Locally Homogeneous pp-Waves

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I pp-waves and plane waves

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- timelike if $\mathrm{g}_p(v, v) < 0$.

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A plane-fronted gravitational wave propagates with light-speed in the x-direction. Its wave vector (in spacetime) is the parallel light-like vector field $\partial_t + \partial_x$.

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Locally, pp-waves can be defined by the existence of $n+2$ coordinates $(x^+, x_1, \ldots, x_n, x^-)$ such that

$$g = 2dx^+ dx^- + 2H(x^+, \mathbf{x})(dx^+)^2 + d\mathbf{x}^2,$$

where $H(x^+, \mathbf{x})$ is an abritrary profile function.

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History:

- Brinkmann, 1925: Einstein manifolds which are conformally equivalent.
- Einstein & Rosen, 1937: Gravitational waves.
- Today: Supergravity backgrounds with "many" symmetries.

II Locally homogeneous pp-waves

Local homogeneity

We want to study locally homogeneous pp-waves.
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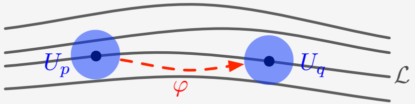
M is locally $V^\perp$-homogeneous if for all points p, q in a leaf $\mathcal{L}$ there is a local isometry $\varphi : U_p \to U_q$ mapping p to q, where U_p, U_q are neighbourhoods in M.



Jordan, Ehlers, Kundt (1960):
Let (M, g) *be a Ricci-flat (= vacuum) pp-wave of dimension* 4*.*
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We want to prove that a locally homogeneous pp-wave of dimension $n + 2$ is a plane wave.

Counterexample:
Consider $M = \mathbb{R}^3$ with the metric

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + 2\mathrm{e}^{2ax}(\mathrm{d}x^+)^2 + \mathrm{d}x^2.$$

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- The curvature operator R has rank > 1 almost everywhere.
- (M, g) is strongly indecomposable, meaning there is no neighbourhood of some $p \in M$ on which g is a product metric.

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III Analysing the Killing algebra

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After a good choice of coordinates and some labour, the Killing equation for a pp-wave is found to be

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A generic solution takes the form

$$X = (c - ax^- - \dot{\Psi}^\top \mathbf{x})\partial_- + (\Psi + F\mathbf{x})^i \partial_i + (ax^+ + b)\partial_+.$$

$V^{\perp}$-homogeneous solutions

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In suitable coordinates and for a suitable choice of $X_1, \ldots, X_n$:

$$X_k|_p = E_k,$$
$$\nabla_{E_j} X_k|_p \in \mathbb{R}E_-,$$
$$\nabla_{E_+} X_k|_p = a_k E_+ + \ldots$$

for some numbers $a_1, \ldots, a_n$.

Integrability conditions

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This entails the following integrability condition:

$$\nabla_X \mathrm{R} = (\nabla X)\mathrm{R}.$$

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Therefore, the plane wave condition is satisfied.

IV Symmetries of plane waves

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This equation admits solutions $\Psi_1, \ldots, \Psi_n$, $\Phi_1, \ldots, \Phi_n$ with

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Killing equation of a plane wave

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Then the Killing fields

$$X_i = \Psi_i^k \partial_k - \mathbf{x}^\top \dot{\Psi}_i \partial_-, \quad Y_i = \Phi_i^k \partial_k - \mathbf{x}^\top \dot{\Phi}_i \partial_-.$$

satisfy

$$[X_i, Y_j] = \delta_{ij}\partial_-.$$

So they generate a Heisenberg algebra $\mathfrak{hei}_{2n+1}$.

Homogeneous plane waves

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Proposition:
Homogeneous plane waves are reductively homogeneous.

Symmetric plane waves

If (M, g) is homogenous of the first type and the matrix S is constant, then (M, g) is a Lorentzian symmetric space (also: Cahen-Wallach spaces).

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In this case, ∂_+ is a Killing field transversal to $V^\perp$, and

$$\partial_+, \quad X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n, \quad \partial_-$$

span a $2n+2$-dimensional oscillator algebra $\mathbb{R} \ltimes \mathfrak{hei}_{2n+1}$.

- M. Blau, M. O'Loughlin,
  Homogeneous plane waves, Nuclear Phys. B 654, 2003
- H.W. Brinkmann,
  Einstein spaces which are mapped conformally to each other, Math. Ann. 94, 1925
- J. Ehlers, W. Kundt,
  Exact solutions of the gravitational field equations in *Gravitation: An introduction to current research*, ed. L. Witten, Wiley 1962
- A. Einstein, N. Rosen,
  On gravitational waves, J. Franklin Inst. 223, 1937
- P. Jordan, J. Ehlers, W. Kundt,
  Strenge Lösungen der Feldgleichungen der allgemeinen Relativitätstheorie, Akad. Wiss. Mainz Math.-Nat. Kl., 1960
- R. Sippel, H. Goenner,
  Symmetry classes of pp-waves, Gen. Relativity Gravitation 18, 1986