Étale representations of reductive algebraic groups

Wolfgang Globke

School of Mathematical Sciences

THE UNIVERSITY of ADELAIDE

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Motivation: Left-symmetric algebras

Left-symmetric products

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$$x(yz) - (xy)z = y(xz) - (yx)z.$$

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Question

Given a Lie algebra $\mathfrak{g}$, does its Lie product come from a left-symmetric product on $\mathfrak{g}$?

- Semisimple $\mathfrak{g}$ does not admit a left-symmetric product.
- Many (not all) solvable/nilpotent $\mathfrak{g}$ admit left-symmetric products.
- Some reductive $\mathfrak{g}$ admit left-symmetric products.

Let $\varrho : \mathfrak{g} \to \mathfrak{aff}(V)$ be a finite-dimensional representation of $\mathfrak{g}$.

ϱ or (ϱ, V) is called étale if there exists $v_0 \in V$ such that

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Conversely, a left-symmetric product on $\mathfrak{g}$ defines an étale representation ϱ with $v_0 = 0 \in \mathfrak{g}$ via

$$\varrho(x) = \begin{pmatrix} L_x & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(\mathfrak{g}).$$

Prehomogeneous modules and relative invariants

Let G be an algebraic group, V a finite-dimensional $\mathbb{C}$-vector space, and $\varrho : G \to \mathrm{GL}(V)$ a rational representation such that G has a Zariski-open orbit. Then (G, ϱ, V) is a prehomogeneous module.

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Clearly,

$$\dim G \geq \dim V .$$

If " $=$ ", then $\varrho' : \mathfrak{g} \to \mathfrak{gl}(V)$ is a linear étale representation.

Castling

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we obtain another prehomogeneous module, its castling transform

$$\big(G \times \mathrm{GL}_{m-n},\ \varrho^* \otimes \omega_1,\ V^{m*} \otimes \mathbb{C}^{m-n}\big).$$

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Example
Identify a prehomogeneous module (G, ϱ, V) with $m = \dim V \geq 2$ with $(G \times \mathrm{SL}_1, \varrho \otimes \omega_1, V)$, and obtain a new prehomogeneous module $(G \times \mathrm{SL}_{m-1}, \varrho \otimes \omega_1, V \otimes \mathbb{C}^{m-1})$. Repeat to obtain

$$(G \times \mathrm{SL}_{m-1} \times \mathrm{SL}_{m^2-m-1}, \varrho \otimes \omega_1 \otimes \omega_1, V \otimes \mathbb{C}^{m-1} \otimes \mathbb{C}^{m^2-m-1}),$$

$$(G \times \mathrm{SL}_{m^2-m-1} \times \mathrm{SL}_{m^3-m^2-2m+1}, \varrho \otimes \omega_1 \otimes \omega_1, V \otimes \mathbb{C}^{m^2-m-1} \otimes \mathbb{C}^{m^3-m^2-2m+1}),$$

$$(G \times \mathrm{SL}_{m-1} \times \mathrm{SL}_{m^2-m-1} \times \mathrm{SL}_{m^4-2m^3+m-1}, \varrho \otimes \omega_1 \otimes \omega_1 \otimes \omega_1, V \otimes \mathbb{C}^{m-1} \otimes \mathbb{C}^{m^2-m}$$

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Proposition

(G, ϱ, V) *is prehomogeneous if and only if any absolute invariant is constant.*

Fact

G reductive: Every étale representation ϱ is linear.
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Certain classification results for "castling-reduced" reductive prehomogeneous modules by Sato, Kimura et al. are known:

- Sato, Kimura 1977:
  Irreducible, G reductive.
- Kimura 1983:
  Non-irreducible, $G = \mathrm{GL}_1^k \times S$ and S simple.
- Kimura et al. 1988:
  Non-irreducible, $G = \mathrm{GL}_1^k \times S_1 \times S_2$ with S_1, S_2 simple, Type I and Type II.

Regular (reductive) prehomogeneous modules

Given a relative invariant f, define

$$\varphi_f : V \backslash V_{\text{sing}} \to V^*, \quad x \mapsto \operatorname{grad} \log f(x).$$

If the image of φ_f is Zariski-dense, then (G, ϱ, V) is called a regular prehomogenous module.

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Theorem (Sato & Kimura 1977)

Let (G, ϱ, V) be a reductive prehomogeneous module. The following are equivalent:

1. (G, ϱ, V) *regular.*
2. $V_{\text{sing}} = \{v \in V \mid \operatorname{Hess} \log f(x) = 0\}$ *is a hypersurface.*
3. *The open orbit* $V \backslash V_{\text{sing}}$ *is an affine variety.*
4. *Each stabilizer* G_v *for* $v \in V \backslash V_{\text{sing}}$ *is reductive.*

Étale representations

Corollary

Let G be a reductive algebraic group.
If (G, ϱ, V) is étale, then it is a regular prehomogeneous module.

Proof:

- Stabilizer G_v is finite, hence reductive.
- Now use previous theorem. $\square$

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Proposition

Let G be an algebraic group with trivial rational character group.
Then G does not admit rational étale representations.

Proof:

- Trivial characters means only absolute invariants exist.
- Contradiction to existence of a relative invariant of degree $\dim V$ (Sato & Kimura). $\square$

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Corollary

Unipotent and semisimple algebraic groups do not admit rational linear étale representations.

Theorem

Let $k \geq 2$ and $(\mathrm{GL}_1 \times S, \varrho_1 \oplus \ldots \oplus \varrho_k, V_1 \oplus \ldots \oplus V_k)$ be an étale module, where

- *S semisimple,*
- *(ϱ_i, V_i) irreducible.*

Then each $(\mathrm{GL}_1 \times S, \varrho_i, V_i)$ is a non-regular prehomogeneous module.

