Holonomy groups of flat pseudo-Riemannian homogeneous spaces

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Discrete Groups and Geometric Structures V

I Basic definitions and facts

Definition

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Let $\mathbb{R}^n_s$ denote $\mathbb{R}^n$ with a symmetric non-degenerate bilinear form represented by

$$\begin{pmatrix} \mathrm{I}_{n-s} & 0 \\ 0 & -\mathrm{I}_s \end{pmatrix},$$

where s is the signature (and $n - s \geq s$).

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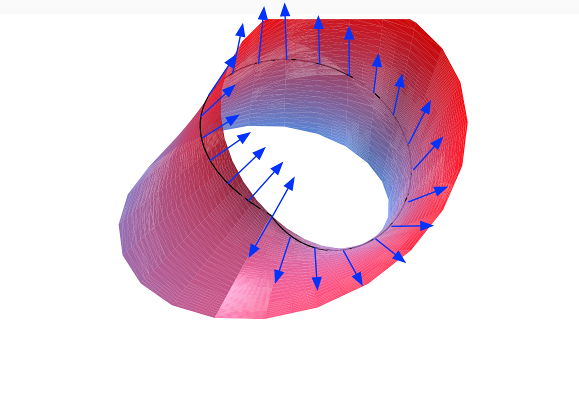
M geodesically complete:

- $\mathfrak{D} = \mathbb{R}^n_s$ (Killing-Hopf Theorem)
- Γ is the fundamental group

Linear holonomy



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II. $\Gamma_1 \cong \Gamma_2 \quad \Leftrightarrow \quad \Gamma_1$ *and* Γ_2 *affinely equivalent.*

III. *For given dimension n, there exist only finitely many (affine equivalence classes of) crystallographic groups.*

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- Reduction to compact case not possible for $s > 0$.
- Γ not virtually abelian (though often virtually polycyclic).
- $\rightsquigarrow$ study M with special properties.

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- *$[\gamma_1, \gamma_2] = (\mathrm{I} + 2A_1A_2, 2A_1v_2)$.*

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- $[\gamma_1, \gamma_2] = (\mathrm{I} + 2A_1A_2, 2A_1v_2)$.
- Γ *abelian in signatures* 0, 1, 2.

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4. And what about the compact case?

Baues, 2010:

- Examples of non-abelian Γ with abelian $\mathrm{LIN}(\Gamma)$.
- Compact M always has abelian $\mathrm{LIN}(\Gamma)$.

II Non-abelian holonomy groups

Matrix representation

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Theorem

The holonomy group $\mathbf{Hol}(M)$ *can be represented as*

$$\textsc{lin}(\gamma) = \begin{pmatrix} \mathrm{I}_k & -B^\top \tilde{\mathrm{I}} & C \\ 0 & \mathrm{I}_{n-2k} & B \\ 0 & 0 & \mathrm{I}_k \end{pmatrix},$$

where $C \in \mathfrak{so}_k$*, and* $-B^\top \tilde{\mathrm{I}} B = 0$*, where* $\tilde{\mathrm{I}}$ *defines a non-degenerate bilinear form on a certain subspace of* $\mathbb{R}^n$*.*

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where $C \in \mathfrak{so}_k$, *and* $-B^\top \tilde{\mathrm{I}} B = 0$, *where* $\tilde{\mathrm{I}}$ *defines a non-degenerate bilinear form on a certain subspace of* $\mathbb{R}^n$.

$\mathbf{Hol}(M)$ *is abelian* $\quad \Leftrightarrow \quad B = 0$ *for all* $\gamma \in \Gamma$.

Dimensions bounds I

Theorem

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$$s \geq 4.$$

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- Together: Subspace of signature $\geq 2 + 2 = 4$ exists. □

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So both dimension bounds are sharp.

III Fundamental groups of complete flat pseudo-Riemannian homogeneous spaces

Malcev hull

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G is called the Malcev hull of Γ.

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- Set $G = H \ltimes_{\mathrm{Ad}^*} \mathfrak{h}^*$ and define a flat bi-invariant inner product by $\langle (X,\xi),(Y,\eta)\rangle = \xi(Y) + \eta(X)$.

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- The action of $\gamma \in \Gamma$ on G by $\gamma.(h,\xi) = (\gamma h, \mathrm{Ad}^*(\gamma)\xi)$ is isometric.
- So $M = G/\Gamma$ is a flat pseudo-Riemannian homogeneous manifold. □

IV Incomplete pseudo-Riemannian homogeneous spaces

Translational isotropy

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$\mathfrak{D} \subset \mathbb{R}^n_s$ is called translationally isotropic if

$$T^\perp \subset T.$$

Classification of incomplete manifolds

Theorem (Duncan-Ihrig, 1992)
Every translationally isotropic domain $\mathfrak{D} \subseteq \mathbb{R}^n_s$ *is of the form*

$$\mathfrak{D} = \mathbb{R}^k \times \mathbb{R}^{n-2k} \times \mathfrak{A},$$

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Theorem (Duncan-Ihrig, 1993)
Classification of $M = \mathfrak{D}/\Gamma$ in signature 2 with translationally isotropic $\mathfrak{D}$.

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Proof:

- $U = \sum_{A \in \mathrm{LIN}(\Gamma)} \operatorname{im} A$ is totally isotropic and $U + \mathfrak{D} = \mathfrak{D}$.

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- Abelian holonomy $\Leftrightarrow$ $U^\perp = \bigcap_{A \in \mathrm{LIN}(\Gamma)} \ker A$ centralises Γ.

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- Abelian holonomy $\Leftrightarrow$ $U^{\perp} = \bigcap_{A \in \mathrm{LIN}(\Gamma)} \ker A$ centralises Γ.
- $\mathfrak{D}$ is open orbit of the centraliser of Γ, so $U^{\perp} + \mathfrak{D} = \mathfrak{D}$.
- So if $v + \mathfrak{D} \not\subset \mathfrak{D}$, then $v \notin U^{\perp}$. Then $v \not\perp U \subset T$, so T is translationally isotropic. □

Full classification

Corollary

The Duncan-Ihrig classification is the full classification of flat homogeneous spaces in signature 2.

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