Density of subgroups and invariance properties of bilinear forms on Lie groups

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- Special case: M is compact.
  The pseudo-Riemannian metric automatically provides a finite measure.

What is interesting?

- Spaces with finite G-invariant measure are “almost classifiable” (Gromov).
- The finite invariant measure puts sufficient constraints on G and H to hope for reasonable structure theorems.
- Non-compact Lie groups G that preserve a finite measure have interesting algebraic and dynamical properties.

Density of subgroups

Finite G-invariant measure

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Two ingredients:

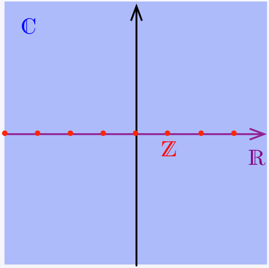
- A good notion of density ("Zariski density").
- Its relation to subgroups of finite covolume.

Zariski topology

Example

Consider $\mathbb{Z}$ as a subset of $\mathbb{C}$.

- $\mathbb{Z}$ is a "small" subset of $\mathbb{C}$ (zero measure).
- But every polynomial function on $\mathbb{C}$ is completely determined by its values on $\mathbb{Z}$.
- Analogous: Any continuous function is completely determined by its values on a dense subset.



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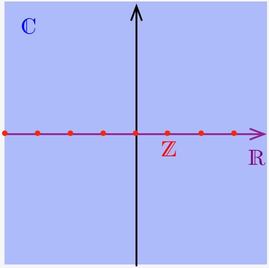

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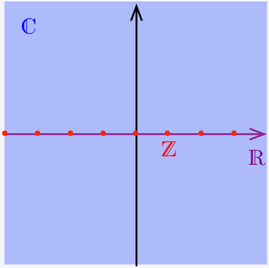

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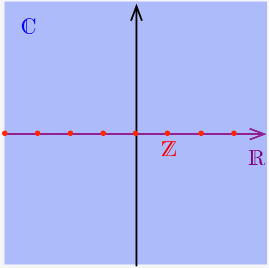

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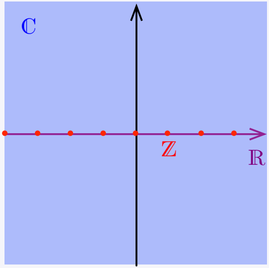

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- Zariski-closed sets have positive codimension.
- Zariski-open sets are "large".

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- $\mathrm{O}(n, \mathbb{C})$, zero set of equations $AA^\top = I_n$.
- $\mathrm{GL}(n-1, \mathbb{C})$, zero set of equations $\det(A)a_{nn} - 1 = 0$, $a_{in} = a_{ni} = 0$ for embedding $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}$ into $\mathrm{GL}(n, \mathbb{C})$.

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Fact

For any subgroup H of G, its (real) Zariski closure $\overline{H}^{Z}$, the smallest Zariski-closed subset in G (in G) containing H, is a (real) algebraic subgroup of G (of G).

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We say $X \subseteq H$ is Zariski-dense in H if $\overline{X}^{\mathrm{Z}} = \overline{H}^{\mathrm{Z}}$ in G (or in G).

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Theorem (Malcev, 1951)

Let N be a simply connected nilpotent Lie group.

- *N contains a lattice if and only if its Lie algebra has a basis with rational structure constants.*
- *Any lattice in N is Zariski-dense.*
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However, if G is solvable, then there may exist lattices that are not Zariski-dense.
For example, $\mathbb{Z}^n$ is a lattice in $G = \mathrm{S}^1 \ltimes \mathbb{R}^n$, but $\overline{\mathbb{Z}^n}^{Z} = \mathbb{R}^n \neq G$.

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(Note: H is not necessarily discrete.)

Furstenberg Lemma

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- For simplicity, assume $G \subseteq \mathrm{GL}(V)$, $\sigma = \mathrm{id}$. Write $G^* = \overline{G}^{\mathrm{z}}$, $H^* = \overline{H}^{\mathrm{z}}$.

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- $H^* \supseteq \ker(G^* \to \varrho(G^*) \to \overline{\varrho}(G^*))$. □

Invariance properties of symmetric bilinear forms

Induced symmetric bilinear form

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Ingredients:

- Metric g induces a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of G.
- Density properties of the stabilizer H imply the peculiar nil-invariance of $\langle\cdot,\cdot\rangle$.
- Understand how nil-invariance and the usual invariance of bilinear forms on Lie algebras are related.

Invariant bilinear forms

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Example

On every Lie algebra, the Killing form κ is invariant,

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

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An invariant Lorentzian scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{osc}_4$ is defined by

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   These (and higher-dimensional analogues) are the only non-abelian solvable Lie algebras with invariant Lorentzian scalar product.

