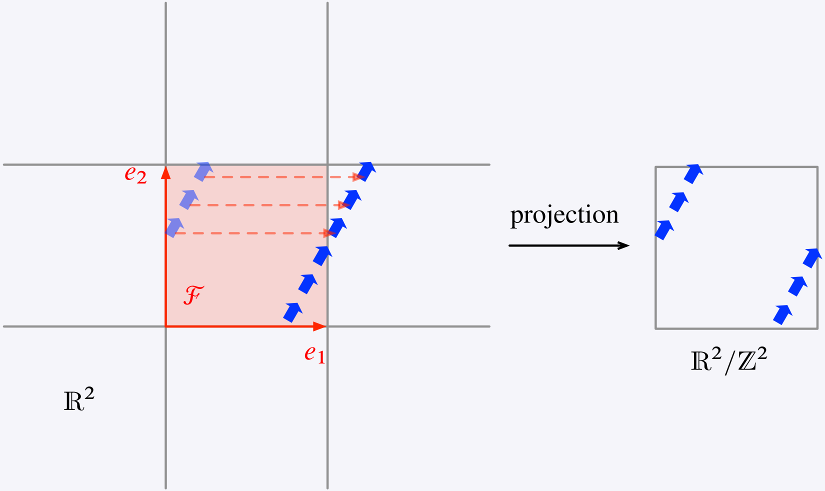

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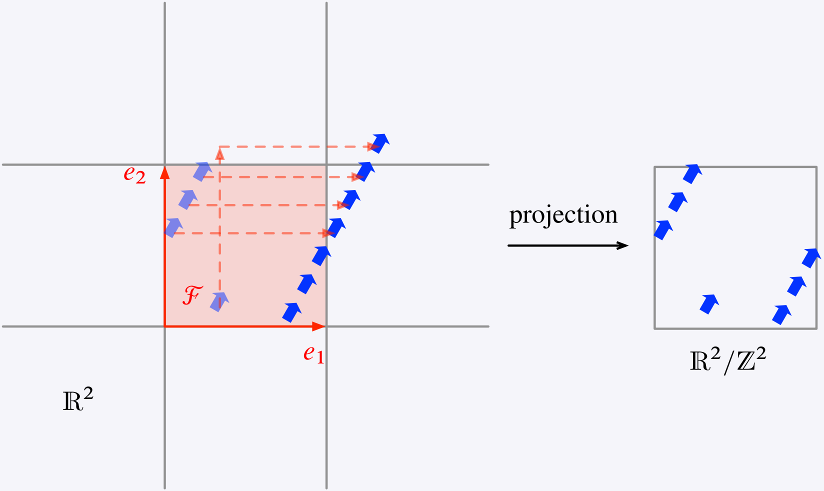

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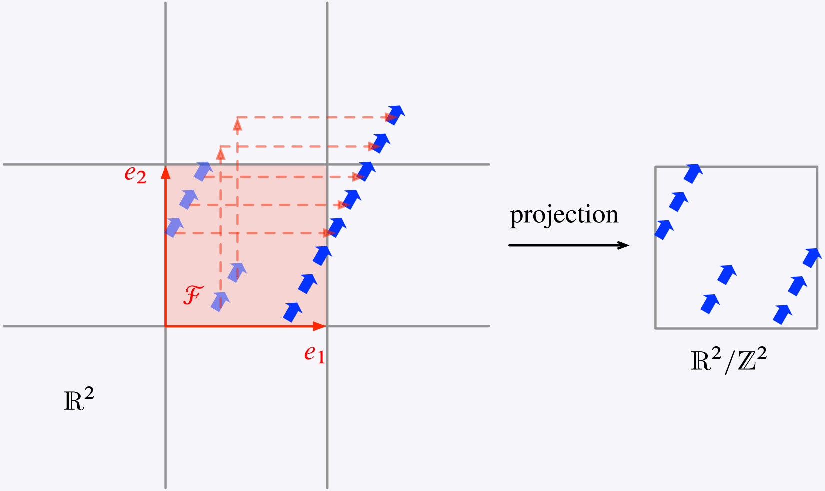

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1. *Let M be a compact complete connected flat Riemannian manifold. Then the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ is a finite Riemannian cover of M, and the holonomy group Θ of M is finite.*

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2. *Let $M_1 = \mathbb{R}^{n_1}/\Gamma_1$ and $M_2 = \mathbb{R}^{n_2}/\Gamma_2$ be a compact complete connected flat Riemannian manifolds.*
   Then $\Gamma_1 \cong \Gamma_2$ if and only if M_1 and M_2 are affinely equivalent.

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3. *For a given dimension n, there are only finitely many equivalence classes of compact complete connected flat Riemannian manifolds.*

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Examples of Lie groups

$$\mathbb{R}^n \quad \text{(commutative, translations)}$$

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) \mid \det(A) = 1\} \quad \text{(volume-preserving, linear)}$$

$$\mathrm{O}_n = \{A \in \mathrm{Mat}_n(\mathbb{R}) \mid AA^\top = I_n\} \quad \text{(rigid, linear)}$$

$$R = \left\{ \begin{pmatrix} \lambda_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix} \;\middle|\; \lambda_i \in \mathbb{R}^\times \right\}, \quad N = \left\{ \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix} \right\} \subset R.$$

Let G be a Lie group.
The commutator measures the failure of the group product to be commutative,

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Main types of Lie groups:

- Semisimple: Maximally non-commutative, $[G, G] = G$ (modulo discrete subgroups).
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Levi decomposition

For an arbitrary Lie group G,

$$G = SR$$

with a semisimple subgroup S and a solvable normal subgroup R.