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- $W \subseteq \pi^{-1}(\{0\})$, where $\pi : V \to V /\!\!/ S$ is the algebraic quotient, $V /\!\!/ S \cong \mathbb{C}$ and $\mathbb{C}[V]^S$ is generated by an irreducible non-constant polynomial f (Baues 1999).

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- So $\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}[W]^S = 0$, and $\dim W = \max\{\varrho(S)w \mid w \in W\}$ (Rosenlicht 1963). This means W is a prehomogeneous S-module.

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- $\mathbb{C}[W]^S = \mathbb{C}$ implies that there are no non-constant relative invariants for $\mathrm{GL}_1 \times S$. Therefore, W is non-regular. $\square$

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Example 2

The non-irreducible module

$$(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\textstyle\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^2)$$

is étale. The first irreducible component, $\omega_2 \otimes \omega_1$, is a regular irreducible module (by Sato-Kimura classification).

Classification results and families of examples

Étale modules from Sato & Kimura

Sato and Kimura (1977) classified irreducible and castling-reduced reductive prehomogeneous modules.

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By checking for $G_v = \{1\}$, we find the following étale modules:

- $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\mathbb{C}^2)$.
- $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathbb{C}^2)$.
- $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^4)$.

Étale modules from Kimura

Kimura (1983) classified non-irreducible prehomogeneous modules for reductive groups with one simple factor.
By checking for $G_v = \{1\}$, we find the following étale modules:

- $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\mathbb{C}^n)^{\oplus n})$.
- $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\mathbb{C}^n)^{\oplus n+1})$.
- $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\mathbb{C}^n)^{\oplus n} \oplus \mathbb{C}^{n*})$.
- $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\mathbb{C}^2 \otimes \mathbb{C}^2)$.

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Example

Let $\varrho : G \to \mathrm{GL}(V^m)$ be a rational representation $n \in \mathbb{N}$ with $\dim V = m \leq n$.
Then $(G \times \mathrm{GL}_n, \varrho \otimes \omega_1, V \otimes \mathbb{C}^n)$ is prehomogeneous.
Identify $V^m \otimes \mathbb{C}^n$ with $\mathrm{Mat}_{m,n}$. We see that the action of $\{1\} \times \mathrm{GL}_n$ is sufficient to generate an open orbit. Such a module is called a trivial prehomogeneous module.

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Let $G = \mathrm{GL}_1^k \times S_1 \times S_2$ with S_1, S_2 simple.
If there is at least one non-trivial irreducible component, then (G, ϱ, V) is of Type I.
Otherwise, (G, ϱ, V) is of Type II.

Étale modules from Kimura et al., Type I

Kimura et al. (1988) classified non-irreducible prehomogeneous modules for reductive groups with two simple factors, and not all irreducible components a trivial prehomogeneous modules (Type I).
By checking for $G_v = \{1\}$, we find the following étale modules:

- $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2))$.
- $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^2)$.
- $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^2) \oplus \mathbb{C}^{5*} \oplus \mathbb{C}^{5(*)})$.
- $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\mathbb{C}^4 \otimes \mathbb{C}^3) \oplus V^5 \oplus \mathbb{C}^3)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (V^5 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^2)$.
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Theorem

If $(\mathrm{GL}_1^k \times S, \varrho, V)$ *for* $k \geq 1$ *and a simple group* S *is an étale module, then* $S = \mathrm{SL}_n$ *for some* $n \geq 1$.

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Theorem

There are no étale modules for $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$*, with* $n \geq 2$.

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If $(\mathrm{GL}_1^k \times S, \varrho, V)$ *for* $k \geq 1$ *and a simple group* S *is an étale module, then* $S = \mathrm{SL}_n$ *for some* $n \geq 1$.

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There are no étale modules for $\mathrm{GL}_1 \times \mathrm{SL}_n \times \mathrm{SL}_n$, *with* $n \geq 2$.

Theorem (Burde)
There are no étale modules for $\mathrm{GL}_1 \times \mathrm{SL}_n \times \overset{d}{\dots} \times \mathrm{SL}_n$, *with* $n \geq 2$ *and* $d \geq n^2 + 1$.

Conjecture
There are no étale modules for $\mathrm{GL}_1 \times \mathrm{SL}_n \times \overset{d}{\dots} \times \mathrm{SL}_n$, *with* $n \geq 2$ *and any* $d \in \mathbb{N}$.

Étale modules from Kimura et al., Type II

Kimura et al. (1988) classified non-irreducible prehomogeneous modules for reductive groups with two simple factors, and all irreducible components trivial prehomogeneous modules (Type II).

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…after several technical lemmas …and distinguishing several subclasses …find several unwieldy lists of étale representations of Type II.

Observation

Among all the preceding classifications, there are only three étale modules for groups with a simple factor other than SL_n:

- $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\mathbb{C}^4 \otimes \mathbb{C}^3) \oplus V^5 \oplus \mathbb{C}^3)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (V^5 \otimes \mathbb{C}^2) \oplus \mathbb{C}^4 \oplus \mathbb{C}^2)$.
- $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (V^5 \otimes \mathbb{C}^4) \oplus \mathbb{C}^4 \oplus \mathbb{C}^4)$.

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Conjecture

Sp_2 *is the only group other than* SL_m, $m \in \mathbb{N}$, *that appears as a simple factor in a reductive group which admits and étale representation.*

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