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Example

The oscillator algebra $\mathfrak{osc}_4$ is a double extension of $\mathfrak{g} = \mathbb{R}^2$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \langle \cdot, \cdot \rangle_{\text{std}}$, by $\mathfrak{b} = \mathbb{R}a$, $\mathfrak{b}^* = \mathbb{R}z$, with

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Classification (Kath & Olbrich, 2003-2006)

- a general classification scheme for Lie algebras with invariant scalar product (however, somewhat impractical for concrete application),
- complete classifications for metric signature $(n-2,2)$ and $(n-3,3)$,
- complete classification for nilpotent Lie algebras with invariant scalar product in dimension ≤ 10.

The induced bilinear form

Recall our original geometric motivation:
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The induced bilinear form

Recall our original geometric motivation:
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- $\langle \cdot, \cdot \rangle$ is H-invariant (and thus $\mathfrak{h}$-invariant),

$$\begin{aligned} \langle \mathrm{Ad}_{\mathfrak{g}}(h)x, \mathrm{Ad}_{\mathfrak{g}}(h)y \rangle &= \langle x, y \rangle, \quad &&\text{for all } h \in H \\ \langle \mathrm{ad}_{\mathfrak{g}}(h')x, y \rangle + \langle x, \mathrm{ad}_{\mathfrak{g}}(h')y \rangle &= 0 \quad &&\text{for all } h' \in \mathfrak{h}. \end{aligned}$$

Nil-invariance

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- In general, nil-invariant $\neq$ invariant.

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$M = G/H$ is a pseudo-Riemannian homogeneous space of finite volume, and $G \subseteq \mathrm{Iso}(M, \mathrm{g})$. Let $\langle \cdot, \cdot \rangle$ denote the bilinear form on $\mathfrak{g}$ induced by g.

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  This implies that every nilpotent element in the Lie algebra of $\overline{\mathrm{Ad}_{\mathfrak{g}}(G)}^{\mathrm{z}}$ is skew-symmetric for $\langle\cdot,\cdot\rangle$. □

Nil-invariant forms on solvable Lie algebras

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Nil-invariant forms on arbitrary Lie algebras

Let $\mathfrak{g}$ be a Lie algebra with a nil-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$, and consider a Levi decomposition

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- For $\mathfrak{r}$, invariance follows from the previous theorem.

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2. *$\langle \cdot, \cdot \rangle$ is invariant by the adjoint action of $\mathfrak{s} \ltimes \mathfrak{r}$.*

Proof

- For $\mathfrak{r}$, invariance follows from the previous theorem.
- For $\mathfrak{s}$-invariance, essentially use the fact that $\mathfrak{s}$ is generated by ad-nilpotent elements.

Nil-invariant forms on arbitrary Lie algebras

Let $\mathfrak{g}$ be a Lie algebra with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, and consider a Levi decomposition

$$\mathfrak{g} = (\mathfrak{k} \times \mathfrak{s}) \ltimes \mathfrak{r},$$

where

- $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$,
- $\mathfrak{k}$ is a semisimple subalgebra of compact type,
- $\mathfrak{s}$ is a semisimple subalgebra without ideals of compact type.

Theorem (Baues, Globke & Zeghib, 2018)

Let $\mathfrak{g}$ be a Lie algebra with a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. Then:

1. *$\langle\cdot,\cdot\rangle$ restricted to $\mathfrak{s} \ltimes \mathfrak{r}$ is invariant by the adjoint action of all of $\mathfrak{g}$.*
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Proof

- For $\mathfrak{r}$, invariance follows from the previous theorem.
- For $\mathfrak{s}$-invariance, essentially use the fact that $\mathfrak{s}$ is generated by ad-nilpotent elements.
- Some more tricky arguments do the rest. □

Theorem (Baues, Globke & Zeghib 2018)

Let $\mathfrak{g}$ *be a Lie algebra with a* nil-invariant *symmetric bilinear form* $\langle\cdot,\cdot\rangle$ *of* signature $(n-s, s)$ *with* $s \leq 2$. *Then:*

1. $\mathfrak{g} = \mathfrak{k} \times \mathfrak{s} \times \mathfrak{r}$.
2. $\ker\langle\cdot,\cdot\rangle \subseteq \mathfrak{k} \times \mathfrak{z}(\mathfrak{r})$ *and* $\ker\langle\cdot,\cdot\rangle \cap \mathfrak{r} = \mathbf{0}$.

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Application
Classification of Lie algebras with nil-invariant $\langle\cdot,\cdot\rangle$ in signatures $(n-1,1)$ and $(n-2,2)$.

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Remark
Counterexamples (non-trivial!) show that the above theorem does not generalize to $(n-3,3)$.