Lattices in Lie groups

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Definition

A lattice in a Lie group G is a subgroup Γ such that

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As with crystallographic group, we may ask for arithmeticity and rigidity properties.

A Lie group N is nilpotent if for large enough k

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Example (Heisenberg group)

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & \frac{k}{a} & \frac{n}{c} \\ 0 & 1 & \frac{m}{b} \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; k, m, n \in \mathbb{Z} \right\} \text{ for fixed } a, b, c \in \mathbb{Z} \setminus \{0\}.$$

Γ is a matrix group over $\mathbb{Z}[\frac{1}{abc}]$, with finite index subgroup $N(\mathbb{Z}) = N \cap \mathrm{Mat}_3(\mathbb{Z})$.

Theorem (A.I. Malcev, 1949)

Lattices in nilpotent Lie groups are rigid and arithmetic:

1. If a lattice Γ_1 in N_1 is isomorphic to a lattice Γ_2 in N_2, then this isomorphism extends to an isomorphism of N_1 and N_2.

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3. A lattice in N exists if and only if its Lie algebra has a basis with structure constants in $\mathbb{Q}$.

(Technical remark: Assume the Lie groups are connected and simply connected.)

A Lie group R is solvable if there exists a sequence of normal subgroups

$$R = R_0 \supsetneq R_1 \supsetneq \ldots \supsetneq R_k = \{1\}$$

such that $[R_{i-1}, R_{i-1}] \subset R_i$ for $i = 1, \ldots, k$. (In particular, nilpotent $\Rightarrow$ solvable.)

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with group product for $r \in \mathbb{R}$, $n \in N$ given by

$$r \cdot n = r \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cos(2\pi r)x - \sin(2\pi r)y & z \\ 0 & 1 & \sin(2\pi r)x + \cos(2\pi r)y \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

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Note: $\Gamma_1 \cong \Gamma_2$ but $R_1 \not\cong R_2$.

Lattices in solvable Lie groups

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The group R_2 is called the oscillator group. A lattice in R_2 is

$$\Gamma_2 = \mathbb{Z} \times N(\mathbb{Z}).$$

Note: $\Gamma_1 \cong \Gamma_2$ but $R_1 \ncong R_2$. Lattices in solvable Lie groups are not rigid in general!

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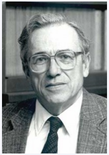

Theorem (G.D. Mostow, 1954)

Let R be a connected solvable Lie group with lattice Γ.

1. R/Γ *is compact.*
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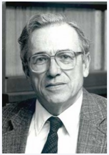

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Lattices in solvable Lie groups have a certain "arithmeticity property":

Theorem (G.D. Mostow, 1970)

If R is a connected and simply connected solvable Lie group with lattice Γ, then R has an injective matrix representation such that Γ is represented by integer matrices.

Lattices in semisimple Lie groups

A Lie group S is simple if its only connected normal subgroups are $\{1\}$ and G itself.
A Lie group G is semisimple if it is the product

$$G = S_1 \cdots S_m$$

of simple Lie groups $S_1, \ldots, S_m$, where $[S_i, S_k] = \{1\}$ and $S_i \cap S_k$ finite if $i \neq k$.

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Examples
$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) \mid \det(A) = 1\}$ and $\mathrm{SO}_n = \{A \in \mathrm{SL}_n(\mathbb{R}) \mid AA^\top = I_n\}$ are both simple Lie groups.

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- SO_n is itself compact, so every discrete (hence finite) subgroup is a lattice in SO_n.

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Definition

Let Γ be a lattice in a semisimple Lie group G. Let $G(\mathbb{Z}) = G \cap \mathrm{Mat}_n(\mathbb{Z})$. Then Γ is called arithmetic if

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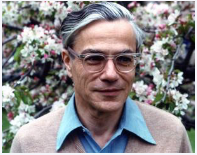

Theorem (A. Borel, 1963)
Let G be a non-compact semisimple Lie group. Then there exist both cocompact and non-cocompact arithmetic lattices in G.

Technical condition: The rank (dimension of a maximal real-diagonalizable subgroup) of G is ≥ 2.

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Theorem (G.A. Margulis, 1984)
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Theorem (G.D. Mostow, 1973, G.A. Margulis, 1975)
Let G_1, G_2 be connected semisimple Lie groups. Assume:

- *G_1 and G_2 both have trivial center and no compact simple factors,*
- *$\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ are lattices,*
- *Γ_1 is irreducible and rank $G_1 \geq 2$.*

Then any isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ extends to an isomorphism $\tilde{\varphi} : G_1 \to G_2$.

Literature

- O. Baues, B. Klopsch, Deformations and rigidity of lattices in solvable Lie groups, Journal of Topology 6, 2013 (4)
- A. Borel, Compact Clifford-Klein forms of symmetric spaces, Topology 2, 1963 (2)
- A.I. Malcev, On a class of homogeneous spaces, American Mathematical Society Translations 39, 1951
- G.A. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer 1991
- G.D. Mostow, Factor Spaces of Solvable Groups, Annals of Mathematics 60, 1954 (1)
- G.D. Mostow, Representative Functions of Discrete Groups and Solvable Arithmetic Subgroups, American Journal of Mathematics 92, 1970 (1)
- A.L. Onishchik, E.B. Vinberg (Eds.), Lie Groups and Lie Algebras II, Encyclopedia of Mathematical Sciences, Springer 2000
- M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer 1972
- A. Starkov, Rigidity problem for lattices in solvable Lie groups, Proceedings of the Indian Academy of Sciences 104, 1994
- W.P. Thurston, Three-dimensional geometry and topology, Princeton University Press 1997
- H. Weyl, Symmetry, Princeton University Press 1952
- R. Zimmer, Ergodic Theory and Semisimple Groups, Birkhäuser 1984